Glassy Transition in the three-dimensional Random Field Ising Model

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June 17, 1994

PACS numbers : 05.20 - Statistical Mechanics
64.60 - Order-Disorder and Statistical Mechanics of Model Systems

Abstract

The high temperature phase of the three dimensional random field Ising model is studied using replica symmetry breaking framework. It is found that, above the ferromagnetic transition temperature T_f, there appears a glassy phase at intermediate temperatures $T_f < T < T_b$ while the usual paramagnetic phase exists for $T > T_b$ only. Correlation length at T_b is computed and found to be compatible with previous numerical results.

Although a great deal of work has been devoted to the understanding of the random field Ising model (RFIM) [1], some aspects still need to be cleared up. It is now well-known that, in dimension $D = 3$, long range order is present at sufficiently low temperature and weak random fields with non trivial critical exponents [2], the upper critical dimension of the RFIM being $D = 6$. Nevertheless, perturbation theory leads to dimensional reduction (critical exponents are incorrectly predicted to be equal to those of the corresponding pure model in dimension $D - 2$) [3] and therefore does not succeed in describing the critical behaviour of the RFIM. The reason of this failure presumably stems from the very complicated energy landscape due to the quenched disorder, and more precisely, from the existence of a huge number of local minima of the free energy in the space of local magnetisations that usual perturbative expansions do not take into account [4]. Numerical simulations and solutions of the mean-field equations

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corroborate this picture [5, 6]. Above the ferromagnetic transition temperature T_f, there seems to appear an intermediate "glassy" regime for $T_f < T < T_b$ where many solutions of the local magnetisations mean-field equations coexist, while only one of them persists in the paramagnetic phase $T > T_b$. From the theoretical point of view, it was suggested that the techniques of replica symmetry breaking (RSB), which proved to be successful in the mean-field theory of spin glasses [7] where such complicated free energy landscapes arise, could also be applied to the RFIM [8]. Experiments made on diluted anti-ferromagnets also found an irreversibility line above the critical temperature where the anti-ferromagnetic order appears [9]. Recently, Mézard and Young, referred to in the following as M−Y, proposed a variational approach of the RFIM [10] based on a self-consistent expansion in $1/N$ (where N is the number of spin components) due to Bray [11]. They found that replica symmetry, which gives back dimensional reduction, must be broken at the ferromagnetic transition $T = T_f$ and that RSB solution leads to sensible results for the critical exponents in agreement with already known results [1, 10].

In this paper, using M−Y's framework, we concentrate upon the non ferromagnetic regime (i.e. $T > T_f$). We find that there exist indeed two different phases : a paramagnetic phase at high temperatures $T > T_b$ and a glassy phase at intermediate temperatures $T_f < T < T_b$. The value of the correlation length at $T = T_b$ where the RSB transition occurs is computed and compared to predictions obtained from numerical solution of mean-field equations [6].

The model we consider is a N-component version of the RFIM on a three-dimensional lattice including L^3 spins $\mathbf{\Phi}_{\boldsymbol{i}} = (\Phi^1_{\boldsymbol{i}}, ..., \Phi^N_{\boldsymbol{i}})$, where $\boldsymbol{i} = (i_1, i_2, i_3)$ and $0 \leq i_1, i_2, i_3 \leq L-1$,

$$\mathcal{H}(\mathbf{\Phi}, \mathbf{h}) = \frac{1}{2}\sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle} \left(\mathbf{\Phi}_{\boldsymbol{i}} - \mathbf{\Phi}_{\boldsymbol{j}}\right)^2 + \frac{r}{2}\sum_{\boldsymbol{i}} (\mathbf{\Phi}_{\boldsymbol{i}})^2 + \frac{1}{4N}\sum_{\boldsymbol{i}} \left((\mathbf{\Phi}_{\boldsymbol{i}})^2\right)^2 - \sum_{\boldsymbol{i}} \mathbf{h}_{\boldsymbol{i}}.\mathbf{\Phi}_{\boldsymbol{i}} \tag{1}$$

where $\mathbf{h}_{\boldsymbol{i}}$ is a quenched random field, the distribution of which is Gaussian, uncorrelated at differents sites, with mean $\overline{h^\mu_{\boldsymbol{i}}} = 0$ and variance $\overline{h^\mu_{\boldsymbol{i}}.h^\lambda_{\boldsymbol{j}}} = \Delta\ \delta^{\mu\lambda}\ \delta_{\boldsymbol{i}\boldsymbol{j}}$. Following the standard procedure [7], we introduce n replicas of the spins $\mathbf{\Phi}^a$, $a = 1...n$, and average over the quenched disorder $\mathbf{h}$ to obtain the effective Hamiltonian

$$\mathcal{H}(\{\mathbf{\Phi}^a\}) = \frac{1}{2}\sum_{\langle \boldsymbol{i},\boldsymbol{j}\rangle, a} \left(\mathbf{\Phi}^a_{\boldsymbol{i}} - \mathbf{\Phi}^a_{\boldsymbol{j}}\right)^2 + \frac{r}{2}\sum_{\boldsymbol{i},a} \left(\mathbf{\Phi}^a_{\boldsymbol{i}}\right)^2 - \frac{\Delta}{2}\sum_{\boldsymbol{i},(a,b)} \mathbf{\Phi}^a_{\boldsymbol{i}}.\mathbf{\Phi}^b_{\boldsymbol{i}} + \frac{1}{4N}\sum_{\boldsymbol{i},a} \left(\left(\mathbf{\Phi}^a_{\boldsymbol{i}}\right)^2\right)^2 \tag{2}$$

The replica correlation functions $\langle \Phi^{\mu,a}_{\boldsymbol{k}} \Phi^{\lambda,b}_{-\boldsymbol{k}}\rangle = \delta^{\mu\lambda}\ G^{ab}(\boldsymbol{k})$ with the effective Hamiltonian (2) are related to the disconnected and connected correlation functions, $\overline{\langle \Phi^\mu_{\boldsymbol{k}}\rangle_h.\langle \Phi^\lambda_{-\boldsymbol{k}}\rangle_h} = \delta^{\mu\lambda}\ G_{dis}(\boldsymbol{k})$ and $\overline{\langle \Phi^\mu_{\boldsymbol{k}}.\Phi^\lambda_{-\boldsymbol{k}}\rangle_h - \langle \Phi^\mu_{\boldsymbol{k}}\rangle_h.\langle \Phi^\lambda_{-\boldsymbol{k}}\rangle_h} = \delta^{\mu\lambda}\ G_{con}(\boldsymbol{k})$, where $\langle . \rangle_h$ denotes the average over the Gibbs measure induced by (1) [10]. For the replica symmetric assumption $G^{ab}(\boldsymbol{k}) = \tilde{G}(\boldsymbol{k})\ \delta^{ab} + G(\boldsymbol{k})$, the correspondance is simply $G_{dis}(\boldsymbol{k}) = G(\boldsymbol{k})$ and $G_{con}(\boldsymbol{k}) = \tilde{G}(\boldsymbol{k})$.

Using Bray's self-consistent screening approximation [11] which is exact to order $1/N$, one finds that the propagators $G^{ab}(\boldsymbol{k})$ are given by the saddle-point of the free energy

$$\begin{aligned}\mathcal{F}\left(\{G^{ab}(\boldsymbol{k})\}\right) &= \sum_{\boldsymbol{k},a}\left(6 - 2\sum_{d=1}^{3}\cos\left(\frac{2\pi k_d}{L}\right) + r + \frac{1}{2L^3}\sum_{\boldsymbol{q}} G^{aa}(\boldsymbol{q}) - \Delta\right) G^{aa}(\boldsymbol{k}) \\ &- 2\Delta\sum_{\boldsymbol{k},a<b} G^{ab}(\boldsymbol{k}) - \sum_{\boldsymbol{k},a}\Big(\log G(\boldsymbol{k})\Big)^{aa} + \frac{1}{N}\sum_{\boldsymbol{k},a}\Big(\log\left[1 + \Pi(\boldsymbol{k})\right]\Big)^{aa}\end{aligned} \tag{3}$$

where

$$\Pi^{ab}(\boldsymbol{k}) = \frac{1}{L^3} \sum_{\boldsymbol{q}} G^{ab}(\boldsymbol{k}-\boldsymbol{q})\; G^{ab}(\boldsymbol{q}) \tag{4}$$

Within the replica symmetric hypothesis, the correlation functions are therefore solutions of the following set of $2L^3$ implicit equations

$$\frac{G(\boldsymbol{k})}{[\tilde{G}(\boldsymbol{k})]^2} = \Delta + \frac{2}{NL^3} \sum_{\boldsymbol{q}} \frac{G(\boldsymbol{k}-\boldsymbol{q})\; \Pi(\boldsymbol{q})}{[1+\tilde{\Pi}(\boldsymbol{q})]^2} \tag{5}$$

$$\begin{aligned} \frac{1}{\tilde{G}(\boldsymbol{k})} &= 6 - 2\sum_{d=1}^{3} \cos\left(\frac{2\pi k_d}{L}\right) + r + \frac{1}{L^3}\sum_{\boldsymbol{q}} \left(\tilde{G}(\boldsymbol{q}) + G(\boldsymbol{q})\right) \\ &+ \frac{2}{NL^3} \sum_{\boldsymbol{q}} \left[\frac{\tilde{G}(\boldsymbol{k}-\boldsymbol{q}) + G(\boldsymbol{k}-\boldsymbol{q})}{1+\tilde{\Pi}(\boldsymbol{q})} - \frac{\tilde{G}(\boldsymbol{k}-\boldsymbol{q})\; \Pi(\boldsymbol{q})}{[1+\tilde{\Pi}(\boldsymbol{q})]^2}\right] \end{aligned} \tag{6}$$

where

$$\tilde{\Pi}(\boldsymbol{k}) = \frac{1}{L^3}\sum_{\boldsymbol{q}} \left[\tilde{G}(\boldsymbol{k}-\boldsymbol{q})\; \tilde{G}(\boldsymbol{q}) + G(\boldsymbol{k}-\boldsymbol{q})\; \tilde{G}(\boldsymbol{q}) + \tilde{G}(\boldsymbol{k}-\boldsymbol{q})\; G(\boldsymbol{q})\right] \tag{7}$$

$$\Pi(\boldsymbol{k}) = \frac{1}{L^3}\sum_{\boldsymbol{q}} G(\boldsymbol{k}-\boldsymbol{q})\; G(\boldsymbol{q}) \tag{8}$$

In order to determine where the transition to RSB occurs, we have studied the local stability of the free energy $\mathcal{F}(G^{ab})$ around the symmetric saddle-point $(\tilde{G}, G)$. Repeating de Almeida–Thouless's calculations [12] and taking into account the $\boldsymbol{k}$-dependence of the order parameters $\tilde{G}$ and G, we have found that the replica symmetric solution is locally stable if and only if the lowest eigenvalue Λ of the matrix

$$\mathcal{M}(\boldsymbol{k},\boldsymbol{l}) = \frac{\delta_{\boldsymbol{k},\boldsymbol{l}}}{[\tilde{G}(\boldsymbol{k})]^2} - \frac{2}{NL^3}\frac{\Pi(\boldsymbol{k}-\boldsymbol{l})}{[1+\tilde{\Pi}(\boldsymbol{k}-\boldsymbol{l})]^2} - \frac{4}{NL^6}\sum_{\boldsymbol{q}} \frac{G(\boldsymbol{k}-\boldsymbol{q})\; G(\boldsymbol{l}-\boldsymbol{q})}{[1+\tilde{\Pi}(\boldsymbol{q})]^2} \tag{9}$$

is strictly positive.

When $N \to \infty$, Bray's partial resummation reduces to the Hartree-Fock approximation. $\tilde{G}(\boldsymbol{k})$ is thus equal to the bare propagator with a renormalised squared mass $\tilde{m}^2$ solution of the gap equation (5, 6). From the expression of the AT matrix (9), one obtains $\Lambda = \tilde{m}^4 > 0$. The replica symmetric solution is therefore always stable. As soon as N becomes finite, the corrections appearing in (9) may lead to instabilities. In our three-dimensional system, however, the self-consistent screening approximation induces no ferromagnetic transition for large N ($\tilde{m}$ never vanishes for finite bare temperatures r) [10, 11]. Hereafter, we choose $N = 1$ (Ising case) which allows for the existence of long range order at finite $r < 0$.

For every size L of the lattice, we fix a value of r and solve for the propagators $\tilde{G}, G$ by an iteration of equations (5, 6). Using rotational and translational symmetries, only $\tilde{G}(\boldsymbol{k})$ and $G(\boldsymbol{k})$ with $0 \le k_1 \le k_2 \le k_3 \le Int(\frac{L}{2})$ are to be found. Once a fixed point is reached, we estimate the mass $\tilde{m}$ and the correlation length $\tilde{\xi} = \frac{1}{\tilde{m}}$ from the low-momentum behaviour of $\tilde{G}$

$$\tilde{G}(0,0,0) \simeq \frac{\tilde{a}}{\tilde{m}^2} \qquad , \qquad \tilde{G}(0,0,1) \simeq \frac{\tilde{a}}{2 - 2\cos(\frac{2\pi}{L}) + \tilde{m}^2} \tag{10}$$

Expression (10) is exact for $N = \infty$ [2]. Moreover, from perturbation theory which is thought to be correct above the AT transition, we expect $\tilde{G}(\boldsymbol{k})$ to have a single pole. A similar calculation gives m and $\xi = \frac{1}{m}$, assuming that $G(\boldsymbol{k})$ has a double pole [3]. The lowest eigenvalue Λ is then computed by diagonalising the AT matrix. This highly time consuming task may be simplified by observing that $\mathcal{M}$ is invariant under the three symmetry operators $\mathcal{S}_d : k_d \to L - k_d$. In the base of their eigenvectors, $\mathcal{M}$ reduces to eight diagonal blocs of size roughly equal to $\left(\frac{L}{2}\right)^3 \times \left(\frac{L}{2}\right)^3$. One can check that the eigenvector corresponding to Λ belongs to the "physical" subspace, i.e. the one spanned by eigenvectors of $\mathcal{S}_d$ of eigenvalues $+1$. The process is repeated until the value r_L of the bare temperature r where Λ vanishes and the corresponding correlation lengths $\tilde{\xi}_L$ and ξ_L are bracketed with a sufficient precision. The final uncertainties on $\tilde{\xi}_L$ and ξ_L are lower than $\pm 5.10^{-4}$ for lattice sizes running from $L = 2$ up to $L = 20$.

The numerical values of the correlations lengths at the AT transition are displayed Fig. 1. Although it seems difficult to extrapolate to $L \to \infty$, reliable information on the thermodynamical limit may be obtained since the correlations lengths are relatively small as compared to the lattice size ($\tilde{\xi}_L < \xi_L < L/3$ for $L = 20$). From finite size effect theory [13], we expect indeed that, if the mass $\tilde{m}_L$ converges to a finite value $\tilde{m}_\infty > 0$ at the thermodynamical limit, then its asymptotic behaviour obeys

$$\tilde{m}_L - \tilde{m}_\infty \;\simeq\; \tilde{C} \, . \, e^{-L/\tilde{\xi}_L} + O\left(e^{-2L/\tilde{\xi}_L}\right) \tag{11}$$

where $\tilde{C}$ is a constant (the same identity holds for m and ξ_L with a different constant C). Fig. 2 shows the dependence of $\tilde{m}_L$ and m_L upon $e^{-L/\tilde{\xi}_L}$ and e^{-L/ξ_L} respectively. The linear law (11) is very well verified with proportionality factors of order one ($\tilde{C} \simeq 1.83$, $C \simeq 1.37$) [4]. Linear extrapolations to $L \to \infty$ provide the values of the correlation lengths at the thermodynamical limit

$$\tilde{\xi}_\infty \;\simeq\; \xi_\infty \;\simeq\; 7.7 \pm 0.2 \tag{12}$$

The equality between the correlation lengths defined from the disconnected and the connected correlation functions is a self-consistent check of our analytical and numerical results. It is indeed predicted by perturbation theory [3, 6] and therefore holds for high temperatures down to the RSB transition.

In this letter, we have argued that the non-ferromagnetic phase of the three-dimensional random field Ising model is composed of a paramagnetic phase at high temperatures and a spin-glass phase at lower temperatures. The onset of this glassy phase therefore occurs at a finite correlation length for both correlation and susceptibility functions which was found to be

[2] For e.g. $N = 4$, long range order is absent and both mass $\tilde{m}$ and eigenvalue Λ are always non zero. However, from numerical resolution of the saddle-point equations (5, 6), we found that using definition (10) of $\tilde{m}$, the relation $\Lambda \simeq \tilde{m}^4$ is roughly correct.

[3] This stems from perturbation theory which is thought to be valid above T_b [3] and can be checked up when $N \to \infty$ on (5, 6). For the Ising system, the one dimensional correlations functions in the real space, $\tilde{g}(x)$ and $g(x)$ (which are respectively the Fourier transforms of $\tilde{G}(k,0,0)$ and $G(k,0,0)$) can be fitted with a very good agreement by $\tilde{g}(x) \simeq \tilde{a}\cosh(x/\tilde{\xi}) + \tilde{b}$ and $g(x) \simeq a(1 + x/\xi)\cosh(x/\xi) + b$ [6]. The discrepancy between these values of the correlation lengths and the ones defined in the text seems to vanish for increasing lattice sizes.

[4] Although scaling relation (11) should also be valid for the correlation length itself, i.e. $\tilde{\xi}_\infty - \tilde{\xi}_L \;\simeq\; \tilde{D} \, . \, e^{-L/\tilde{\xi}_L}$, one sees from $\tilde{D} = -\tilde{C} . \, \xi_\infty^2 \simeq -100$ that it cannot be directly observed with lattice sizes lower than $L = 20$.

in the range $7.5 < \xi < 8$. Although such a result might be an artefact due to the $1/N$ approach used here, it is in qualitative agreement with previous numerical studies which found that the mean field equations of the RFIM begin to have more than one solution, and thus that the perturbative approach ceases to be correct, for $\xi > 4.5$. Such a behaviour is to be expected in the low temperature phase too. Since at very low T, only the two states where all spins are aligned along the same direction remain, replica symmetry has to be restored at a temperature T_s with $0 < T_s < T_f$. It would be interesting to extend the calculation we have presented here to verify explicitely this conjecture.

We are particularly grateful to G. Parisi for numerous and fruitful discussions and suggestions. We also wish to thank M. Aizenman, C. De Dominicis, T. Garel, D.J. Lancaster and H. Orland for interesting discussions. This study was partly funded by the EEC and the French Ministry of Research.

References

[1] T. Nattermann, J. Villain, *Phase Transitions* **11**, 817 (1988)
D.P. Belanger, A.P. Young, *J. Magn. Magn. Mater.* **100**, 272 (1991)

[2] Y. Imry, S.K. Ma, *Phys. Rev. Lett.* **35**, 1399 (1975)
J. Bricmont, A. Kupiainen, *Phys. Rev. Lett.* **59**, 1829 (1987)
M. Aizenman, J. Wehr, *Phys. Rev. Lett.* **62**, 2503 (1989)

[3] A. Aharony, Y. Imry, S.K. Ma, *Phys. Rev. Lett.* **37**, 1364 (1976)
A.P. Young, *J. Phys. A* **10**, L257 (1977)
G. Parisi, N. Sourlas, *Phys. Rev. Lett.* **43**, 744 (1979)

[4] G. Parisi, *Proceedings of Les Houches 1982*, edited by J.B. Zuber and R. Stora (North Holland, Amsterdam, 1984)
A.J. Bray, A.J. MacKane, *Europhys. Lett.* **1**, 427 (1986)

[5] H. Yoshizawa, D. Belanger, *Phys. Rev. B* **30**, 5220 (1984)
C. Ro, G. Grest, C. Soukoulis, K.Levin, *Phys. Rev. B* **31**, 1681 (1985)
U. Nowak, K.D. Usadel, *Phys. Rev. B* **44**, 7426 (1991)

[6] M. Guagnelli, E. Marinari, G. Parisi, *Rome University preprint*, (1993)

[7] M. Mézard, G. Parisi, M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987)

[8] J.R.L. de Almeida, R. Bruinsma, *Phys. Rev. B* **35**, 7267 (1987)

[9] F.C. Montenegro, A.R. King,V. Jaccarino, S.J. Han, D.P. Belanger, *Phys. Rev. B* **44**, 2155 (1991)
D.P. Belanger, W.E. Murray, F.C. Montenegro, A.R. King, V. Jaccarino, R.W. Erwin, *Phys. Rev. B* **44**,2161 (1991)

[10] M. Mézard, A.P. Young, *Europhys. Lett.* **18**, 653 (1992)

[11] A.J. Bray, *Phys. Rev. Lett.* **32**, 1413 (1974)

[12] J. de Almeida, D. Thouless, *J. Phys. A* **11**, 983 (1978)

[13] E. Brézin, *J. Physique* **43**, 15 (1982)

Figure Captions

Figure 1 : the correlation lengths $\tilde{\xi}_L$ (empty dots) and ξ_L (full dots) at the onset on the spin glass phase for different lattice sizes $L = 2$ up to $L = 20$ (the total number of spins is L^3).

Figure 2 : the masses $\tilde{m}_L = \frac{1}{\tilde{\xi}_L}$ (empty dots) and $m_L = \frac{1}{\xi_L}$ (full dots) plotted vs. the "finite size" factors $\exp(-L/\tilde{\xi}_L)$ and $\exp(-L/\xi_L)$ respectively. The dashed lines are the best linear fits from the last twelve points (sizes $L = 9$ to $L = 20$).

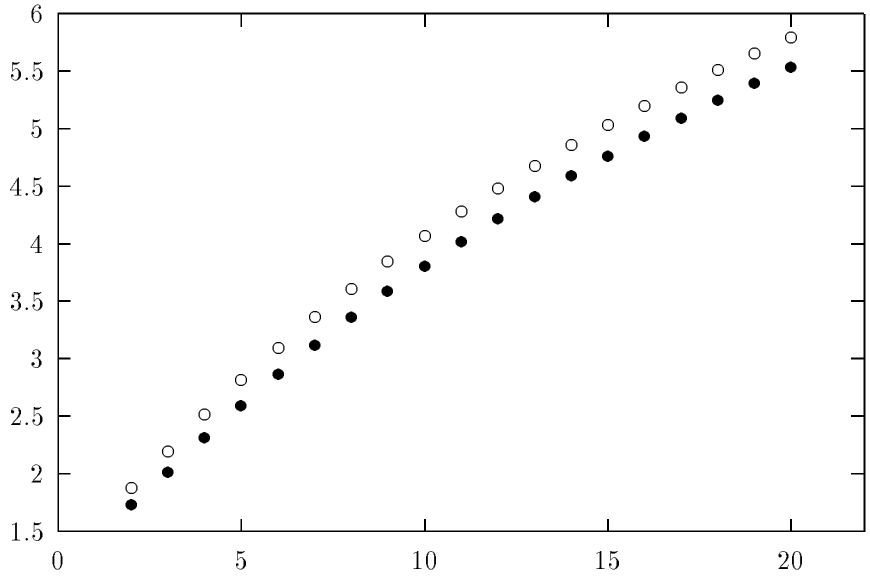

Figure 1: correlation lengths $\tilde{\xi}_L$ (empty dots) and ξ_L (full dots) at the RSB transition for differents lattice sizes L.

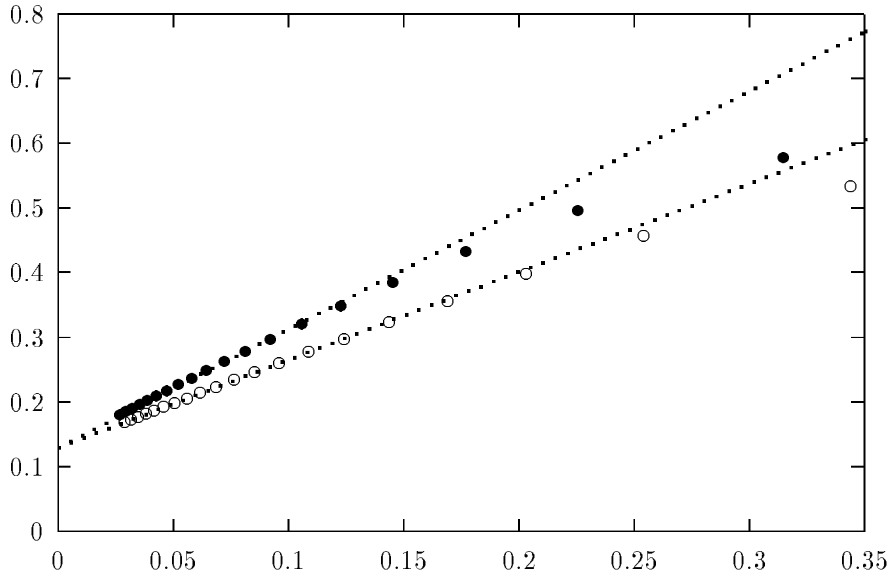

Figure 2: masses $\tilde{m}_L = \frac{1}{\tilde{\xi}_L}$ (empty dots) and $m_L = \frac{1}{\xi_L}$ (full dots) vs. $e^{-L/\tilde{\xi}_L}$ and e^{-L/ξ_L} respectively. The dashed lines are the best linear fits from the last twelve points (sizes $L = 9$ to $L = 20$).