Mean Field Dynamical Exponents in Finite-Dimensional Ising Spin Glass

G. Parisi, P. Ranieri, F. Ricci-Tersenghi and J. J. Ruiz-Lorenzo

Dipartimento di Fisica, Università di Roma *La Sapienza*

and INFN, Sezione di Roma I

P. A. Moro 2, 00185 Roma (Italy)

parisi,ranieri@roma1.infn.it   ricci,ruiz@chimera.roma1.infn.it

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Abstract

We have studied numerically the remanent magnetization in the six and eight dimensional Ising spin glass and we have compared it with the behavior observed in the SK model, that we have also computed analytically. We also report the value of the dynamical critical exponent z in six dimensions measured in three different ways: from the behavior of the energy and the susceptibility as a function of the Monte Carlo time and by studying the overlap-overlap correlation function as a function of space and time. These three results are in very good agreement with the Mean Field prediction $z = 4$. Finally we have checked numerically the analytical prediction, obtained by assuming spontaneously broken replica symmetry, for the most singular part of the propagator in the spin glass phase. This last result supports the existence of spontaneously broken replica symmetry in finite dimensional spin glasses.

1 Introduction

One can identify two main problems in the current spin glass theoretical research. The first one is the nature of the spin glass phase in finite dimensions. The droplet theory can not to explain a great quantity of numerical data which it is possible to interpretate in the framework of a theory like Mean-Field (i.e. with spontaneously broken replica symmetry) [1]. In the last part of this paper we will see that the numerical data point, in particular one of the propagators of the theory, clearly to a low temperature phase of the Mean-Field type.

We note that in a recent paper [2] it was shown that if the infinite volume is done in a given (weak) topology, the infinite volume overlap probability distribution $P(q)$ does not depend on the couplings (always in this weak product topology). This result is not in contradiction with our findings that point towards the correctness of the Mean Field expressions for the propagators.

The second main problem is the analytical calculation of critical exponents (static and dynamical) in three dimension in order to be able to confront the theory with the experiments. The first part of this paper is devoted to study the value of the upper critical dimension for the decay exponent of the remanent magnetization. This knowledge is essential for a further calculation of the value of this exponent in finite dimensions (e.g. using the ϵ-expansion). This exponent has a great physical importance, mainly because it gives us information on the early regimes of the dynamics and because it is often measured experimentally [3].

The remanent magnetization is defined as follows: we put the system under a large magnetic field, turn it off and follow the decay of the magnetization of the system. . The magnetization decays like

$$M(t) \sim t^{-\lambda}, \tag{1}$$

which defines the λ exponent [4]. Following the results of D. Fisher and H. Sompolinsky [5] we should expect that λ is equal to the MF value only for $d \geq 8$, i.e. for exponents of observables which are reminiscent of t he "magnetic field" the upper critical dimension is eight, not six.

We check this fact numerically in the present paper obtaining the value of the λ exponent in six, eight and in infinite dimensions (the SK model) that we have computed analytically. These results show that for this observable the upper critical dimension is eight instead of six.

Moreover we have found numerically, for the SK model, that the dependence of λ with the temperature is discontinuous at the critical point.

Another related issue addressed in this paper is the numerical evaluation of the dynamical critical exponent in six dimensions. In the past Wang and Young [6] checked that in six dimension the static critical exponent are those of Mean Field, but such a check still lacks for the dynamical side of the problem. Hence, another of the goals of this paper is to show that the upper critical dimension is six for dynamics as well.

To do this, we have developed different techniques that yield accurate determinations of the dynamical exponent.

We will plan a further use of these techniques in five dimensions in order to check the recent analytical calculation by two of the authors (G. Parisi and P. Ranieri)[8]. They have been able to compute the one loop correction to the dynamical critical exponent, z, whose Mean Field (MF) value (the base of the ϵ-expansion) is 4. They found

$$z(\epsilon) = 4\left(1 - \frac{\epsilon}{12}\right), \tag{2}$$

where $\epsilon = 6 - d$.

In particular we have obtained the z exponent [10] using three different off-equilibrium methods: the decay of the energy, the growth of the non linear susceptibility and the scaling of the overlap-overlap correlation function. These three methods provide us with three estimates of z or ratios of z to the static critical exponents which are in a very good agreement with the Mean Field predictions ($\nu_{\rm MF} = 1/2$, $\eta_{\rm MF} = 0$ and $z_{\rm MF} = 4$).

Whereas calculations of z have been done at the critical temperature of the system we have performed numerical simulations inside the cold phase to monitor the "expected" dependence on the temperature of the exponent z (as obtained in three and four dimensions [11, 12]) and to check the predictions of De Dominicis *et al.* (by monitoring the growth of the susceptibility) that imply that the propagator restricted to the $q = 0$ ergodic component behaves like p^{-4}, where p is the momenta of the propagator. Obviously at the critical point we expect the usual dependence on the momenta, i.e. p^{-2}.

We remark that the analytical prediction of De Dominicis *et al.* was done assuming that the spontaneous breaking of the replica symmetry is that of Mean Field. Hence, our numerical results are a further test that the spin glasses in finite dimensions follow the picture of Mean Field.

2 Numerical simulation and observables

We have simulated the six (eight) dimensional Ising spin glass whose Hamiltonian defined in a hypercube of volume L^d with periodic boundary conditions is

$$\mathcal{H} = -\sum_{<i,j>} S_i J_{ij} S_j, \tag{3}$$

where $< i, j >$ denotes nearest neighbor pairs, $J_{ij} = \pm 1$ (with the same probability) are quenched variables and $S_i = \pm 1$ are spin variables.

The static of this model was studied by Wang and Young [6]. Simulating lattice sizes up to $L = 8$ they found that the static critical exponents were compatible with the Mean Field predictions ($\nu_{\rm MF} = 1/2, \eta_{\rm MF} = 0$) and that there were logarithmic corrections to the Mean Field exponents, an effect linked to the upper critical dimension. Their estimate for the critical temperature was $T_c = 3.035 \pm 0.01$.

We will study the decay of the remanent magnetization defined as

$$M(t, t_w) = \frac{1}{L^d}\sum_{i=1}^{L^d} \sigma_i(t)\sigma_i(t_w), \quad t \gg t_w. \tag{4}$$

This observable decays like

$$M(t, t_w) \propto t^{-\lambda}. \tag{5}$$

We find that the Mean Field prediction for this exponent is $\lambda = 5/4$.

Another aim of this paper is to measure the dynamical critical exponent z in order to compare it with the Mean Field results ($z_{\rm MF} = 4$). To do this we have measured the behavior of the energy and susceptibility as a function of the Monte Carlo time

$$E(t) - E_\infty \propto t^{-\delta} \tag{6}$$

$$\chi(t) \propto t^h \tag{7}$$

and the $q - q$ correlation function.

We will also examine the dependence of the energy on the Monte Carlo time. We assume that at the critical point (and only at the critical point) it is possible to connect the approach to equilibrium of the energy and of the equal time correlation functions to the equilibrium static and dynamical exponents. For example, in the case of the energy we find:

$$E(t) - E_\infty \propto t^{-\dim(E)/z}, \quad T = T_c, \tag{8}$$

where z is the dynamical critical exponent, $\dim(E) \equiv d - 1/\nu$ is the dimension of the energy operator and d is the dimension of space. Assuming $d = 6$ and $\nu = \nu_{\rm MF} = 1/2$ we have that the exponent of the energy decay at $T = T_c$ is $\delta = 4/z$.

Analogously for the non linear susceptibility

$$\chi(t) = L^d \overline{\langle q^2(t) \rangle}, \quad q(t) = \frac{1}{L^d} \sum_i \sigma_i(t) \tau_i(t), \tag{9}$$

where σ and τ are two real replicas with the same quenched disorder, and we can write the following dependence on the Monte Carlo time [7]

$$\chi(t) \propto t^{(2-\eta)/z}, \quad T = T_c, \tag{10}$$

for $t \ll \tau_{\rm eq}(L)$, where $\tau_{\rm eq}(L)$ is the equilibration time, which should diverge as $\tau_{\rm eq}(L) \propto L^z$. Here we have used that $\dim(\chi) = 2 - \eta$. By assuming the Mean Field value for η we obtain for the exponent of the susceptibility: $h = 2/z$ at $T = T_c$.

From these formulæ it is possible to calculate the dynamical exponent via two different observables. In the six dimensional case, if $z = z_{\rm MF} = 4$ we should see a behavior like t^{-1} for the energy and $t^{1/2}$ for the non linear susceptibility.

Finally, a third way to calculate the dynamical exponent is to use the overlap-overlap correlation length at distance x and time t defined by

$$G(x,t) \equiv \frac{1}{V} \sum_i \overline{\langle \sigma_{i+x} \tau_{i+x} \sigma_i \tau_i \rangle}, \tag{11}$$

where again σ and τ are two real replicas with the same disorder.

In the simulations we start from a random configuration ($T = \infty$) and suddenly quench the system to T_c or below. Then the system begins to develop internal correlations and we can define a time dependent off-equilibrium correlation length, $\xi(T,t)$, as the typical distance over which the system has already developed correlations different from zero, i.e. $G(x,t) \simeq 0$ for $x > \xi(T,t)$.

The growth of this correlation length with the Monte Carlo time defines the dynamical exponent z trough

$$\xi(T,t) \propto t^{1/z(T)}. \tag{12}$$

We have seen that in three and four dimensions [11, 12] the data fit very well the following functional form

$$G(t,x) = \frac{A(T)}{x^\alpha} \exp\left\{ -\left(\frac{x}{\xi(T,t)}\right)^\gamma \right\}, \tag{13}$$

Thereby, this will be the third way to obtain the dynamical critical exponent. This third estimate of z is independent of the values of the static critical exponents.

Moreover, the equilibrium overlap-overlap correlation function constrained to $q = 0$ was obtained by De Dominicis *et al.* [9], and in six dimensions reads

$$C_{\mathrm{SRSB}}(x)|_{q=0} \sim \begin{cases} x^{-4} & \text{if } T = T_c, \\ x^{-2} & \text{if } T < T_c. \end{cases} \tag{14}$$

or in momenta space

$$C_{\mathrm{SRSB}}(p)|_{q=0} \sim \begin{cases} p^{-2} & \text{if } T = T_c, \\ p^{-4} & \text{if } T < T_c. \end{cases} \tag{15}$$

The fact that the equilibrium correlation function $C(x)$ has a power law decay also for $T < T_c$, implies that spin glasses are always critical in the glassy phase (i.e. below T_c) and therefore we can relate the off-equilibrium behavior of the correlation function to the equilibrium critical exponents. Since the susceptibility is the integral of the correlation function

$$\chi = \int \mathrm{d}^6 x \; C(x) \tag{16}$$

so that in the region where the susceptibility grows (following a power law with time, the overlap remains very small), we can substitute equations (14) in equation (16) obtaining

$$\chi(t) \sim \begin{cases} t^{1/2} & \text{if } T = T_c, \\ t^{4/z(T)} & \text{if } T < T_c. \end{cases} \tag{17}$$

If we take the limit $T \to T_c$ in the above equation we obtain that $h(T)$, the exponent of the growth of the susceptibility, must be discontinuous at the critical point (i.e. $h(T_c^-) = 1$ while $h(T_c^+) = 1/2$).

Moreover, if we assume that $1/z(T)$ is proportional to the temperature (this happens in three and four dimensions [11, 12]) we must obtain a linear dependence on the temperature for $h(T)$ in the low temperature phase.

3 Remanent magnetization

The first part of this work has been dedicated to the decay of the remanent magnetization.

We prepare the system with all the spins up ($M(t=0) = 1$) and then we let it evolve towards equilibrium where, in absence of any external field, no magnetization should remain. In the cold phase we expect the decay of the remanent magnetization to be algebraic (see equation (1)); in particular we are interested in the exponent of such a decay at the critical temperature (which hereafter will be called simply λ), to compare it with the same exponent of the SK model.

We report in the following subsections our calculation of the λ exponent in the MF approximation (see reference [13] for another calculation of the λ exponent in the spherical spin-glass model), together with the numerical verification and the estimates of such an exponent in the finite dimensional cases ($d = 6$ and $d = 8$) and also for the SK model.

3.1 Analytical results in the SK model

To study analytically the dynamical properties of the SK model we define the following soft-spin Hamiltonian:

$$\beta\mathcal{H} = -\beta \sum_{ij} J_{ij} s_i s_j + \frac{1}{2} \sum_i s_i^2 + \frac{1}{4!} g \sum_i s_i^4, \tag{18}$$

where s_i ($i = 1, ..., N$), are one-dimensional real variables and J_{ij} is a symmetric matrix with independent elements following a Gaussian distribution with zero mean and variance proportional to $1/N$. From the random matrix theory [14] we know that, in the thermodynamic limit (i.e. N goes to infinity), the probability distribution for the eigenvalues of J_{ij} is given by the semi-circle law:

$$\sigma(\mu) = \frac{1}{2\pi}(4 - \mu^2)^{1/2} \qquad |\mu| < 2. \tag{19}$$

A relaxation dynamics is introduced by means of the following Langevin equation:

$$\partial_t s_i(t) = -\frac{\partial(\beta\mathcal{H})}{\partial s_i(t)} + \xi_i(t). \tag{20}$$

where $\xi_i(t)$ are Gaussian noises with zero mean and variance $\langle \xi_i(t)\xi_j(t')\rangle = 2\delta_{ij}\delta(t-t')$. To study the dynamical evolution of this model, we diagonalize the J_{ij} matrix and we consider the dynamics of the projections $s^n(t)$ of the spins $s_i(t)$ on the eigenvector directions ψ_i^n (with eigenvalues μ_n), such that $s_i(t) = \sum_n s^n(t)\psi_i^n$, where $n = 1, .., N$ is the eigenvector index. The properties of independence and orthonormality of the eigenvectors, [14] let us define the following Langevin equation for the component s^n:

$$\partial_t s^n(t) = (\beta\mu_n - 1)s^n(t) - \frac{g}{3!} \sum_{\alpha\beta\gamma} s^\alpha(t) s^\beta(t) s^\gamma(t) \sum_i \psi_i^\alpha \psi_i^\beta \psi_i^\gamma \cdot \psi_i^n + \xi^n(t). \tag{21}$$

where $\xi^n(t)$ are the components of $\xi_i(t)$ in the eigenvectors basis.

As usual we take into account the non linear term perturbatively. The dynamical response function $G(t, t')$ is given by the Dyson equation:

$$G^{-1}(t, t') = G_0^{-1}(t, t') + \Sigma(t, t') \ , \tag{22}$$

where $G_0^{-1}(t, t')$ is the inverse bare response function and $\Sigma(t, t')$ is the self-energy. In analogy with the equilibrium dynamics, we suppose that the leading term for large N and large t is given by the Hartree-Fock approximation[1]:

$$\Sigma(t,t') = C(t,t)\delta(t-t') \tag{23}$$

where $C(t,t)$ is the dynamical auto-correlation:

$$C(t,t') = \overline{\langle s_i(t)s_i(t')\rangle}, \tag{24}$$

evaluated at $t = t'$. We represent the average over the thermal noise as $\langle(\cdots)\rangle$, while $\overline{(\cdots)}$ indicates the average over the disorder as usual. With respect to the eigenvalues the auto-correlation function can be written as follows

$$C(t,t') = \int \mathrm{d}\mu\, \sigma(\mu)\langle s^\mu(t)s^\mu(t')\rangle. \tag{25}$$

The Langevin equation, (21), becomes

$$\partial_t s^n(t) = (\beta\mu_n - 1)s^n(t) - \frac{g}{2}C(t,t)s^n(t) + \xi^n(t) \tag{26}$$

The self-consistent solution of eq. (26) is

$$s^n(t) = s^n(0)e^{(\beta\mu_n-1)t}e^{-g/2\int_0^t \mathrm{d}t'\, C(t',t')} + \int_0^t \mathrm{d}t''\, e^{(\beta\mu_n-1)(t-t'')}e^{-g/2\int_{t''}^t \mathrm{d}t'\, C(t',t')}\xi^n(t''), \tag{27}$$

where $t = 0$ is the initial time.

We want to analyze the evolution of the system at $T = T_c$ from a uniform initial condition: $s^n(0) = 1$, $\forall n$. From (27) we obtain the following self-consistent equation for $C(t,t)$:

$$C(t,t) = H^2(t)\int \mathrm{d}\mu\, \sigma(\mu)e^{2(\beta\mu-1)t} + 2\int_0^t \mathrm{d}t'' \left(\frac{H^2(t)}{H^2(t'')}\right)\int \mathrm{d}\mu\, \sigma(\mu)e^{2(\beta\mu-1)(t-t'')} \tag{28}$$

where

$$H(t) = e^{-g/2\int_0^t \mathrm{d}t'\, C(t',t')}. \tag{29}$$

Let us suppose for $H(t)$, at $T = T_c$, a time-dependent asymptotic behavior ($t \to \infty$) like

$$H(t) \sim t^\rho e^{-gt}, \tag{30}$$

where ρ can be determined self-consistently from eq.(28). This hypothesis implies for $C(t,t)$, at large t, the behavior

$$C(t,t) \sim (2 + \Delta T_c) - \frac{a}{t}. \tag{31}$$

where a is an appropriate constant. The critical temperature of the Hamiltonian (18) is given by $T_c = T_c^0 + \Delta T_c = 2 + \Delta T_c$ ($T_c^0 = 2$ is the critical temperature of (18) when $g = 0$) and a perturbative calculation gives $\Delta T_c = -2g$ so that $2\beta_c = 1 + g$ [15].

[1]We have verified that the results obtained are consistent with this assumptions.

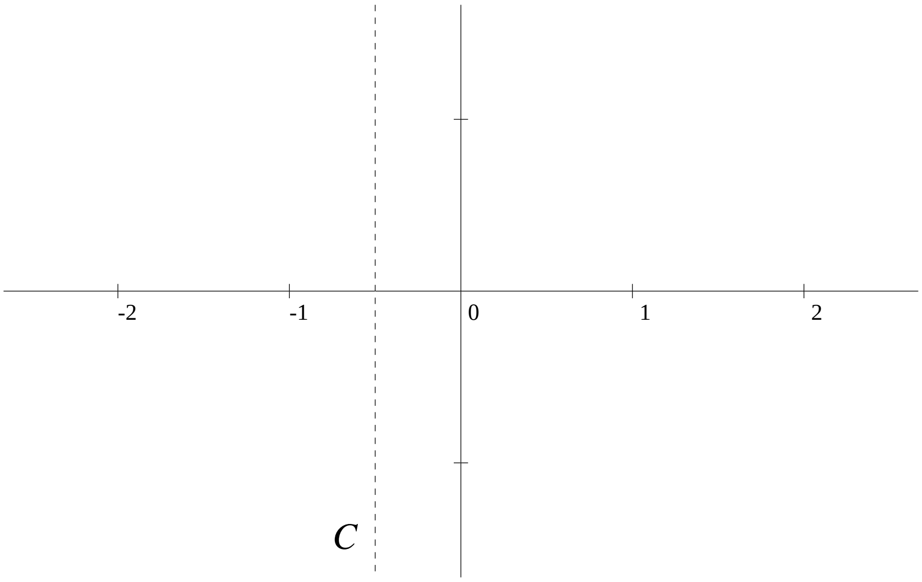

Figure 1: Integration path.

We recall that for $T \sim T_c$ the largest contribution to the dynamical relaxation of the spins comes from the region of the maximum eigenvalue, $\mu = 2$.

At the critical point, the first term on the right side of the eq. (28) scales like a power law $t^{-3/2+2\rho}$. To be consistent with the hypothesis (30) and (31) we should have $\rho \leq 1/4$.

Now we have to estimate the second term, which is proportional to:

$$2\int_0^t \mathrm{d}t'' \int \mathrm{d}\mu\, \sigma(\mu) \frac{t^{2\rho}}{t''^{2\rho}} e^{-2\beta_c(2-\mu)(t-t'')} \tag{32}$$

Let us consider $t' = t - t''$ and we define the exponential as an integral in the complex plane,

$$e^{-2\beta_c(2-\mu)t'} = \sum_k \frac{(-1)^k}{k!} \left[2\beta_c(2-\mu)t'\right]^k = \int_{\mathcal{C}} \frac{\mathrm{d}s}{2\pi i}\, \Gamma(-s) \left[2\beta_c(2-\mu)t'\right]^s , \tag{33}$$

where $\mathcal{C}$ is the path shown in figure 1 (i.e. $s = s_0 + ir$, where $r \in \mathcal{R}$, and s_0 is an arbitrary real number in $(-1, 0)$) $\Gamma(s)$ is the Euler Gamma function. We obtain:

$$2\int_{\mathcal{C}} \frac{\mathrm{d}s}{2\pi i} \int_0^t \mathrm{d}t'\, \Gamma(-s) \frac{t^{2\rho}}{(t-t')^{2\rho}} t'^s \int \mathrm{d}\mu\, \sigma(\mu)(2\beta_c)^s(2-\mu)^s \ . \tag{34}$$

After the integration over t' and μ we can write:

$$\frac{8}{\sqrt{\pi}} \int_{\mathcal{C}} \frac{\mathrm{d}s}{2\pi i}\, \Gamma(-s) t^{1+s} \frac{\Gamma(s+1)\Gamma(1-2\rho)}{\Gamma(s-2\rho+2)} 4^s (2\beta_c)^s \frac{\Gamma(s+3/2)}{\Gamma(3+s)} \tag{35}$$

To evaluate the integral (35) we analytically continue the function on the left of the path $\mathcal{C}$, i.e. in the region where Re $s < s_0$. Thereby, we have to consider the residues of the poles in this region. The residue of the pole at $s = -1$ gives the constant contribution to the auto-correlation function $C(t,t)$ while the time-dependent behavior, for large t, is determined by the value of ρ. In fact, for $\rho = 1/4$, $\Gamma(s+3/2)$ simplifies to $\Gamma(s-2\rho+2)$, and, for large t, the leading behavior of $C(t,t)$ comes from the residue of the pole at $s=-2$:

$$C(t,t) \sim \mathrm{const} + t^{-1}, \tag{36}$$

consistent with the hypothesis (30) and (31).

For $\rho \neq 1/4$, on the contrary, we can not cancel the pole at $s=-3/2$ and we should obtain

$$C(t,t) \sim \mathrm{const} + t^{-1/2}, \tag{37}$$

in contrast to the previous hypothesis. Thus the only consistent solution for $H(t) \sim t^{\rho} e^{-t}$ is $\rho = 1/4$.

Another way that we can use to obtain the ρ exponent is solving the eq. (28) in Laplace transform space. In terms of the function $g(t)$

$$g(t) \equiv e^{-2gt} \frac{C(t,t)}{\Gamma^2(t)} \ , \tag{38}$$

eq. (28) at $T = T_c$ becomes:

$$g(t) = \int \mathrm{d}\mu\, \sigma(\mu) e^{-2\beta_c(2-\mu)t} + 2\int_0^t \mathrm{d}\,t'' \frac{g(t'')}{C(t'',t'')} \int \mathrm{d}\mu\, \sigma(\mu) e^{-2\beta_c(2-\mu)(t-t'')} \ . \tag{39}$$

By using the asymptotic form (31) of $C(t,t)$, we obtain the following asymptotic equation for the Laplace transform of $g(t)$, that we will denote $\tilde{g}(s)$:

$$\begin{aligned} \tilde{g}(s) &= \int \mathrm{d}\mu\, \sigma(\mu) \left[\frac{1}{s+2\beta_c(2-\mu)}\right] && (40)\\ &+ \int \mathrm{d}\mu\, \sigma(\mu) \left[\frac{1}{s+2\beta_c(2-\mu)}\right] \left[\left(1-\frac{\Delta T_c}{2}\right)\tilde{g}(s) + \frac{a}{2}\int_s^{\infty} \mathrm{d}x\, \tilde{g}(x)\ .\right] && (41) \end{aligned}$$

By averaging over the eigenvalue distribution we obtain:

$$\tilde{g}(s) = \left(\frac{1}{2\beta_c} - \frac{\sqrt{2s}}{4\beta_c^{3/2}}\right) + \left(\frac{1}{2\beta_c} - \frac{\sqrt{2s}}{4\beta_c^{3/2}}\right)\left[\left(1-\frac{\Delta T_c}{2}\right)\tilde{g}(s) + \frac{a}{2}(1-\Delta T_c)\int_s^{\infty} \mathrm{d}x\, \tilde{g}(x)\right] \ . \tag{42}$$

By remembering that $\beta_c = 1/(2+\Delta T_c)$, we can expand the previous equation in ΔT_c. We will also assume that in the limit $s \to 0$; $\int_s^{\infty} \mathrm{d}x\, \tilde{g}(x)$ is negligible with respect to $\tilde{g}(s)$, and we finally obtain

$$\tilde{g}(s) \sim \frac{1}{\sqrt{s}} + \mathrm{O}(1). \tag{43}$$

Thus, for $t \to \infty$, $g(t) \sim 1/t^{1/2}$ and from (38) we obtain $\rho = 1/4$. This solution implies that $\int_s^{\infty} \mathrm{d}x\, \tilde{g}(x) \ll \tilde{g}(s)$.

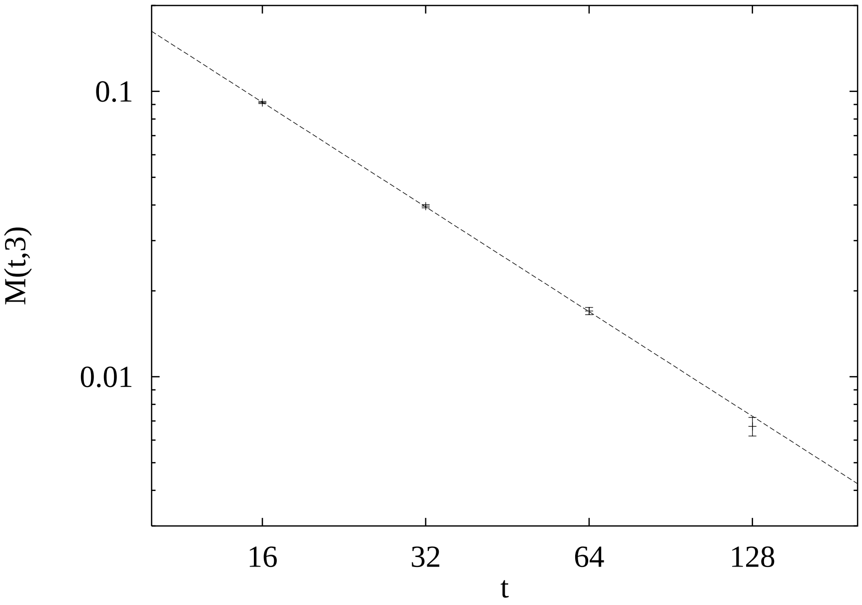

Figure 2: Remanent magnetization in the SK model at $T = T_c$. The line is the best power fit which gives an exponent $\lambda_{\mathrm{MF}} = 1.22 \pm 0.02$.

At this point, we can determine the decay rate of the correlation between the system at time t and the system at $t = 0$, i.e. the scaling decay of the remanent magnetization:

$$M(t) = C(t,0) = \overline{\langle s_i(t)\rangle} = \int \mathrm{d}\mu\, \sigma(\mu)\langle s_\mu(t)\rangle \tag{44}$$

$$= \int \mathrm{d}\mu\, \sigma(\mu) H(t) e^{(\beta\mu-1)t} \sim \frac{1}{t^{3/2}} t^{1/4} \sim t^{-5/4}. \tag{45}$$

Thus, the analytical prediction for the exponent λ, defined by $M(t) \sim t^{-\lambda}$, is, in the Mean Field limit, $\lambda = 5/4$.

This result is quite different from the one obtained in the spherical model at $T = T_c$ where $\lambda = 3/4$ [13].

3.2 Numerical results in six, eight and infinite (SK) dimensions

For a numerical confirmation of this result we have simulated three SK models at criticality ($T_c = 1$) of sizes $N = 480, 992, 2016$ with number of samples of 10000, 5000 and 1000 respectively, obtaining three estimates of the λ_{MF} exponent all compatible with the theoretical prediction. Since the data for the remanent magnetization contain non evident finite-size effects, we have plotted in figure 2 the data averaged over all the simulated samples. The observable that we have measured is $M(t, t_w = 3)$ defined in eq.(4), which follows the same decay of $M(t)$ but has some advantages as will be explained below. The line in figure 2 is the best power fit which gives $\lambda_{\mathrm{MF}} = 1.22 \pm 0.02$.

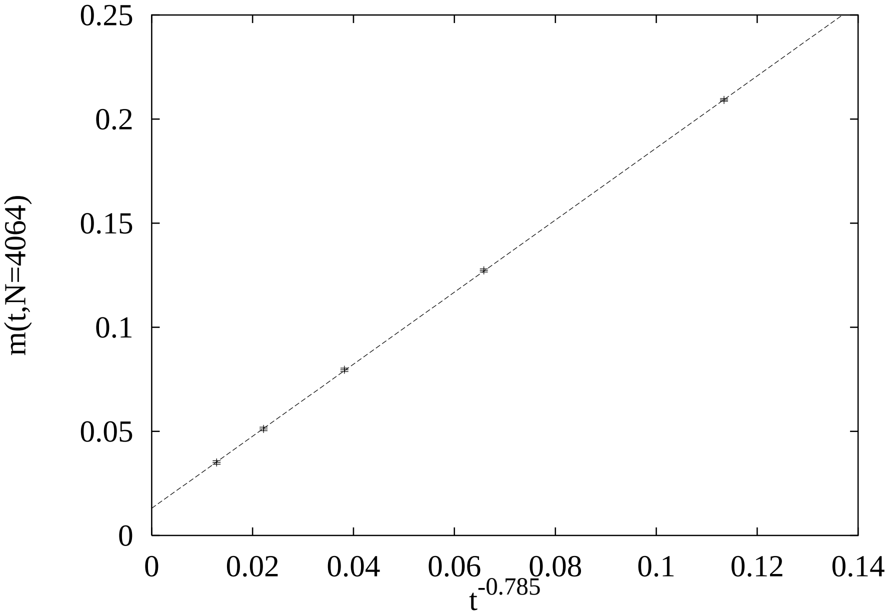

Figure 3: Remanent magnetization in the SK model at $T = 0.8\,T_c$ plotted versus $t^{-\lambda(T)}$, to make more evident the finite size effect ($m(t=\infty) \neq 0$). The line is the linear fit.

We have repeated this numerical calculation for a temperature below the critical one ($T = 0.8\,T_c$) and sizes $N = 480, 992, 2016, 4064$. In this case we expect that the remanent magnetization tends to a non-zero asymptotic value due to the finite size of the system. So we have fitted the data for $M(t, t_w\!=\!3)$ with the following formula

$$m(t, N) = m_\infty(N) + A\, t^{-\lambda(T)} \tag{46}$$

where we let λ depend on the temperature. Via a preliminary three parameter fit we have estimated $\lambda = 0.785(10)$ and found no systematic dependence on the lattice size. Then fixing the value of λ to this previously found value, we extrapolated the value of $m_\infty(N)$ by using a simple linear fit, like the one plotted in figure 3.

Using the values of $m_\infty(N)$ found by the previous analysis we were able to fit them to a power law of the system size: $m_\infty(N) \propto N^{-0.25(1)}$. The data with the best fit are reported in a double-log scale in figure 4 (see ref. [16] for a detailed study).

Assuming a linear dependence of the exponent $\lambda(T)$ with the temperature, which has been observed in [16, 17] for the SK model and in [18] for a spin glass system on quenched ϕ^3 graphs (which should behave like a mean-field SK model), and from the fact that $\lambda(T = 0.8) \simeq 0.8$, we obtain that the λ critical exponent as a function of the temperature is discontinuous at the critical point (i.e. $\lambda(T_c^-) \simeq 1$ while $\lambda(T_c^+) = 5/4$).

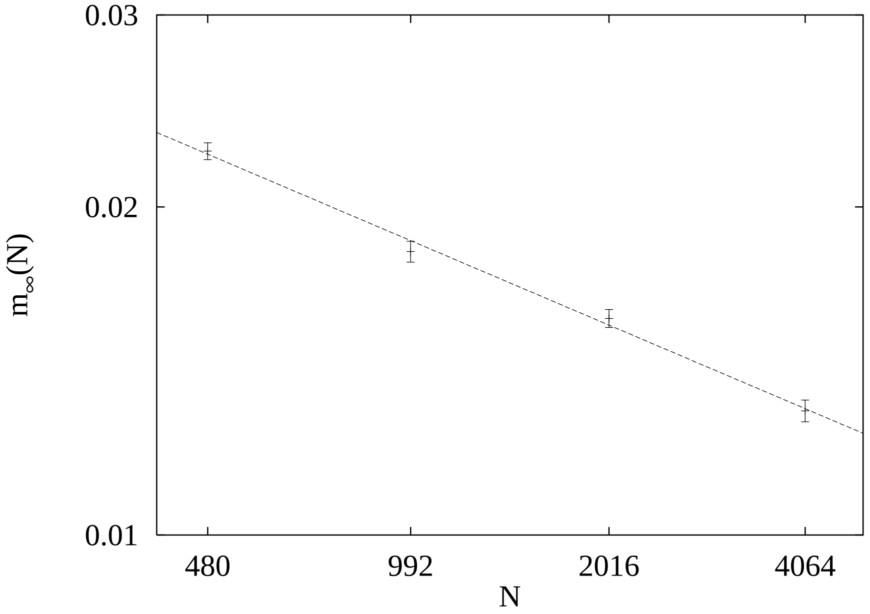

Figure 4: $m_\infty(N)$ vs. N in the SK model at $T = 0.8\,T_c$. The line is the best power fit.

3.3 Numerical estimate in $d = 6$ and $d = 8$

The measurement of the decay rate of the remanent magnetization is not easy because although we know that $M(t=0) = 1$ we try to fit the $M(t)$ data with a power law which diverges at $t = 0$. This effect is evident in a log-log scale where a power fit behaves like a straight line, while the $M(t)$ data tends to the value 1 when $\ln t \to -\infty$; in such a situation we have to discard the first data points to be sure we are measuring the asymptotic behavior. Unfortunately the useless data are just the ones with smaller relative error, while the data fitted are affected by a greater statistical indetermination which makes worst the estimate of λ.

One possible way to overcome this source of error is to measure some other observable that has the same behavior of $M(t)$, but with a stronger and less fluctuating signal. Starting with all the spins up, the magnetization at time t is just the overlap between the configuration at time t and the initial one. If we measure the overlap between the configuration at time t and one at a fixed small time ($t_w = 3$ in our case), we expect that $M(t, t_w)$ behaves like $M(t)$, with similar statistical fluctuations, but with a signal ten times greater.

The results of the simulations of the EA model in 6 dimensions can be found in figure 5 where we have plotted $M(t, t_w = 3)$ versus the simulation time t; the line is the best power fit which gives an exponent $\lambda = 0.995 \pm 0.005$. This value is compatible with 1, but not with the Mean Field value $\lambda_{\rm MF} = 5/4$.

As explained in the introduction we believe that the upper critical dimension becomes $d_u = 8$ for such quantities linked to a magnetic field. In this case we start the simulation with the system totally magnetized, as if it was feeling an infinitely strong magnetic field,

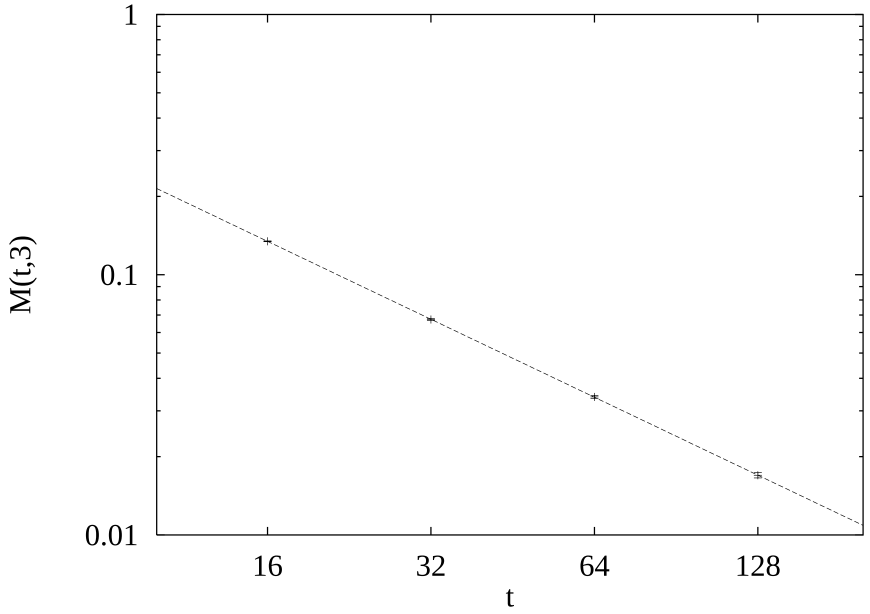

Figure 5: Remanent magnetization in $d = 6$ at $T = T_c$ together with the best power fit, which gives an exponent $\lambda = 0.995 \pm 0.005$.

so the remanent magnetization may be one such quantities.

For this reason we have done some simulations of the Ising Spin Glass model in $d = 8$ (again with the couplings $J = \pm 1$) to see if we recover the Mean Field behavior of the remanent magnetization.

The critical temperature in $d = 8$ that we have used was extrapolated from the critical temperatures in d=3 [19], 4 [20] and 6 [6]. In the limit of $d \to \infty$ the critical temperature diverges like $T_c(d) \simeq \sqrt{2d}$. Moreover, in the Bethe-Peierls approximation, there is an exact formula for the critical temperature [21, 22]

$$(2d-1)\tanh^2(\beta_c) = 1 \tag{47}$$

We have used this formula, which turns out to be valid in the $d \to \infty$ limit, as a starting point and adding to it some terms which may mimic the finite dimensions corrections. In particular we have substitute the r.h.s. of eq.(47) with a fourth order polynomial in $1/d$ (the term of zeroth order being always 1) and we tried to fit the known critical temperatures by fixing two term of the polynomial to zero and leaving free the coefficients of the other two terms. Among the six possible choices we have selected the one with smallest value of χ^2, which is

$$(2d-1)\tanh^2(\beta_c) = 1 + \frac{B}{d^2} + \frac{D}{d^4} \tag{48}$$

with $B = 0.95 \pm 0.14$ and $D = 117 \pm 4$. With this fit we estimate the critical temperature for the eight-dimensional model as $\beta_c(d = 8) = 0.270 \pm 0.001$. The error reported is an underestimate of the real one because there are systematic deviations due to the arbitrary choice of the fitting function.

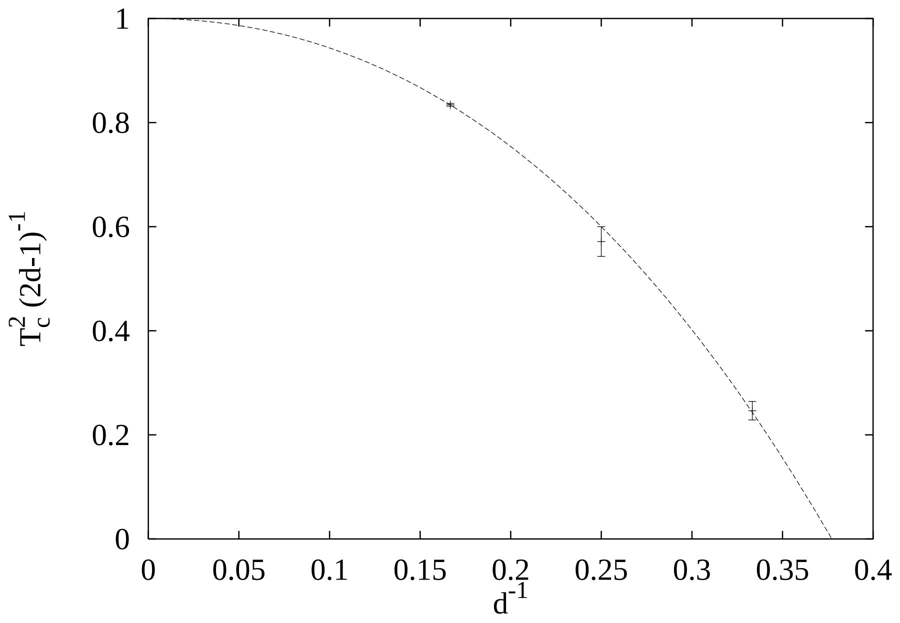

Figure 6: Critical temperatures of the EA model against the dimensionality. The line is a polynomial fit as described in the text.

We have also repeated the analysis looking at the quantity ${T_c}^2/(2d-1)$, which takes values in the range $[0,1]$. Knowing that in the $d \to \infty$ limit

$$\frac{{T_c}^2}{2d-1} = 1 \quad , \tag{49}$$

we have tried to fit the critical temperatures by adding to the r.h.s. of eq.(49) a polynomial in $1/d$, obtaining as the best interpolation, reported in figure 6,

$$\frac{{T_c}^2}{2d-1} = 1 - 5.16(26)\frac{1}{d^2} - 5(1)\frac{1}{d^3} \quad . \tag{50}$$

From the plotted fit we obtain an estimate of $\beta_c(d=8) = 0.270 \pm 0.001$. This error is too small because systematic deviations due to the arbitrarily choice of the fitting function are not taken into account.

Anyway the estimates of $\beta_c(d=8)$ obtained with different interpolations are all in the range $[0.260, 0.270]$ and we think that the true critical temperature is with very good probability in this range; in fact, as one can see in figure 6, the value of $\beta_c(d=8)$ is strongly dependent on the value of $\beta_c(d=6)$, which is known with high accuracy, and on the way $\beta_c(d)$ tends to zero as the dimensionality is increased.

For example if we assume that the successive improvements of the Bethe-Peierls approximation tend to increase the value of $\beta_c(d)$ for each d, then the formula (47) will give a lower bound for the inverse critical temperature; in $d=8$ this lower bound reads $\beta_c(d=8) > 0.264$.

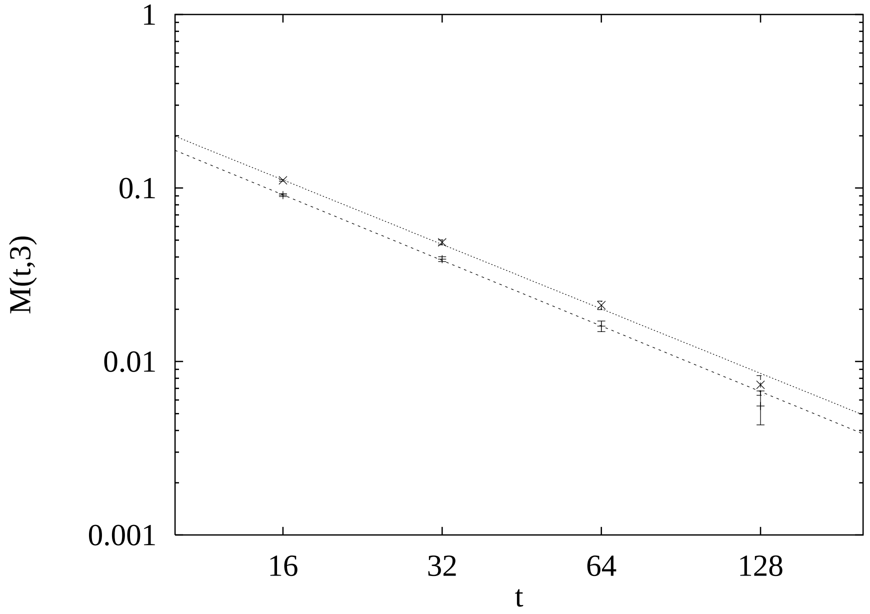

Figure 7: Remanent magnetization in $d = 8$ at two temperatures very near to the critical one. The lines are the power fits which give exponents compatibles with the Mean Field predictions $\lambda_{\rm MF} = 5/4$.

From figure 6 we can also get another important information: the point where the fitting function crosses the d^{-1}-axis may give us an estimate of the lower critical dimension, which turn to be $d_l \simeq 2.65$.

In figure 7 we have plotted the data, with the best power fits, of the remanent magnetization in $d = 8$ at temperature $\beta = 0.260$ and $\beta = 0.270$. From the fits we get the exponents $\lambda(\beta = 0.260) = 1.256 \pm 0.08$ and $\lambda(\beta = 0.270) = 1.235 \pm 0.013$, which are both compatible with the MF result ($\lambda_{\rm MF} = 5/4$).

4 Results in $d = 6$

The major part of the simulation work was done at the critical temperature, chosen as the weighted mean between the one found by Wang and Young [6] ($T_c = 3.035 \pm 0.01$) and the one calculated by series expansion [23] ($T_c = 3.027 \pm 0.005$): $\beta_c = 0.3302 \pm 0.0005$. In particular we have tested that the exponents we measure do not vary significantly if the temperature is changed by an amount of the order of the uncertainty on T_c. To this purpose we have simulated the same system (of volume 8^6) at the inverse temperatures $\beta_1 = 0.330$ and $\beta_2 = 0.331$, checking that the dynamics were compatible.

Having verified that, for the range of time and sizes we have used, the exponents we are interested in do not depend on the precise choice of T_c choice, we have decided to run all the subsequent simulations at $\beta_c = 0.330$. At the critical point we have simulated more than two hundred samples of size 8^6, 13 samples of size 10^6 and also 106 samples of an

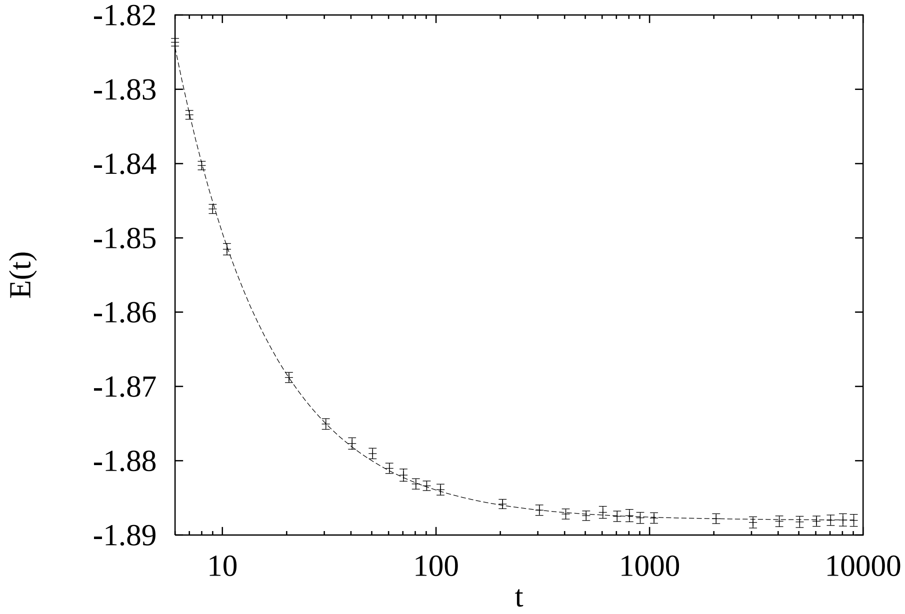

Figure 8: Energy decay in a system of volume 10^6 at the critical temperature. The line plotted is the best power fit, which gives a value for the exponent $\delta = -0.98 \pm 0.01$ compatible with the MF one $\delta_{\mathrm{MF}} = -1$.

asymmetric lattice 12×8^5. The last number of samples may appear too small to average over the disorder; in fact we have used mainly the data from the 8^6 systems to calculate the moments of the distribution of the overlaps. The data from the bigger systems have been used to measure almost-self-averaging quantities like energy whose fluctuations are very small considering that we are working with a system with a million of spins.

The results are shown in figure 8 for the energy decay and in figure 9 for the non linear susceptibility growth.

We have tried to fit the energy decay both to a power law ($E(t) = E_\infty + At^{-\delta}$) and to a logarithmic law ($E(t) = E_\infty + A[\ln(t/\tau)]^{-\delta}$). We are interested in the asymptotic behavior of the decay; then we fit the data in the range $t \in [t_{min}, \infty)$ for various choices of t_{min} and we expect that the parameters of the fit converge quickly when we increase t_{min}. The impossibility of fitting all the data with a single law (for $t < 6$) is due to the existence of an initial short time regime of a few steps during which the dynamic does not follow yet the asymptotic behavior. We find that the logarithmic law does not describe well the data because, even if it has more adjustable parameters, the best values of the parameters depend strongly on t_{min}, they are very correlated and they tend to unphysical values. On the other hand we find that fitting with the power law the values of the parameters E_∞, A and δ converge to a stable value, with t_{min} of order of few Monte Carlo Steps (MCS). In figure 8 we plot the power fit obtained with $t_{min} = 6$ (this is the lowest value for which the fit satisfies the χ^2 test); the best parameters are: $E_\infty = -1.8880 \pm 0.0001$, $A = 0.37 \pm 0.01$ and $\delta = -0.98 \pm 0.01$. We note that the decay exponent is compatible with the mean field

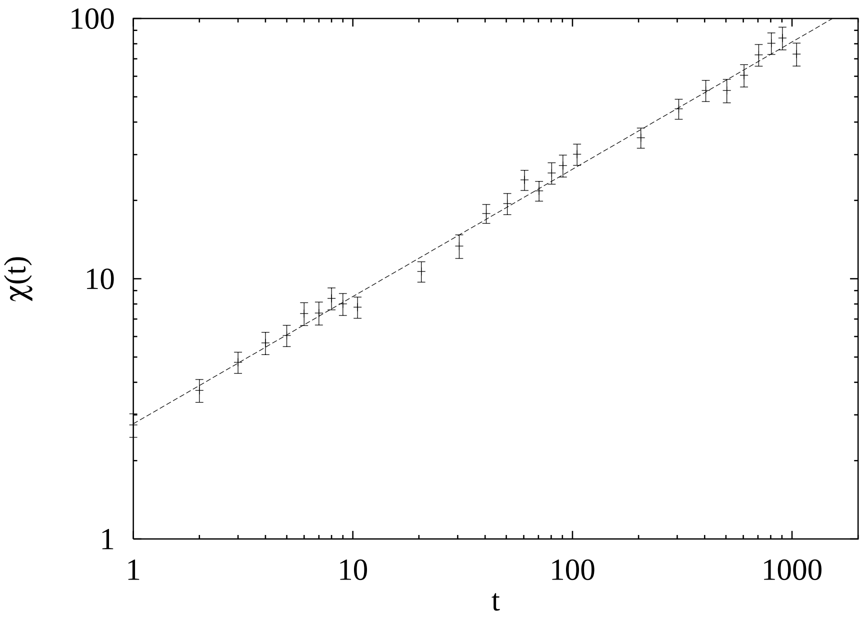

Figure 9: Susceptibility growth in systems of size 8^6 at $T = T_c$. The line is the best power fit which gives an exponent $h = 0.49 \pm 0.01$ in agreement with the MF value $h_{\rm MF} = 1/2$.

value ($\delta_{\rm MF} = -1$).

The line plotted in figure 9 is the best power fit to the susceptibility data. We have to be careful when we try to interpolate in such a way because we know that the susceptibility growth following a power law only in a limited time window; in fact at the beginning of the simulation the dynamics needs some time to reach the asymptotic regime [2] and then because of the finite size of the system they have to converge to some finite value, i.e. the data of figure 9 have to converge to a plateau. These two effects may induce systematic deviations in the estimate of the h exponent: $\chi(t) \propto t^h$. In our case the first transient is almost absent thanks to the sufficiently high temperature and the problems arising from the finite size effects have been solved by fitting the $\chi(t)$ data only in a limited time range far away from the plateau (estimated in a previous longer run). The line reported in figure 9 is the best power law fit to the $\chi(t)$ data (209 samples of an 8^6 system), which gives an estimate of the dynamical exponent $h = 0.49 \pm 0.01$ which is compatible with the mean field value ($h_{\rm MF} = 1/2$).

We also show in figure 10 the results (obtained on the $8^5 \times 12$ lattice) for the h exponent in the low temperature phase ($T/T_c = 0.5, 0.625, 0.75, 0.875$). We also plot in this figure the value obtained at T_c ($h = 0.49$). It is clear that $h(T)$ is a discontinuous function at the critical point and that the limit from below, assuming a linear behavior, ($h(T_c^-) \simeq 0.9$) is almost twice the value of $h(T_c^+) = 0.5$, in very good agreement with the correlation functions (propagators), which are restricted to the $q = 0$ ergodic component, found by De Dominicis *et al.* [9]. The quite small discrepancy can be due to the crossover between the

[2] This initial time increases when the temperature is lowered.

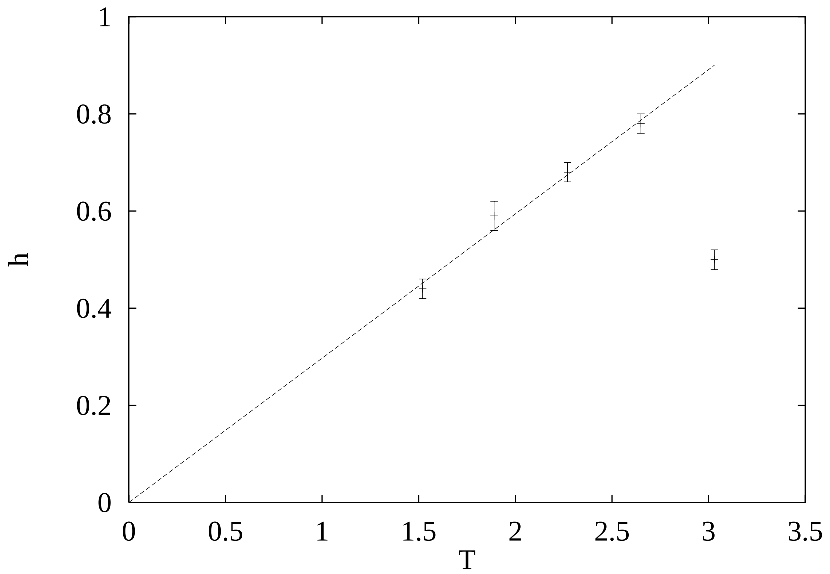

Figure 10: Behavior of the exponent $h(T)$. Note that the linear fit which describes well the data in the cold region tends when $T \to T_c$ to a value almost twice the one measured just at $T = T_c$ and this confirms that the propagators (which are restricted to the $q = 0$ ergodic component) at and below T_c are proportional to p^{-2} and to p^{-4} respectively.

two regimes in a finite lattice or due to logarithmic corrections.

Moreover we can see that the dependence of this exponent is well described by a linear law of the temperature, according to a $z(T)$ proportional to $1/T$.

Finally, we studied the spatial correlations by the following technique (already used with success for the data analysis in three and four dimensions [11, 12]). We expect a functional dependence for this correlation function of the form

$$G(t,x) = \frac{a(T)}{x^\alpha} \exp\left\{ - \left(\frac{x}{\xi(T,t)} \right)^\gamma \right\}, \tag{51}$$

where, as usual, $\xi(T,t) \propto t^{1/z(T)}$ is the dynamical correlation length. For each value of the distance we fit the data of the $q-q$ correlation function to the formula

$$G(x,t) = G_\infty(x) \exp\left[A(x) t^{-B} \right] \qquad \forall \text{ fixed } x, \tag{52}$$

and we verify that the value of the B parameter is almost independent from x and then we fix it during the following study. The dynamical exponent can be expressed as the ratio $z = \gamma/B$, where γ may be estimated by the power law fit $A(x) \propto x^\gamma$. This method yields our third estimate $z = 4.2 \pm 0.2$ which is again compatible with the Mean Field value. In figure 11 we have plotted $\ln G(x=2,t)$ versus t, together with the best fit (eq.(52)).

In the infinite time limit the function $G(x,t)$ converges to $G_\infty(x)$ which must give information on the $q-q$ correlation function at zero overlap, calculated for the SK model by De Dominicis *et al.* [9]. They found that $G_{q=0}(x) \propto x^{-4}$ at the critical temperature.

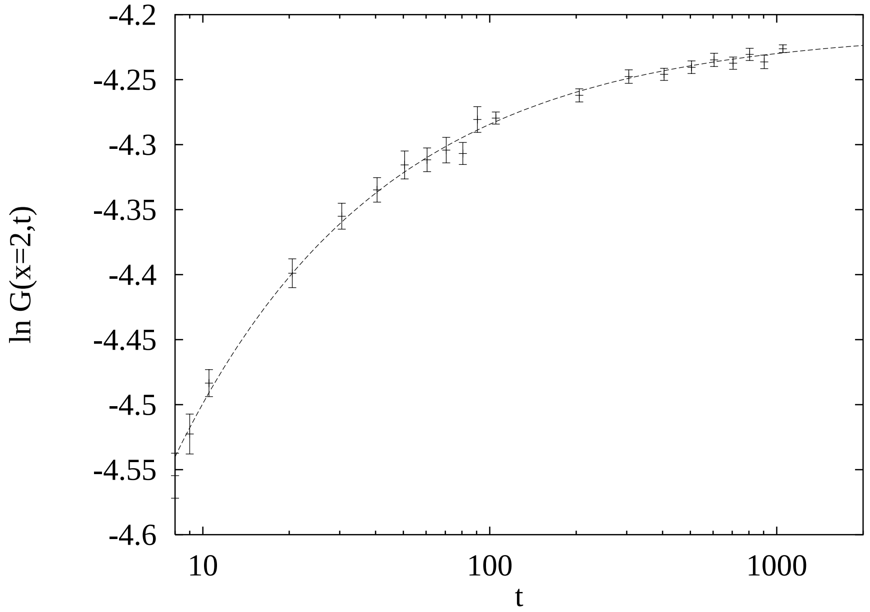

Figure 11: $\ln G(x=2,t)$ vs. t at the critical temperature. The line is the best power fit whose asymptotic value is $\ln G_\infty(x=2)$.

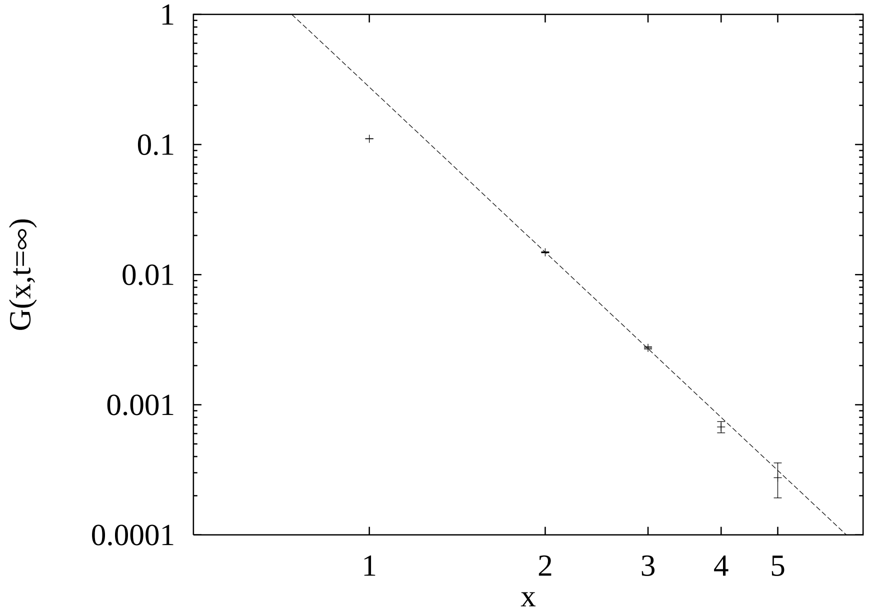

Figure 12: $G_\infty(x)$ at the critical point. The line is the best power fit to the data with $x > 1$, which gives an exponent $\alpha = 4.2 \pm 0.1$.

By our simulation we find the $G_\infty(x)$ plotted in figure 12 together with the best power fit in the range $x \in [2,5]$. We note that the point in $x=1$ if far away from the fit because for $\ln(x) \to -\infty$ the data must converge to 1 (by definition $G(x=0)=1$) while the power fit diverges. Anyway we are interested in the asymptotic behavior which seems to be well described by a power law, $G_\infty(x) \propto x^{-\alpha}$ with an exponent $\alpha = 4.2 \pm 0.1$, in agreement with the Mean Field result previously cited.

5 Conclusions

We have calculated analytically in the Mean Field approximation the exponent of the remanent magnetization and we obtain numerical results that confirm that for this observable the upper critical dimension is eight and not six.

We have calculated numerically for the first time the dynamical critical exponents in six dimensions in three different ways, all compatible within the statistical error.

Thanks to the previous results we can check also the static critical exponents (for instance getting the z value obtained from the scaling of the $q-q$ correlation function), and we obtain values that agree very well with the static critical exponents and the critical temperature found in the literature [6].

Moreover, by monitoring, the growth of the non linear susceptibility in the spin glass phase, we have shown numerical evidence favoring a p^{-4} propagator, constrained to the $q=0$ ergodic component, according with the analytical results obtained assuming spontaneously broken replica symmetry. This numerical result is a further check that it is possible to have spontaneously broken replica symmetry in finite dimensions.

We plan in the future to extend this work to the five dimensional Ising spin glass.

6 Acknowledgments

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