A Field Theory for Finite Dimensional Site Disordered Spin Systems

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We present a new field theoretic approach for finite dimensional site disordered spin systems by introducing the notion of grand canonical disorder, where the number of spins in the system is random but quenched. We analyze this field theory using the variational replica formalism. For a variety of interactions we find a spin glass phase (with continuous replica symmetry breaking) and explicitly discuss a three dimensional system where this occurs. This approximation also suggests that any ferromagnetic transition occur within the spin glass phase.

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Most advances in the field of disordered spin systems have been based on models in which the bonds take random values. However in most experimental systems the positions of the spins are random but the interactions occur through deterministic potentials. Analytic studies of site-disordered spin systems, such as RKKY spin glasses and dilute ferromagnets, have been hampered by the lack of a suitable field theoretic model (however a lattice based formulation has been proposed [1]). By considering a situation in which the number of spins in the system is random but quenched we are able to write a replica field theory for site-disordered systems. This field theory seems to be simpler than many of those coming from bond-disordered and diluted lattice models and should be accessible to a many standard analytical techniques.

Generally, the site-disordered spin systems we are considering may be subject to two kinds of phase transition: ferromagnetic and spin glass (antiferromagnetic order is not possible without a lattice structure). In both cases the role of replica symmetry breaking (RSB) is of great interest, and we hope that our field theory may shed light on the following issues.

- It has been argued [2] that the renormalisation group flow to a ferromagnetic fixed point is disrupted to a RSB form in certain 3 dimensional dilute ferromagnetic models. The precise interpretation of this observation is as yet unclear.
- Although the RSB in the mean field theory for spin glasses is now well understood [3] and related to the proliferation of pure states of the system, in finite dimensions the picture is less clear. Alternative qualitative approaches based on droplets [4] view the spin glass phase as a disguised ferromagnetic phase with only two underlying fundamental states.

Firstly consider a model where the number of spins N is fixed: N spins S_i are placed randomly at positions r_i uniformly throughout a volume V. This type of disorder we refer to as canonical disorder, as the number of particles is the same for each realization of the disorder. The spins interact via a pairwise potential J depending only on the distance between the spins. The Hamiltonian is then given by

$$H = -\frac{1}{2}\sum_{i,j} J(r_i - r_j) S_i S_j. \tag{1}$$

Assuming that J is positive definite, making a Hubbard-Stratonovich transformation expresses the partition function as

$$\begin{aligned} Z_N = &\sum_{S_i}\int \mathcal{D}\phi [det J\beta]^{-\frac{1}{2}} \times \qquad (2)\\ &\exp\left(\frac{-1}{2\beta}\int\int \phi(r)J^{-1}(r-r')\phi(r')\ drdr' + \sum_i^N \phi(r_i)S_i\right)\end{aligned}$$

Employing replicas, we average out the site-disorder by integrating over the positions r_i using the flat measure: $\frac{1}{V^N}\int_V \prod dr_i$.

$$\begin{aligned}\overline{Z_N^n} = &\int \mathcal{D}\phi_a [det J\beta]^{-\frac{n}{2}} \times \\ &\exp\Big(-\frac{1}{2\beta}\int\int \sum_a \phi_a(r)J^{-1}(r-r')\phi_a(r')\ drdr' \\ &+ N\log\frac{1}{V}\int\prod_a^n 2\cosh\phi_a(r)\ dr\Big) \qquad (3)\end{aligned}$$

A field theoretic analysis of the above theory is complicated by the presence of the log term in the action. We overcome this difficulty by making a physically desirable modification to the definition of the disorder. In general one might expect the system to have been taken from a much larger system with a mean concentration of spins per unit volume, ρ. A suitably large subsystem of volume V will thus contain a number of spins N which is random and Poisson distributed: $p(N) = \exp(-\rho V)(\rho V)^N/N!$. This distribution must be used to weight the averaged free energy so we are led to define $\Xi^n = \sum_N p(N)\overline{Z}_N^n$. By analogy with the statistical mechanics of pure systems, we shall call this type of disorder "grand canonical disorder".

The resulting theory is simpler than (3) and is defined by

$$\Xi^n = \exp(-\rho V) \int \mathcal{D}\phi_a [det J\beta]^{-\frac{n}{2}} \times$$
$$\exp\Big(-\frac{1}{2\beta} \int\int \sum_a \phi_a(r) J^{-1}(r-r')\phi_a(r')\ drdr'$$
$$+ \rho \int \prod_a^n 2\cosh\phi_a(r)\ dr\Big). \tag{4}$$

Expanding the cosh one sees that the leading term corresponds to the random temperature or random mass, familiar from bond disordered approaches, and that depending on the choice of interaction one might expect similar renormalisation group results [5,2].

In order to relate this theory to measurable quantities we return to the original formulation of the model in equation (1) and identify physical operators. The spin density operator, $M_a(r) = \sum \delta(r-r_i)S_i^a$ is closely related to the field ϕ_a appearing in the theory. The equations of motion following from the replicated version of (2) show that the physical magnetization density is given by,

$$M = \frac{1}{V}\int dr \langle M_a(r)\rangle = \frac{1}{V}\langle \sum_i^N S_i^a\rangle\ = \frac{\langle\phi\rangle}{\beta\tilde{J}(0)}, \tag{5}$$

and the correlator $\langle M_a(r)M_b(r')\rangle$, is in terms of $G_{ab}(r-r') = \langle\phi_a(r)\phi_b(r')\rangle_c$, obtained as

$$\langle\tilde{M}_a(k)\tilde{M}_b(-k)\rangle_c = \frac{\tilde{G}^{ab}(k)}{\beta^2\tilde{J}^2(k)} - \frac{\delta^{ab}}{\beta\tilde{J}(k)}\ . \tag{6}$$

$M_a(r)$ is not, however, the operator sensitive to spin glass ordering, and it is natural to consider another operator $q_{ab}(r) = \sum_i \delta(r-r_a)S_i^a S_i^b$, related to the non-linear susceptibility. This new operator is composite and does not manifestly appear in the field theory (4), it is for this reason that we must go beyond mean field theory to obtain non-trivial results. Operators involving more spins can be introduced in the same way.

In the remainder of the letter we analyze this field theory with the Gaussian variational method. This method, otherwise known as Hartree Fock, is a truncation of the Schwinger Dyson equations and becomes exact in the limit of many spin components (such an m-component theory is treated in a separate publication [6]). In the context of disordered systems, this method has has success in calculating exponents for random manifolds [7], but one should bear in mind that important effects may occur at higher orders in $1/m$.

We allow the possibility of ferromagnetic order and make the ansatze that $\langle\phi_a(r)\rangle = \bar{\phi}_a$ and $\langle\phi_a(r)\phi_b(r')\rangle_c = G_{ab}(r-r')$ (by translational invariance). The variational free energy is given, up to constant terms, by

$$n\beta F_{var} = -\frac{1}{2}\mathrm{Tr}\log G_{ab} + \frac{1}{2\beta}\mathrm{Tr}\ G_{ab}J^{-1} \tag{7}$$
$$+ \frac{1}{2\beta}\sum_a \bar{\phi}_a^2 \tilde{J}^{-1}(0) - \rho\Omega,$$

where the above traces are both functional and on replica indices and where Ω is defined by

$$\sum_{S_a} \exp\left(\sum_a \bar{\phi}_a S_a + \frac{1}{2}\sum_{ab} G_{ab}(0)S_a S_b\right)\ . \tag{8}$$

As usual we do not expect breaking of replica symmetry on single-index objects and hence set $\bar{\phi}_a = \bar{\phi}$. The variational equations are

$$\bar{\phi}_a = \bar{\phi} = \beta\rho\tilde{J}(0)\Omega_a \tag{9}$$

and

$$\tilde{G}_{ab}^{-1}(k) = \frac{1}{\beta}\delta_{ab}\tilde{J}^{-1}(k) - \rho\Omega_{ab}. \tag{10}$$

where Ω_a and Ω_{ab} are traces of the type (8) containing respectively S_a and $S_a S_b$.

Within this approximation, by introducing a source for the operator $q_{ab}(r)$ and using the FDT theorem, we obtain an equation for the correlation function $Q_{abcd} = \langle q_{ab}(r)q_{cd}(0)\rangle$.

$$\tilde{Q}_{abcd}(k) = \rho\Omega_{abcd} + \frac{\rho}{2}\sum_{gh}\tilde{\Sigma}_{abgh}(k)\tilde{Q}_{ghcd}(k)$$
$$\tilde{\Sigma}_{abgh}(k) = \sum_{ef}\Omega_{abef}\int\frac{d^dp}{(2\pi)^d}\tilde{G}_{eg}(p)\tilde{G}_{fh}(k-p), \tag{11}$$

where Ω_{abcd} is another object of the type (8) containing four S's.

A replica symmetric (RS) anzatz for G leads to a regime specified by two order parameters: the magnetization M (5) and the Edwards Anderson order parameter q_{EA}. These parameters are determined by a pair of equations very similar to the mean field equations for the Sherrington Kirkpatrick (SK) model [8]

$$M = \frac{\rho}{\sqrt{2\pi}}\int d\xi e^{-\frac{\xi^2}{2}}\tanh\left(\beta\tilde{J}(0)M + \xi\sqrt{g_1}\right)$$
$$q = \frac{1}{\sqrt{2\pi}}\int d\xi e^{-\frac{\xi^2}{2}}\tanh^2\left(\beta\tilde{J}(0)M + \xi\sqrt{g_1}\right) \tag{12}$$

Where g_1 (the off diagonal part of $G_{ab}(0)$) is given by

$$g_1 = \rho\beta^2 q\int\frac{d^dk}{(2\pi)^d}\frac{\tilde{J}^2(k)}{\left(1-(1-q)\rho\beta\tilde{J}(k)\right)^2} \tag{13}$$

The simplest solution of these equations yields the high temperature, low density paramagnetic region with $q_{EA}=0$ and $M=0$. In this region, the two index correlator (6) is known and equation (11) can be solved for the most interesting correlator, $\langle q_{ab}(k)q_{ab}(-k)\rangle$

$$\frac{\rho\beta^2}{1-\rho\int\frac{d^dp}{(2\pi)^d}\frac{\tilde J(k)\tilde J(p-k)}{\left(1-(1-q)\rho\beta\tilde J(k)\right)\left(1-(1-q)\rho\beta\tilde J(p-k)\right)}} \tag{14}$$

This correlator is simply related to $\overline{\langle S(k)S(-k)\rangle^2} = \sum'_{ab}\langle q_{ab}(k)q_{ab}(-k)\rangle$ and the divergence in the above formula signals the onset of a spin glass phase. The divergence occurs on a line in the temperature density plane specified by $g_1=q$ and coincides with the AT line as determined by stability considerations [5,9]. Furthermore the phase boundary also coincides with the line on which the RS equations (12) develop solutions with non-zero q. This situation also occurs in the SK model and suggests a continuous breaking of replica symmetry.

It is useful to illustrate these results for specific interaction types and for the purposes of this letter we consider the following examples in d-dimensions

$$\text{Yukawa}_\pm: \quad \tilde J(k)=\pm\frac{1}{\mu^{d-2}(k^2+\mu^2)} \tag{15}$$

$$\text{RKKY}: \ \tilde{\mathrm{J}}(\mathrm{k})=\mu^{-\mathrm{d}}\theta(\mu-|\mathrm{k}|). \tag{16}$$

The two signs correspond to ferromagnetic Yukawa (FY) interactions and antiferromagnetic Yukawa (AY) interactions respectively (the use of a negative definite interaction changes the sign of G_{ab} in Ω). The dimensional constant μ merely sets the scale of the problem and can be set to 1. In 3 dimensions for Yukawa and RKKY respectively we obtain for $\langle q_{ab}(k)q_{ab}(-k)\rangle$

$$\mathrm{Y}_\pm \qquad \frac{\rho}{1-\frac{\rho\beta^2}{4\pi}\frac{1}{k}\sin^{-1}\left(\frac{k}{\sqrt{4(1\mp\beta\rho)+2k^2}}\right)} \tag{17}$$

$$\text{RKKY} \quad \begin{cases}\frac{\rho}{1-\frac{\rho}{96\pi^2}\left(\frac{\beta}{1-\rho\beta}\right)^2(k+4)(k-2)^2}, & \text{for } k<2;\\ \rho, & \text{for } k>2.\end{cases} \tag{18}$$

The spin glass phase boundary is given by $\rho = 2(4\pi T)^2(\mp T+\sqrt{T^2+1/(4\pi)^2})$ for the Yukawa and $\rho = (1+12\pi^2T-\sqrt{1+24\pi^2T})/(12\pi^2)$ for the RKKY case. The exponents associated with these transitions take their mean field values, except that $\eta=1$ for RKKY. Numerical inspection of the RS equations finds stable ferromagnetic solutions at low temperature, we illustrate the phase diagram in the RKKY case in figure 1.

In the context of the approximations made, the spin glass transitions found occur before any ferromagnetic ordering appears, the two index magnetic correlator remaining finite at the spin glass transition. In fact, the usual way of finding a continuous ferromagnetic transition in the RS equations (12) by looking at the derivative of the M equation always fails, and we take this to indicate that any such transition must occur within a RSB phase. Such a situation may be termed "magnetic censorship" and whilst seeming reasonable for RKKY and the AY which are frustrating interactions, the prediction of a spin glass transition before a ferromagnetic transition for a ferromagnetic interaction is potentially controversial. It would not be too surprising if this was an artifact of the Gaussian variational method. Indeed, in 3 dimensions one finds that the spin glass transition is extremely close to the mean field line for the ferromagnetic transition. Nevertheless the physics of this problem is very interesting [2] and we will discuss it at length in a separate publication [5]. These considerations do not affect our conclusions for the two frustrated interaction types RKKY and AY.

We shall look for continuous replica symmetry broken solutions and parameterize the off-diagonal part of the matrix $\tilde G_{ab}(k)$ by a continuous Parisi function $\tilde g(k,u)$ where $u\in[0,1]$, and a diagonal part denoted by $\tilde g_D(k)$. For such a matrix, Ω (8) is very similar to the free energy in the SK model; it cannot be obtained in a closed form and a standard strategy is to work close to the transition line by expanding Ω up to a term of $O(g^4)$ which in the SK model is the first term leading to a breaking of replica symmetry. The expansion is [10]

$$\frac{\Omega-1}{n}\approx\frac{1}{2}g_D(0)-\frac{1}{4}\int_0^1\left(g^2+\frac{1}{6}g^4-\frac{u}{3}g^3-g\int_0^u g^2\right)du \tag{19}$$

The remaining terms in the action are easily computed within the algebra of Parisi matrices [7]. The variational equations one obtains are

$$\begin{aligned}[\tilde g_D(k)]^{-1} &= +\rho\sigma_D(k)=\rho\left(\frac{\tilde J^{-1}}{\beta\rho}-1\right)\\ [\tilde g(u,k)]^{-1} &= -\rho\sigma(u)=2\rho\frac{\delta\Omega}{\delta\tilde g}\end{aligned} \tag{20}$$

Defining

$$\begin{aligned}D_D(k)&=\sigma_D+\langle\sigma\rangle\\ D(u,k)&=\sigma_D+\langle\sigma\rangle+[\sigma](u),\end{aligned} \tag{21}$$

(in the notation of [7]), the equations can be inverted to find

$$\begin{aligned}\tilde g_D(k) &= \frac{1}{\rho D_D(k)}\left(1+\int_0^1\frac{du}{u^2}\frac{[\sigma](u)}{D(u,k)}+\frac{\sigma(0)}{D_D(k)}\right)\\ \tilde g(u,k) &= \frac{1}{\rho D_D(k)}\left(\frac{[\sigma](u)}{uD(u,k)}+\int_0^u\frac{dv}{v^2}\frac{[\sigma](v)}{D(v,k)}+\frac{\sigma(0)}{D_D(k)}\right).\end{aligned} \tag{22}$$

Proceeding by differentiating the second equation of (20) with respect to u one obtains $\sigma'=0$ or

$$\left(1+g^2-ug-\int_u^1 g\right)=\rho\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{D^2}\right)^{-1} \tag{23}$$

Taking a second derivative in some region where equation (23) holds we find:

$$g = \alpha(u)u = \frac{u}{2}\left(1 + 2\rho^2 \frac{\int \frac{d^dk}{(2\pi)^d}\frac{1}{D^3}}{\left(\int \frac{d^dk}{(2\pi)^d}\frac{1}{D^2}\right)^3}\right). \tag{24}$$

For 4 or more dimensions, in the limit in which the short distance cutoff is removed, this equation is simple and we find a scenario similar to that found in the SK model. In general the function $\alpha(u)$ depends on $\sigma(u)$ and one obtains a first order nonlinear differential equation for $g(0,u)$. However in $d=3$ for Yukawa interactions there is a remarkable cancellation and α is a constant. For consistency with our perturbative analysis the region where $g(0,u)$ is non constant must be close to the origin. More precisely there must be a break point u_0 with small value, above which $g(0,u)$ is constant and equal to αu_0 if the solution is to be continuous. The breaking pattern scenario is reminiscent of the random manifold problem with long range disorder [7] and is qualitatively the same as found in the SK model near T_c [11]. The above derivation was for a positive definite type interaction and for AY there are a few sign changes. We find $\alpha = \frac{1}{2}(32\pi^2/\rho\beta^3 \pm 1)$ and for the AY case it is easy to see that below the spin glass transition α is everywhere negative. One finds also that to leading order $u_0 = 4\frac{\delta\beta}{\beta}\frac{(\rho\beta^3)(\rho\beta^3\pm 16\pi^2)}{(\rho\beta^3\pm 32\pi^2)^2}$ In general we expect that close to the origin $g(0,u) \sim u^{\chi}$, where χ, may be deduced from a power series analysis of the nonlinear differential equation about the origin, and that $g(0,u)$ is a constant above some u_0 close to the origin.

Despite having the full structure of the two index correlators $\tilde{g}(k,u)$, leading to non trivial momentum dependence in $\tilde{g}_D(k)$ related to the connected magnetic correlation function (6), the analysis only holds close to the spin glass transition and cannot address the exponents at the ferromagnetic transition itself. Indeed, at the ferromagnetic line, g will be large and the expansion of Ω is manifestly inappropriate. This problem is intimately related to the question of the relative locations of the two transitions in the ferromagnetic case. In addition the four index correlation functions Q_{abcd} contain much of the physics of the spin glass phase: for example the θ-exponent [4] may be extracted from the long distance behavior of such objects. Equation (11) is however an equation carrying four replica indices and the solution in the case of continuous replica symmetry breaking is technically rather formidable requiring extensions of the methods described in [12].

Although we have not been able to compute the exponent θ in the most interesting case where it receives essential contributions from the physics of RSB, we report the results found for m-component Heisenberg spins in the limit $m \to \infty$ [6]. In this case we expect our approximations to be exact, and obtain a similar picture to that described above: namely a high temperature phase separated from a spin glass phase at low temperature. The form of the spin glass phase is RS with q non-zero and the equations for Q_{abcd} may be solved to find that Q_{abab} stays critical below the transition with exponent θ given by $\theta = d-2$ for Yukawa and $\theta = d-1$ for RKKY (the bounds in [4] are for Ising spins).

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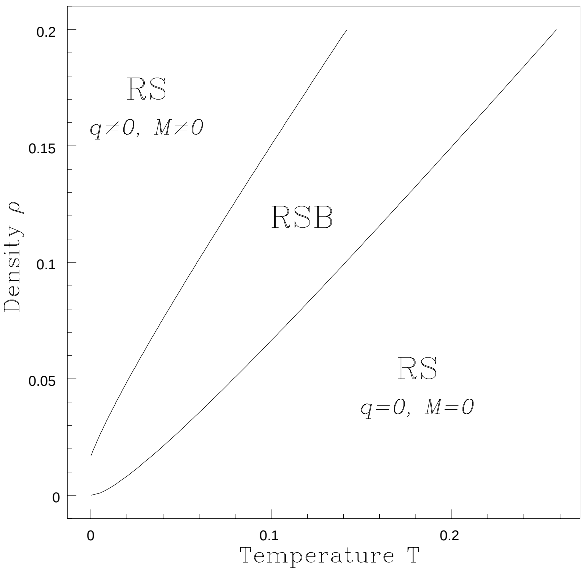

FIG. 1. Phase diagram for the 3 dimensional RKKY interaction using Hartree approximation.