Many-Body Localization, Ergodicity and Multi-fractality

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Critical Phenomena in Random and Complex Systems

Villa Orlandi, Anacapri
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Anderson Localization:

One quantum particle in a random potential

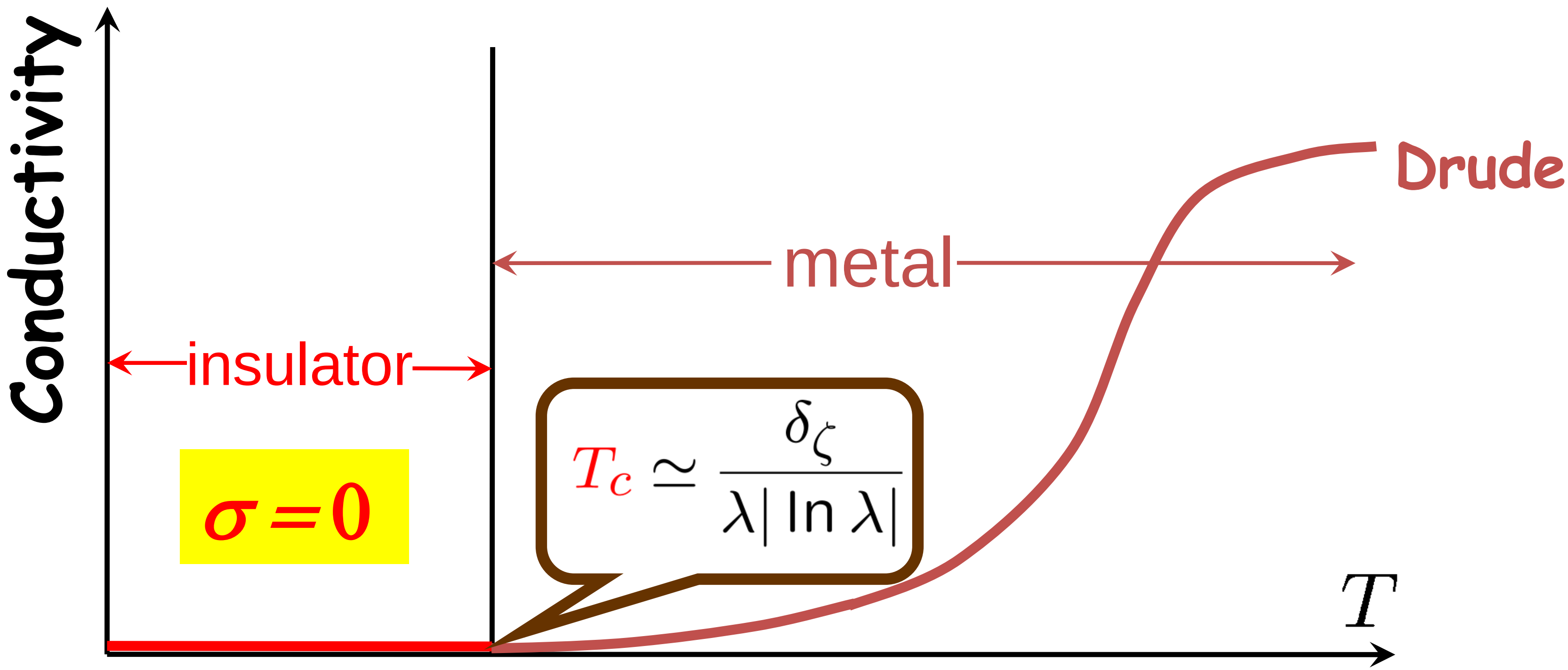
- Strong enough disorder – the eigenstates are localized
- Weak disorder – maybe the eigenstates are extended
- Localization – Delocalization – in real space

Many-Body Localization:

Isolated quantum system, many degrees of freedom

- Close to the integrability – the eigenstates are localized
- Far from the integrability –the eigenstates are extended
- Localized – Extended: – space of quantum numbers $\{\mu\}$

Finite T Metal-Insulator Transition (Basko, Aleiner, BA (2006))

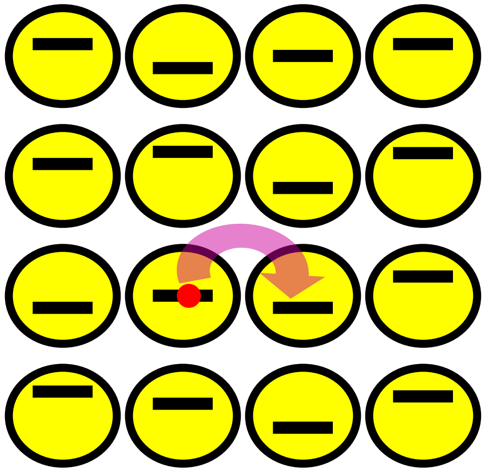


Phononless DC conductivity

1. All one-electron states are localized
2. Electrons interact with each other
3. The system is closed (no phonons)
4. Temperature is low but finite

Conventional Anderson Model

- one particle,
- a lattice in ***d***-dimensional space
- one level per site,
- onsite disorder
- nearest neighbor hoping



basis: $|i\rangle$, i labels sites

Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{V}$

$$\hat{H}_0 = \sum_i \varepsilon_i |i\rangle\langle i| \qquad \hat{V} = \sum_{i,j=n.n.} |i\rangle\langle j| \qquad -\frac{W}{2} < \varepsilon_i < \frac{W}{2}$$

random

Eigenstates $|a\rangle = \sum_i \psi_a(i) |i\rangle$

$\psi_a(i)$ random wave f-ns

Anderson Model for a General Many-Body system

Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{V}$

$$\hat{H}_0 = \sum_{\mu} E_{\mu} |\mu\rangle\langle\mu|$$

Hamiltonian of an integrable system

μ - a set of quantum numbers

$|\mu\rangle$ eigenstates of $\hat{H}_0$,

E_{μ} - eigenenergies

$$\hat{V} = \sum_{\mu,\nu} V_{\mu,\nu} |\mu\rangle\langle\nu|$$

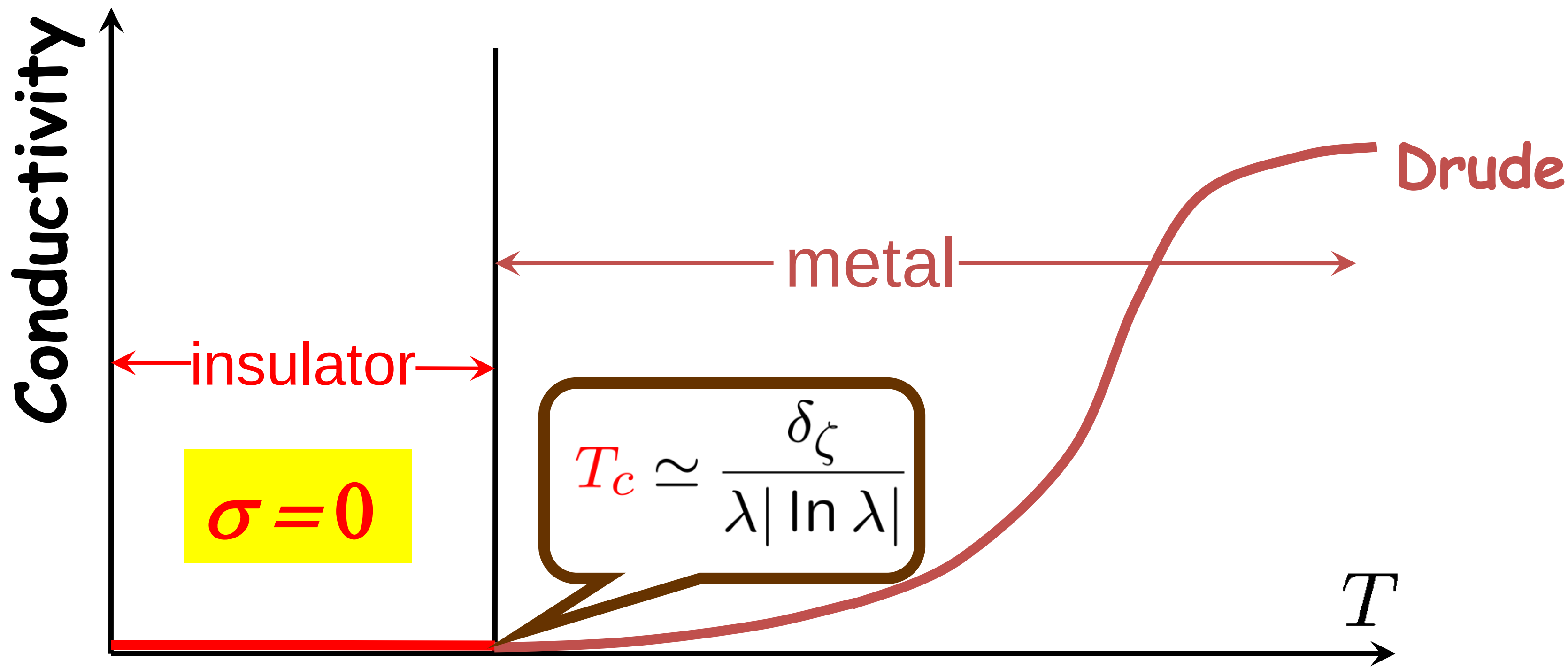
Perturbation, which violates the integrability

μ labels sites of the effective lattice

$V_{\mu,\nu}$ hoping matrix element

Q: What is the geometry/topology of the effective lattice ?

Finite T Metal-Insulator Transition (Basko, Aleiner, BA (2006))



Phononless DC conductivity

- Localized one-electron states
- Electron - electron interaction
- The system is closed
- Finite temperature

$$\hat{H}_0 = \sum_\mu E_\mu |\mu\rangle\langle\mu|$$ One-particle Hamiltonian

$|\mu\rangle$ Many-Body Eigenstates of $\hat{H}_0$

μ Set of the occupation ## of the one-particle eigenstates

$$\hat{V} = \sum_{\mu,\nu} V_{\mu\nu} |\mu\rangle\langle\mu|$$ E-e interactions - Transitions between $|\mu\rangle$ and $|\nu\rangle$

Equipartition – the basic postulate of Statistical Physics

States of a system, which have the same energy are realized with the same probability

A particle located at time t at some particular point after long enough time can be found at any point with the same probability

This is not always correct: Anderson Localization

Many-Body Localization ⇨ No equipartition

Q: Is the equipartition rule always valid for the extended many-body states ?

Localization and Ergodicity – one particle, $2 < d < \infty$

Anderson Model, N sites $\hat{H} = \hat{H}_0 + \hat{V}$

$\hat{H}_0 = \sum_i \varepsilon_i |i\rangle\langle i|$ $\hat{V} = \sum_{i,j=n.n.} |i\rangle\langle j|$ $-(W/2) < \varepsilon_i < (W/2)$ - random

$\psi_a(i)$ random wave functions

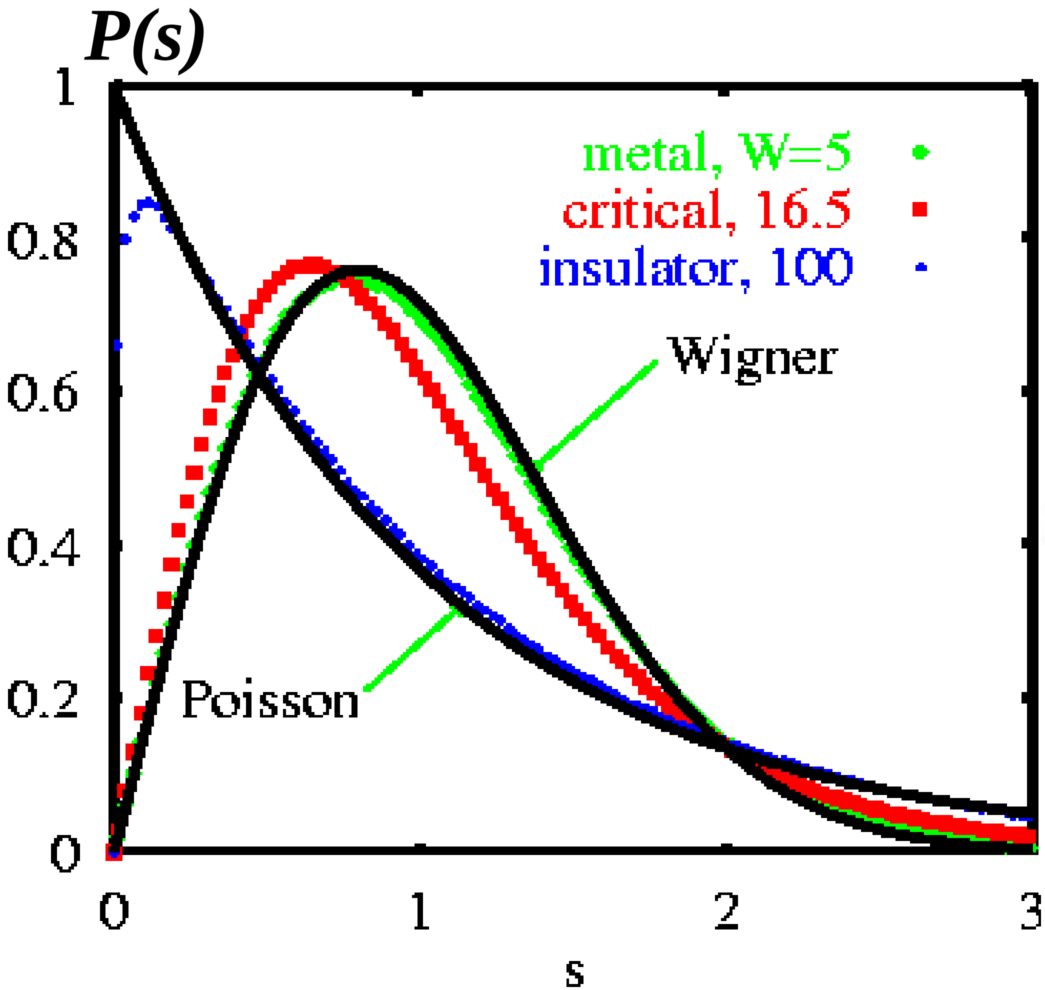
W_c critical disorder

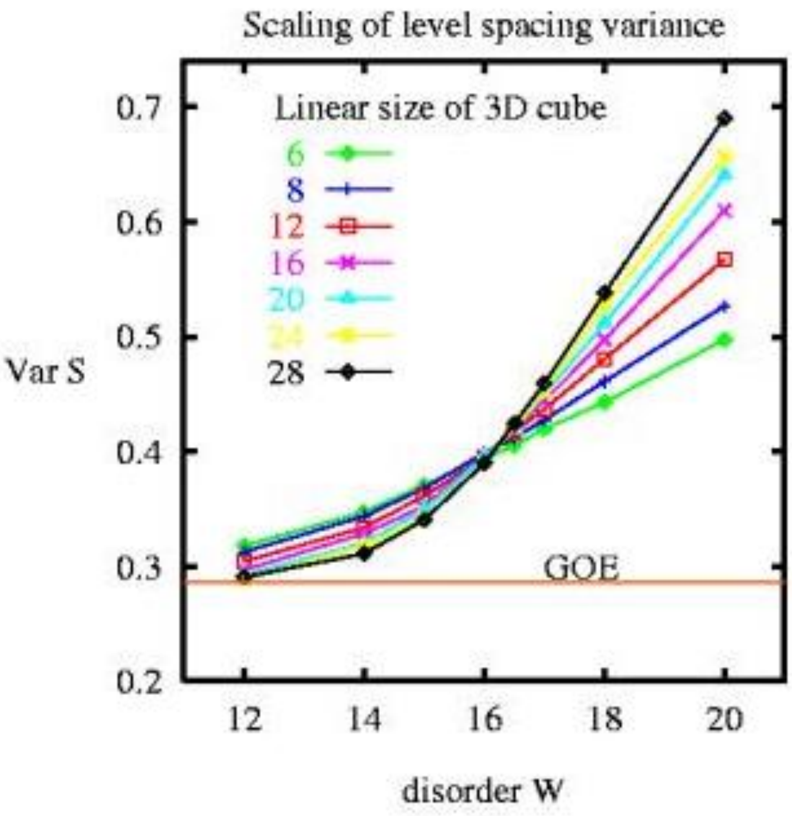
For $N \to \infty$, $W \neq W_c$ fixed, $\psi_a(i)$ are localized if $W > W_c$; extended ergodic if $W < W_c$

Critical behavior: Critical volume: $N_c(W) \xrightarrow[W \to W_c]{} \infty$

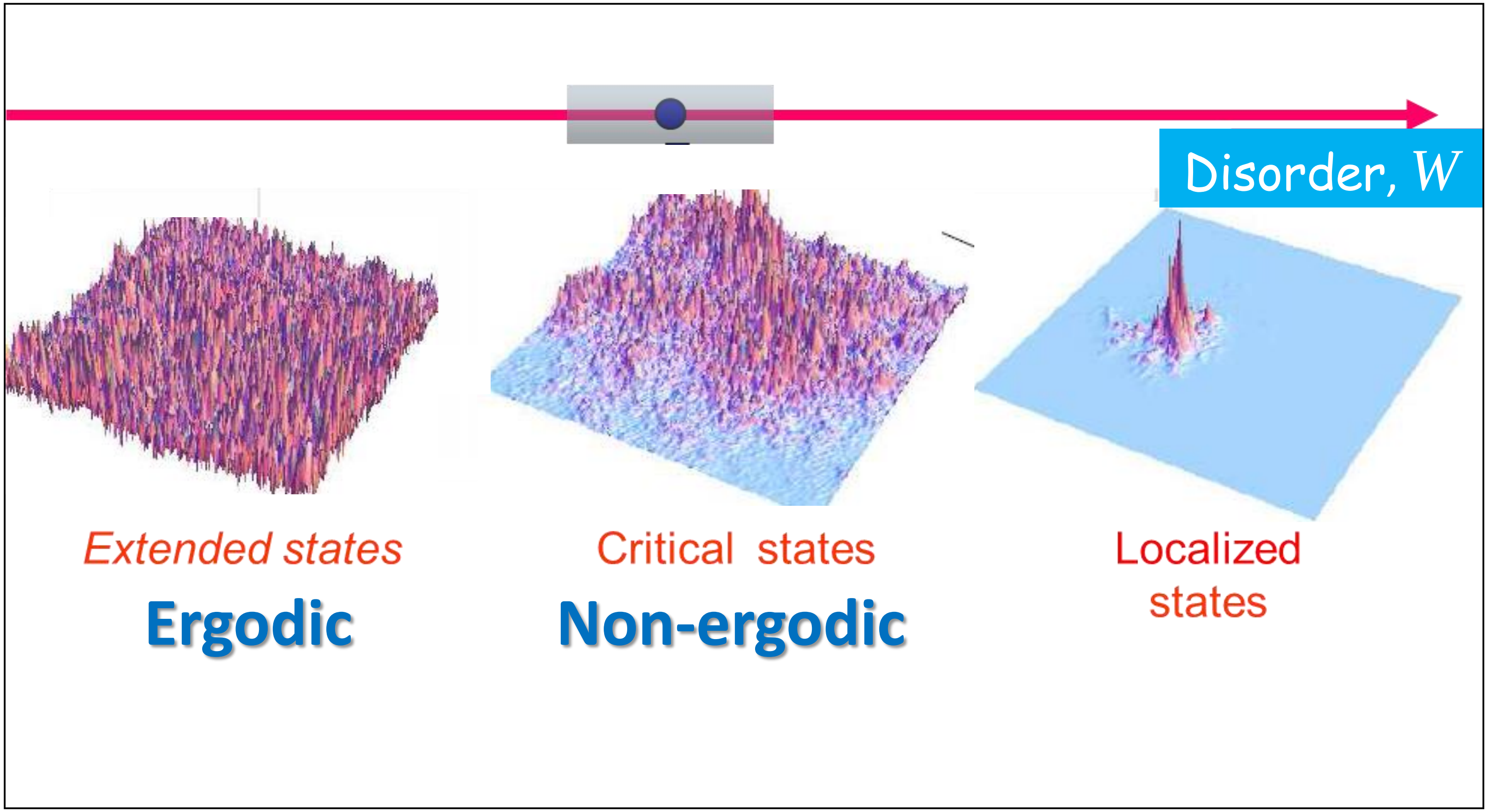
$1 \ll N \ll N_c$ ⇨ $\psi_a(i)$ are extended but non-ergodic $\approx$ multifractal

Anderson transition in terms of level statistics 3D





3D Anderson transition



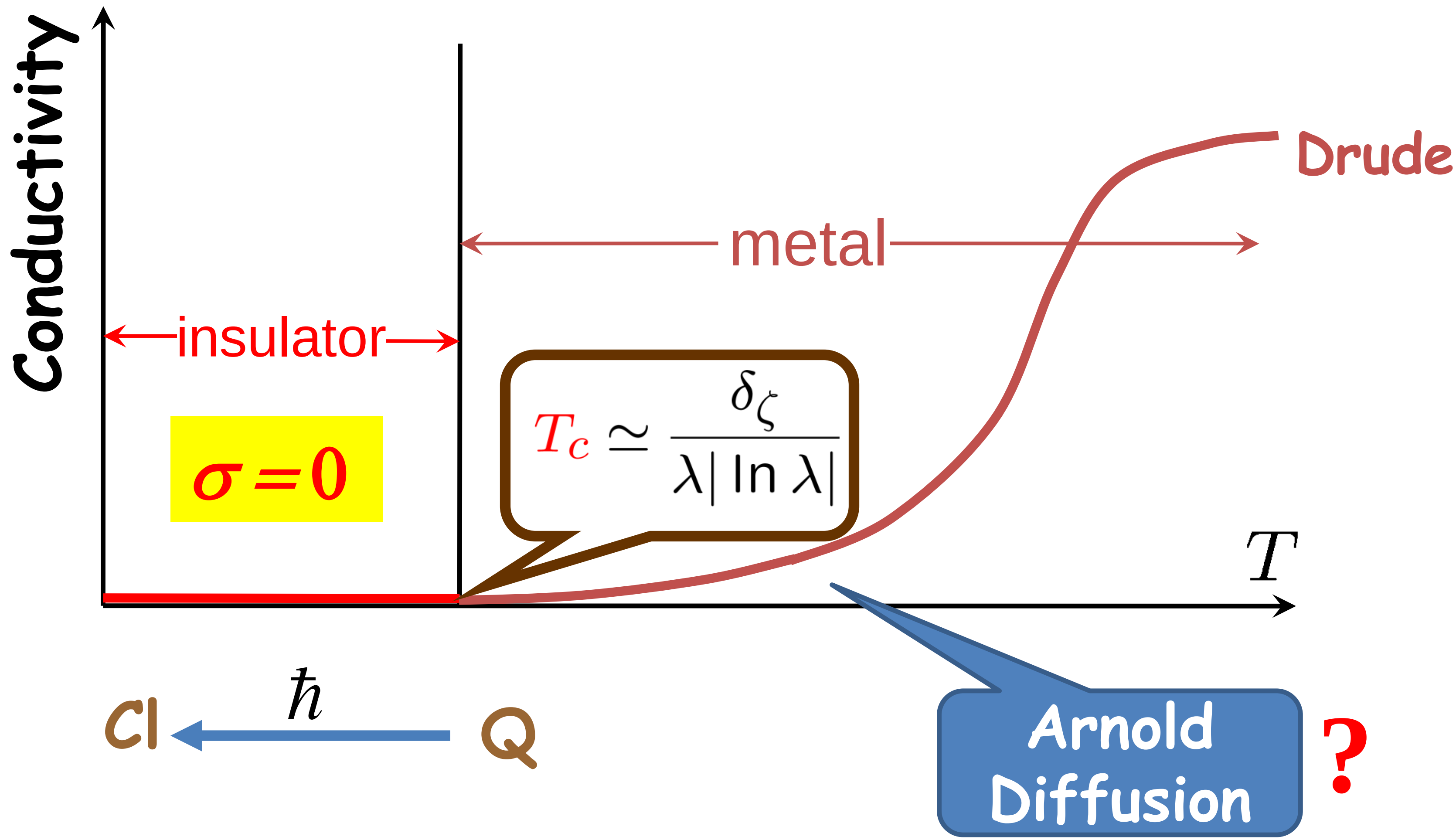
Q: Can extended many-body wave functions be non-ergodic outside the critical region ?

Why this is an interesting question ?

Lack of the ergodicity of the many-body wave functions would imply the violation the equipartition theorem of an isolated macroscopic system and thus

- no thermalization
- glassy like behavior
- etc.

Finite T Metal-Insulator Transition (Basko, Aleiner, BA (2006))



Classical Dynamical Systems

Arnold diffusion is strongly non-ergodic.

Is the dynamics ergodic outside the KAM regime ?

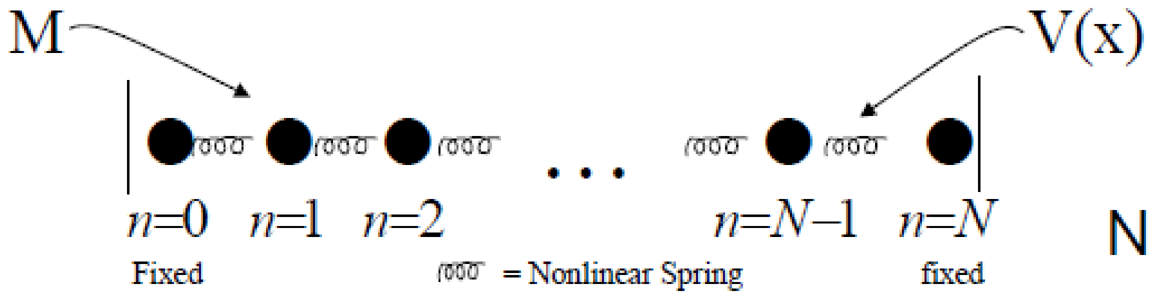
Classical Dynamical Systems:

Are the dynamics ergodic outside the KAM regime?

For some low-dimensional systems one can prove the ergodicity: Sinai billiard, Bunimovich billiard, etc.

At least some systems with high number of dimensions are known to be non-ergodic:

- Solar System
- Fermi-Pasta-Ulam system of connected non-linear oscillators
- ... .

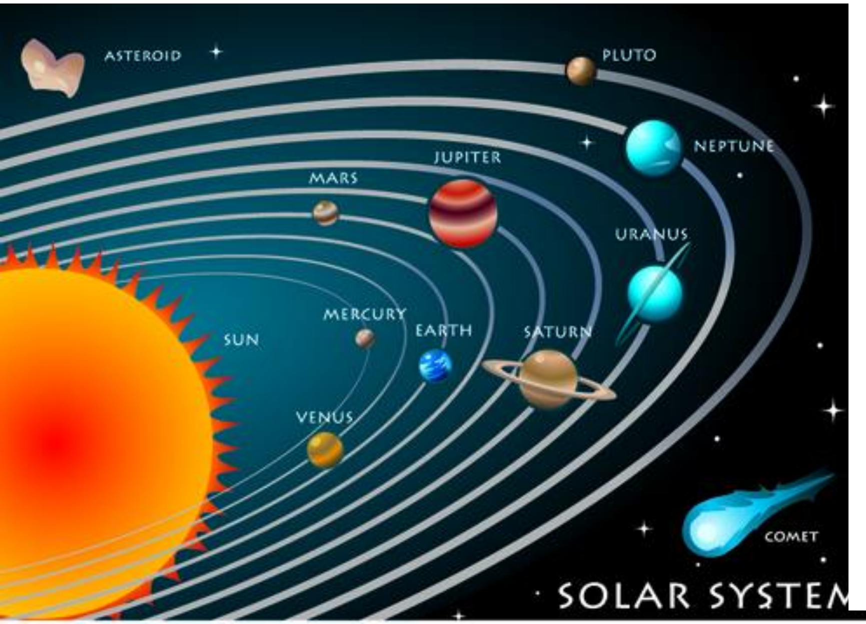


N=32, 64

$$V(x) = \frac{1}{2} kx^2 + \alpha/3\, x^3 + \beta/4\, x^4$$

"The results of the calculations (performed on the old MANIAC machine) were interesting and quite surprising to Fermi. He expressed to me the opinion that they really constituted a little discovery in providing limitations that the prevalent beliefs in the universality of "mixing and thermalization in *non-linear* systems may not always be justified."

[S.Ulam]



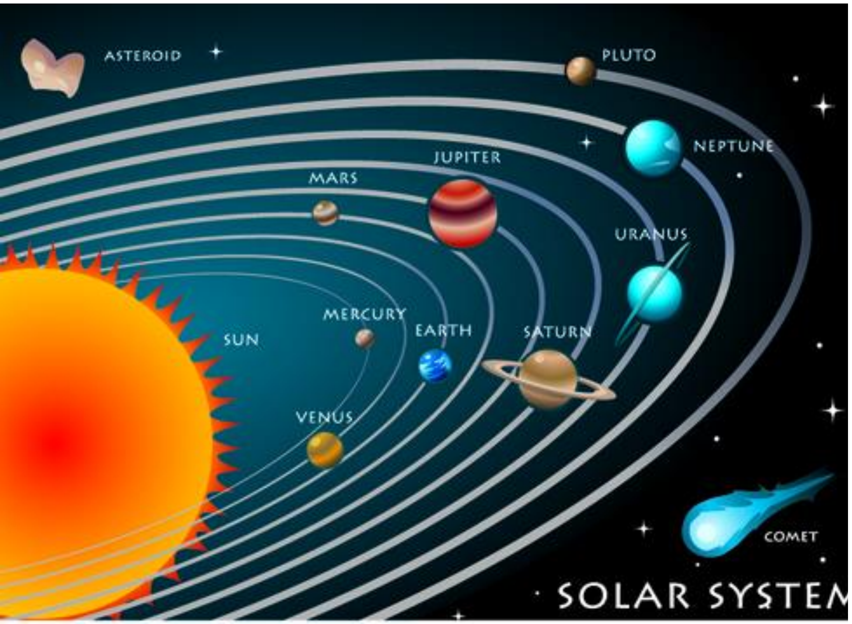
Age: ~4.5 Billion years
Sun dies in ~8 Billion years
Mass 1.0014 Solar masses

Newton:
Motion of a single planet around the Sun. However, there are 8 planets (Newton knew 6). Each one exerts forces on the others – small and periodically varying,.

Newton: "…the Planets move one and the same way in Orbs concentric, some inconsiderable Irregularities excepted, which may have arisen from the mutual Actions of Comets and Planets upon one another, and which will be apt to increase, till this System wants a Reformation.",

God has to intervene continuously to stabilize the world?!

Leibniz sneered at Newton's conception, as being that God so incompetent as to be reduced to miracles in order to rescue his machinery from collapse.



Age: ~4.5 Billion years
Sun dies in ~8 Billion years
Mass 1.0014 Solar masses

Isaac Newton:

Motion of a single planet around the Sun. However, there are 8 planets (Newton knew 6). Each one exerts small and periodically varying forces on the others

- The positions of the planets in >10^8 years are unpredictable: they are too sensitive to initial condition - chaos.
- In 8 billion years (just before the Sun dies) the orbits will most likely be similar to their present ones.
- The unpredictability is mostly in the orbital phases, collisions between planets are unlikely in spite of the chaos.
- Ensemble of solar systems with slightly different parameters at the present time (random shifts ~1mm): ~1% percent probability that Mercury collides with Venus before the death of the Sun.

The solar system is neither absolutely stable nor ergodic

Multifractality

Moments of the inverse participation ratio:

$$I_q(N) \equiv \sum_i |\psi_a(i)|^{2q}$$

$$I_1(N) = 1$$

normalization

Scaling with $N \to \infty$

$$I_q(N) = O\left(N^0\right) N^{-\tau(q)}$$

$$\tau_1 = 0$$

Ergodicity:

$$\tau(q) = q - 1 \Leftrightarrow |\psi_a(i)|^2 = O\left(N^{-1}\right)$$

Exponentially localized states:

$$\tau(q) = 0 \quad \forall q$$

Multifractality

$$D_q \equiv \frac{\tau(q)}{q-1}$$

Fractal dimensions **differ from** 0 and 1

They depend on q

Spectrum of fractal dimensions

Statistics of the onsite values of the eigenfunctions $\left|\psi_a(i)\right|^2$

Distribution function $P(\alpha)$

$$\alpha_i = -\frac{\ln\left|\psi_a(i)\right|^2}{\ln N}$$ random variable

Multifractal ansatz $$P(\alpha) \xrightarrow[N\to\infty]{} O(N^0)\times N^{f(\alpha)-1}$$

$$f(\alpha) \equiv 1 + \lim_{N\to\infty}\left\{\frac{\ln\left[P(\alpha)\right]}{\ln N}\right\}$$

Spectrum of Fractal Dimensions

Legendre transform of τ_q

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Spectrum of Fractal Dimensions

Legendre transform of τ_q

Properties of $f(\alpha)$

- $\int P(\alpha)\,d\alpha = 1$ ⇒ $f_{\max} = 1$
- $\left|\psi_a(i)\right|^2 < 1$ ⇒ $f(\alpha)$ is a convex function

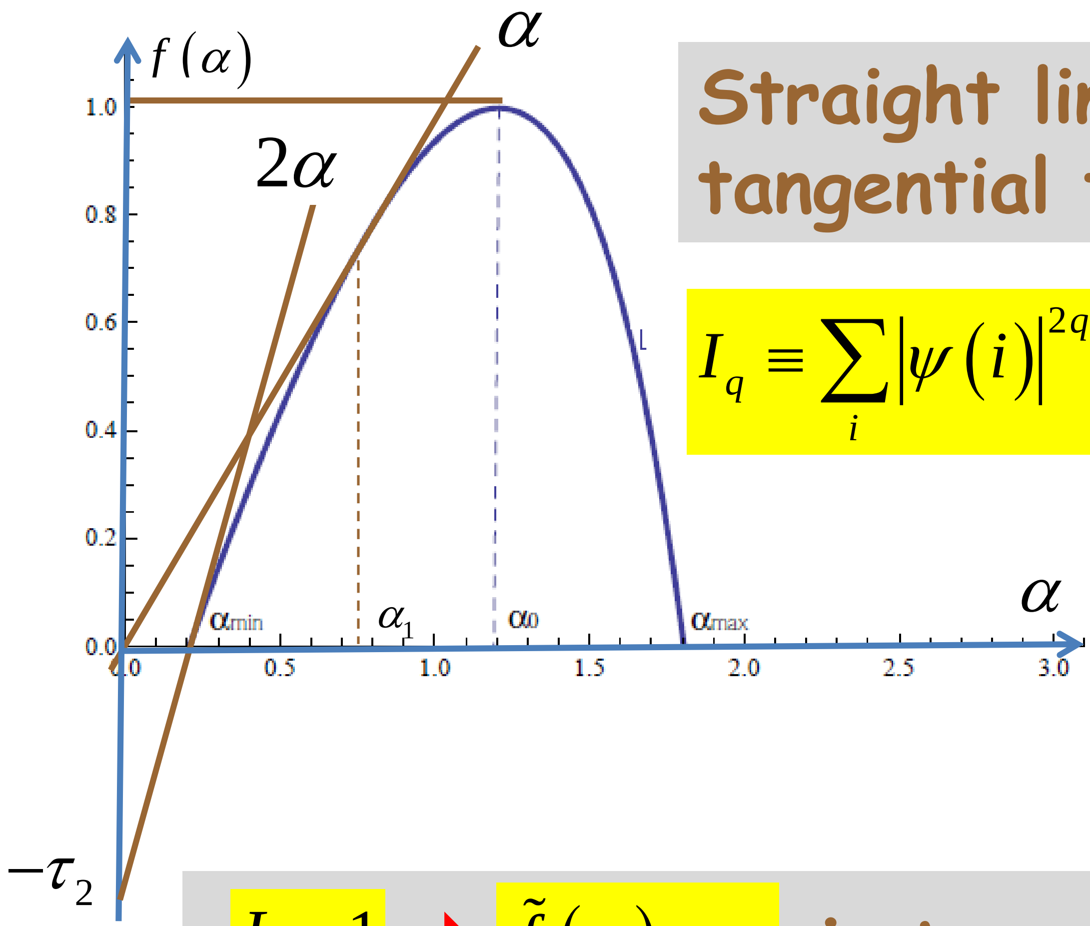
Typical spectrum of fractal dimensions

$$\alpha_q : \left[\frac{\partial f}{\partial \alpha}\right]_{\alpha=\alpha_q} = q$$

$$I_q = N^{-\tau_q}$$

$$\tau_q = q\alpha_q - f(\alpha_q)$$

$$D_q = \frac{q\alpha_q - f(\alpha_q)}{q-1}$$



Straight lines $q\alpha - \tau_q$ are tangential to $f(\alpha)$ at points α_q

$$I_q \equiv \sum_i |\psi(i)|^{2q}$$

$I_0 = N$ ⇨ $\alpha_0 = N$ ⇨ $f_{max} = 1$

$|\psi(i)|^2 < 1$ ⇨ $\partial\tau_q / \partial q > 0$ ⇨ $f(\alpha)$ is convex

$I_1 = 1$ ⇨ $\tilde{f}(\alpha) = \alpha$ is tangential to $f(\alpha)$ at $\alpha = \alpha_1$

Support set exponent

$$D_1 = \left[\frac{\partial \tau_q}{\partial q} \right]_{q=1} = \alpha_1 = f(\alpha_1)$$

Shannon Entropy

$$\sum_i |\psi(i)|^2 \ln\left(|\psi(i)|^2 \right) = \alpha_1 \ln N$$

Renyi Entropies

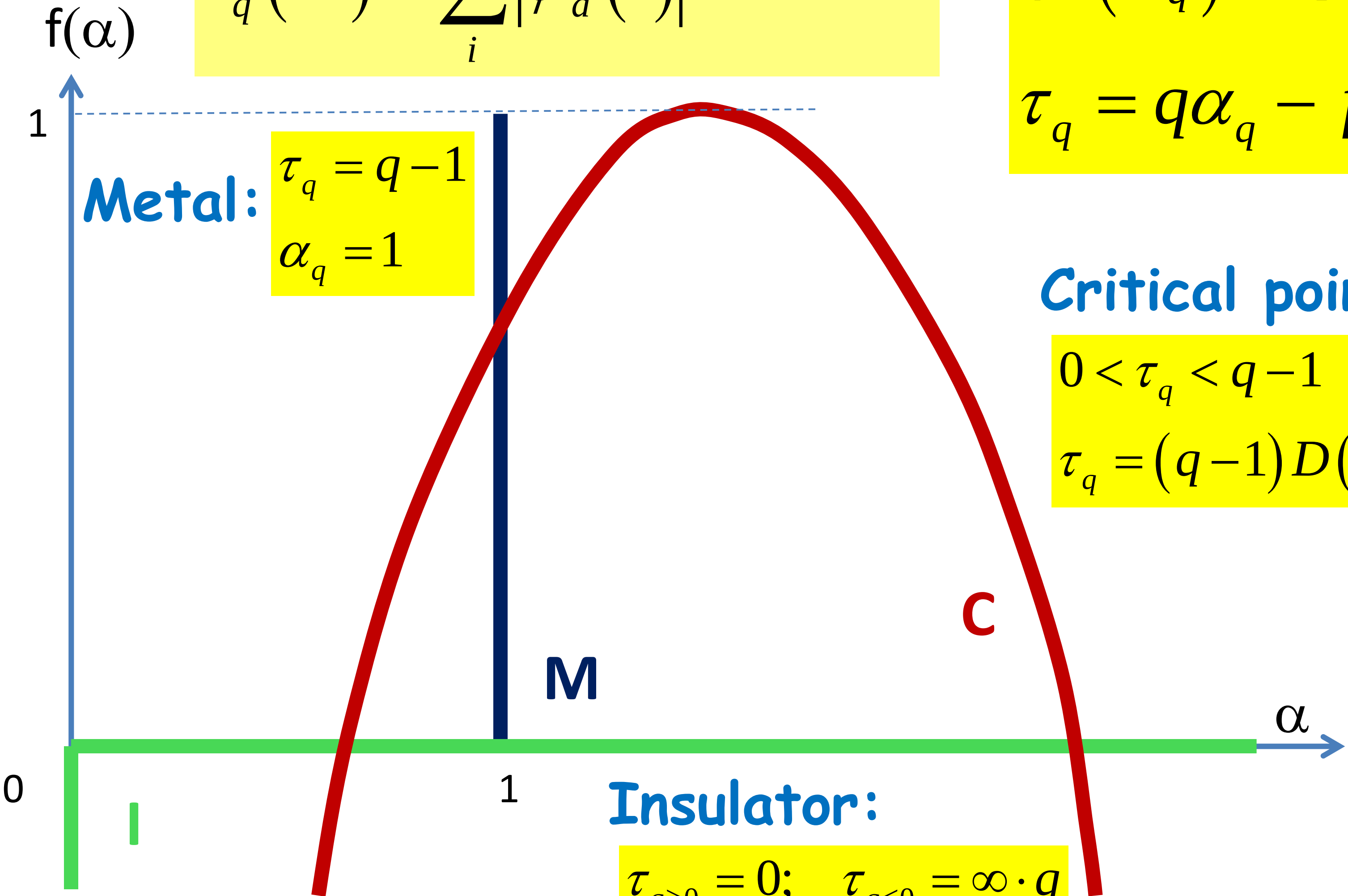
$$D_q \ln N$$

$f(\alpha)$ for a d-dimensional lattice

$$I_q(N) \equiv \sum_i |\psi_a(i)|^{2q} \propto N^{-\tau_q}$$

$$f'(\alpha_q) = q \quad \text{def}$$

$$\tau_q = q\alpha_q - f(\alpha_q)$$



Metal:

$$\tau_q = q - 1$$

$$\alpha_q = 1$$

Critical point:

$$0 < \tau_q < q - 1$$

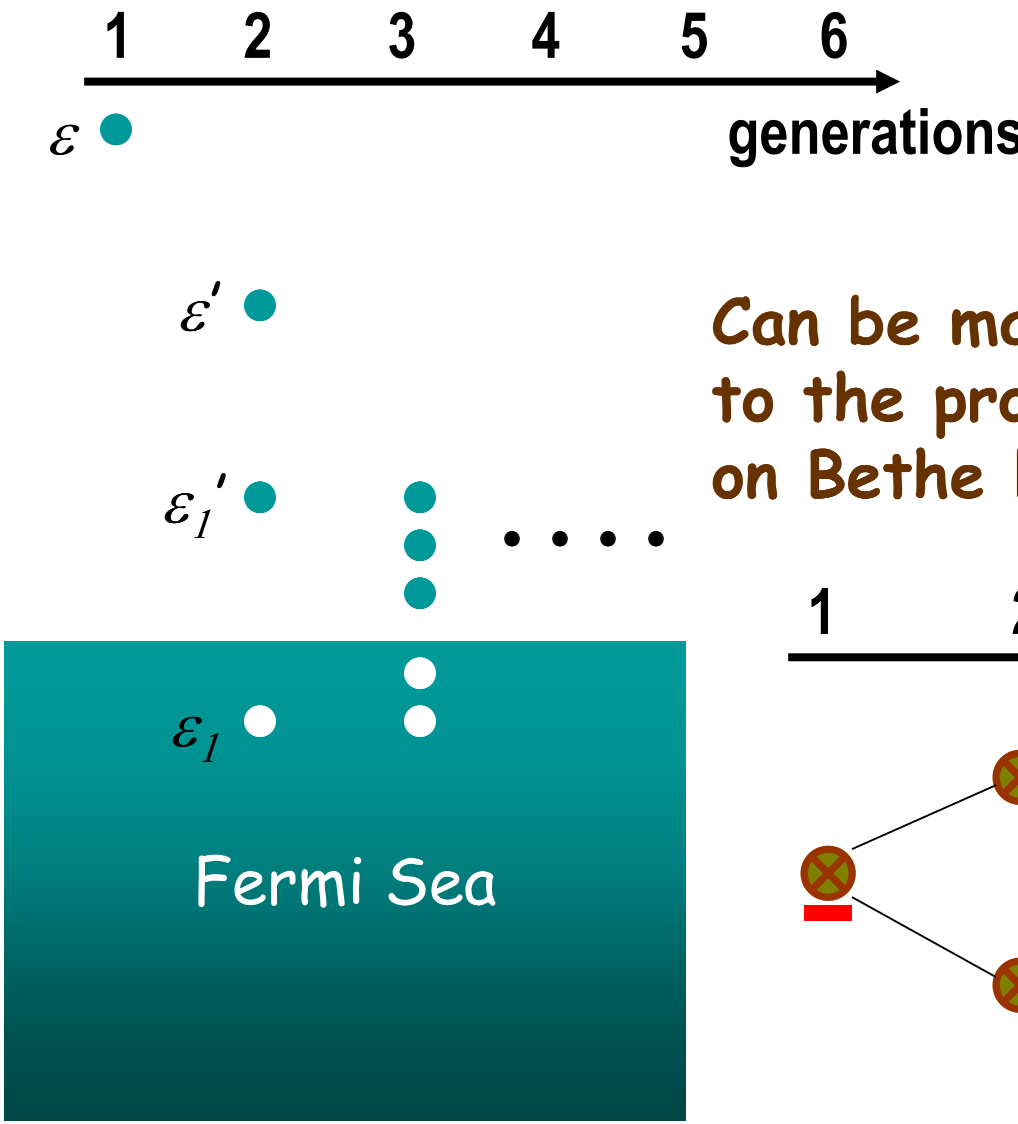
$$\tau_q = (q-1)D(q)$$

Insulator:

$$\tau_{q\geq 0} = 0; \quad \tau_{q\leq 0} = \infty \cdot q$$

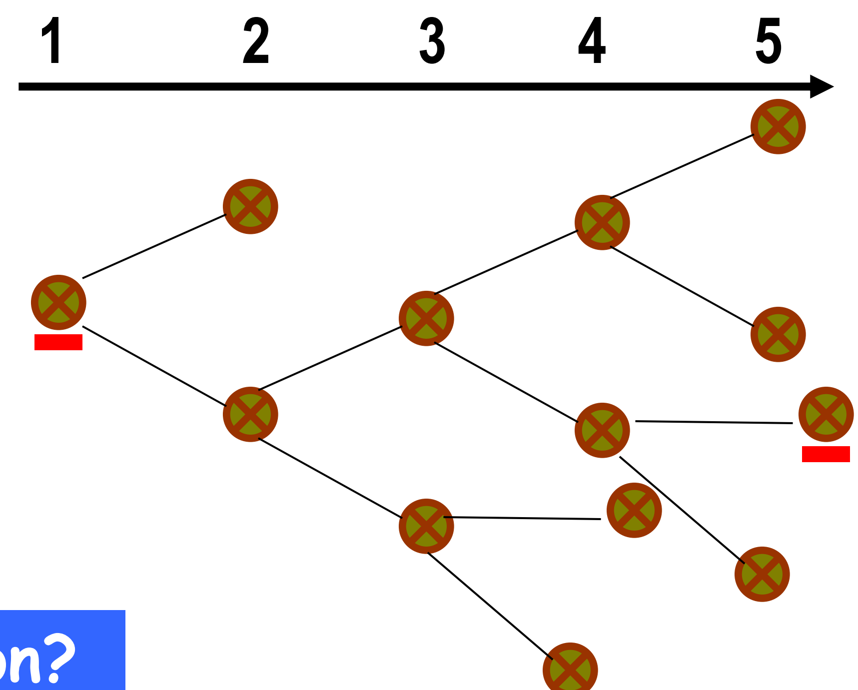
$$\alpha_{q\geq 0} = 0; \quad \alpha_{q\leq 0} = \infty$$

From Many-Body Systems to Bethe Lattice



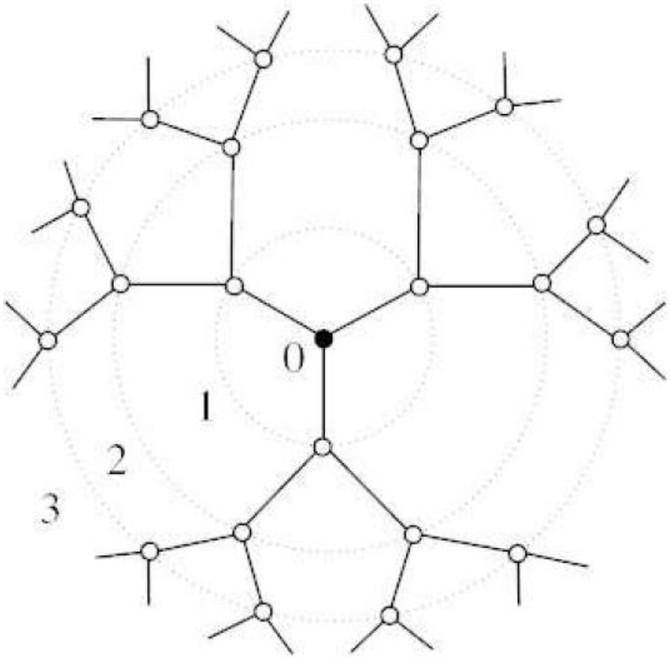
Delocalization in Fock space

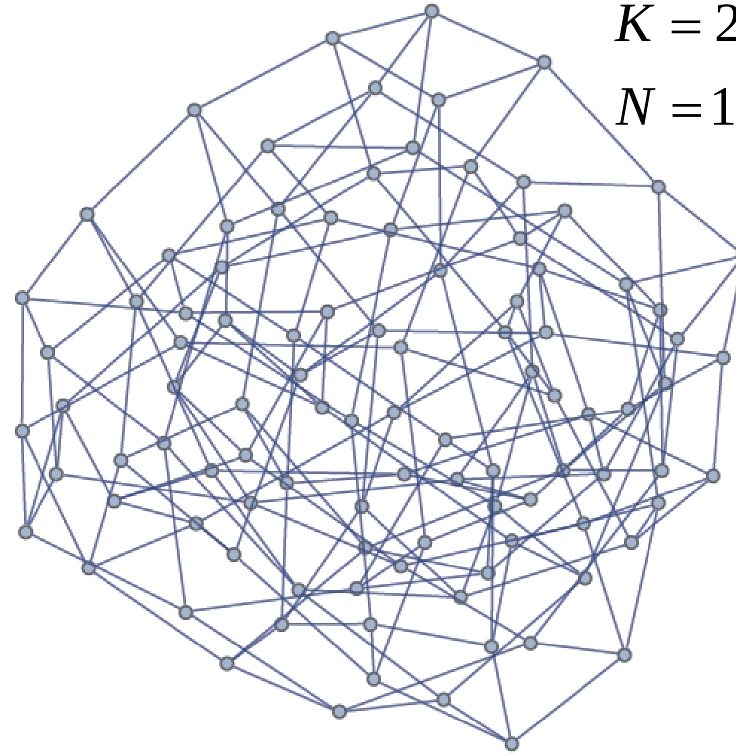
Can be mapped (approximately) to the problem of localization on Bethe lattice



Chaos in Nuclei - Delocalization?

Bethe Lattice





$K = 2,$

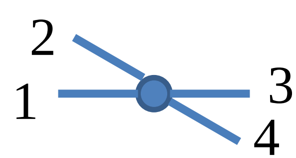
$N = 100$

Cayley tree

not good for numeric: most of the sites are on the boundary

Random Regular Graph

with a fixed connectivity **K+1**

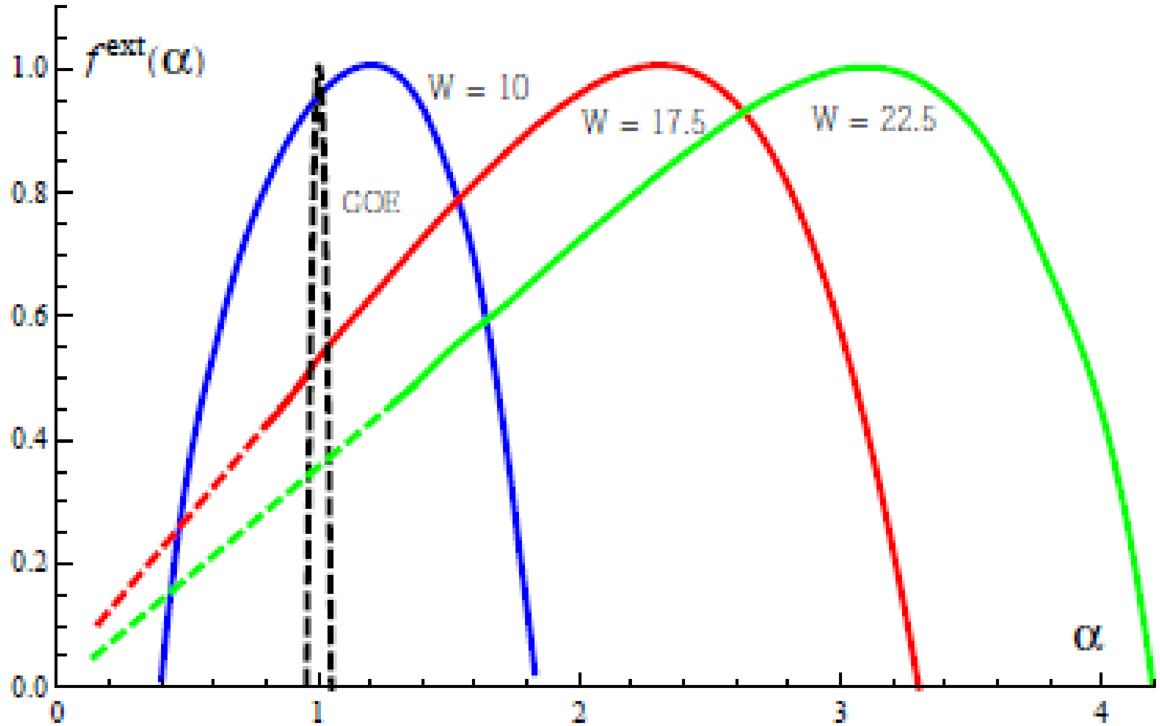


N sites K=3

randomly connected

Q: Can extended eigenstates of the Anderson model on the Bethe-Lattice be non-ergodic outside the critical region **?**

A: YES



Localized states – triangular shape of $f(\alpha)$

Extended states – gradually approach the ergodic limit, but reach it only at $W = 0$

Problem:

N is finite! $N \le 32000$

$$f(\alpha) \equiv 1 + \lim_{N\to\infty} \left\{ \frac{\ln[P(\alpha)]}{\ln N} \right\}$$

We have to deal with

$$f(\alpha, N) = 1 + \frac{\ln[P(\alpha)]}{\ln N}$$

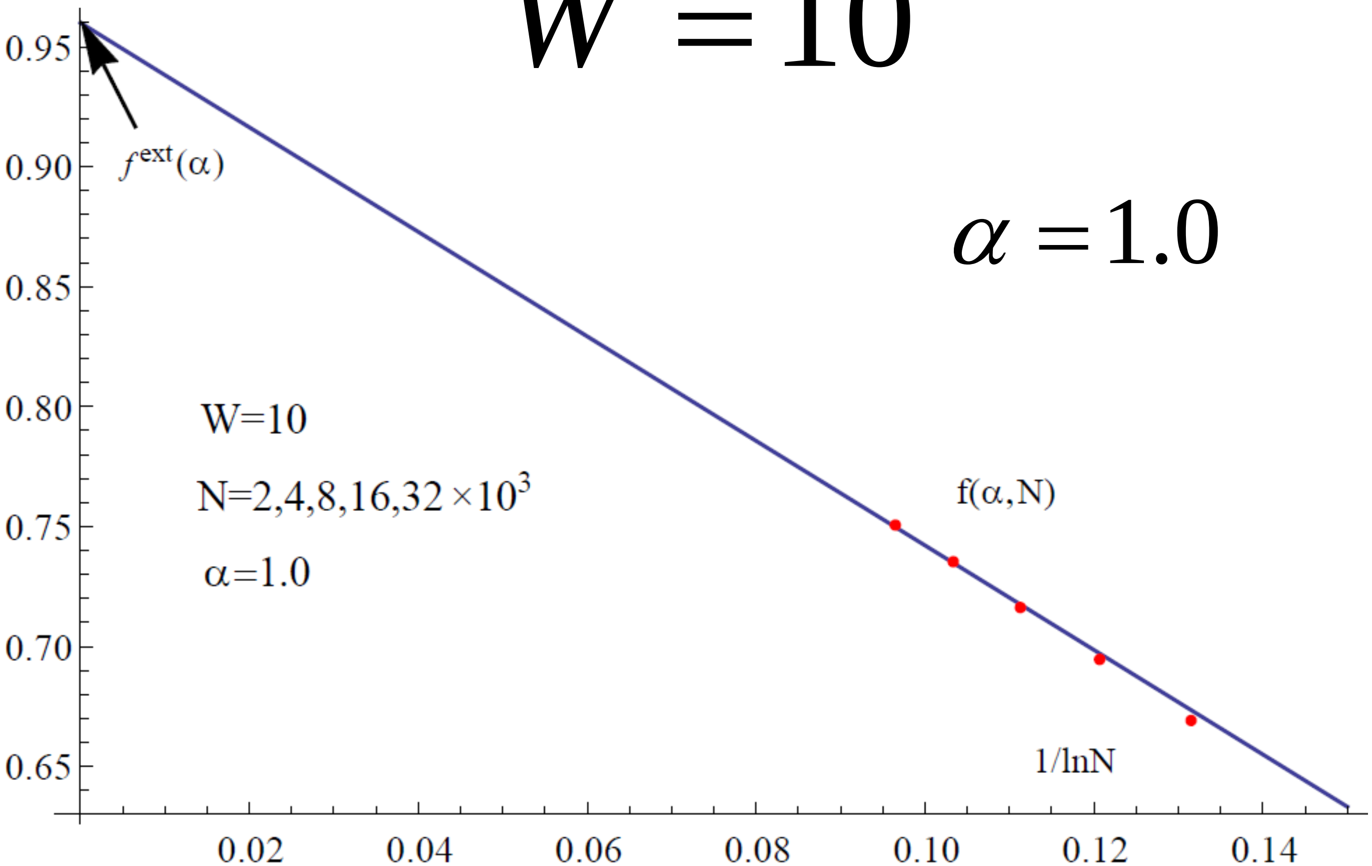
Problem:
very slow convergence

$$f(\alpha, N) = f(\alpha) + O\left(\frac{1}{\ln N}\right)$$

Solution:
extrapolation works surprisingly well

$$f^{ext}(\alpha) = f(\alpha, N) + \frac{A}{\ln N}$$

$W = 10$

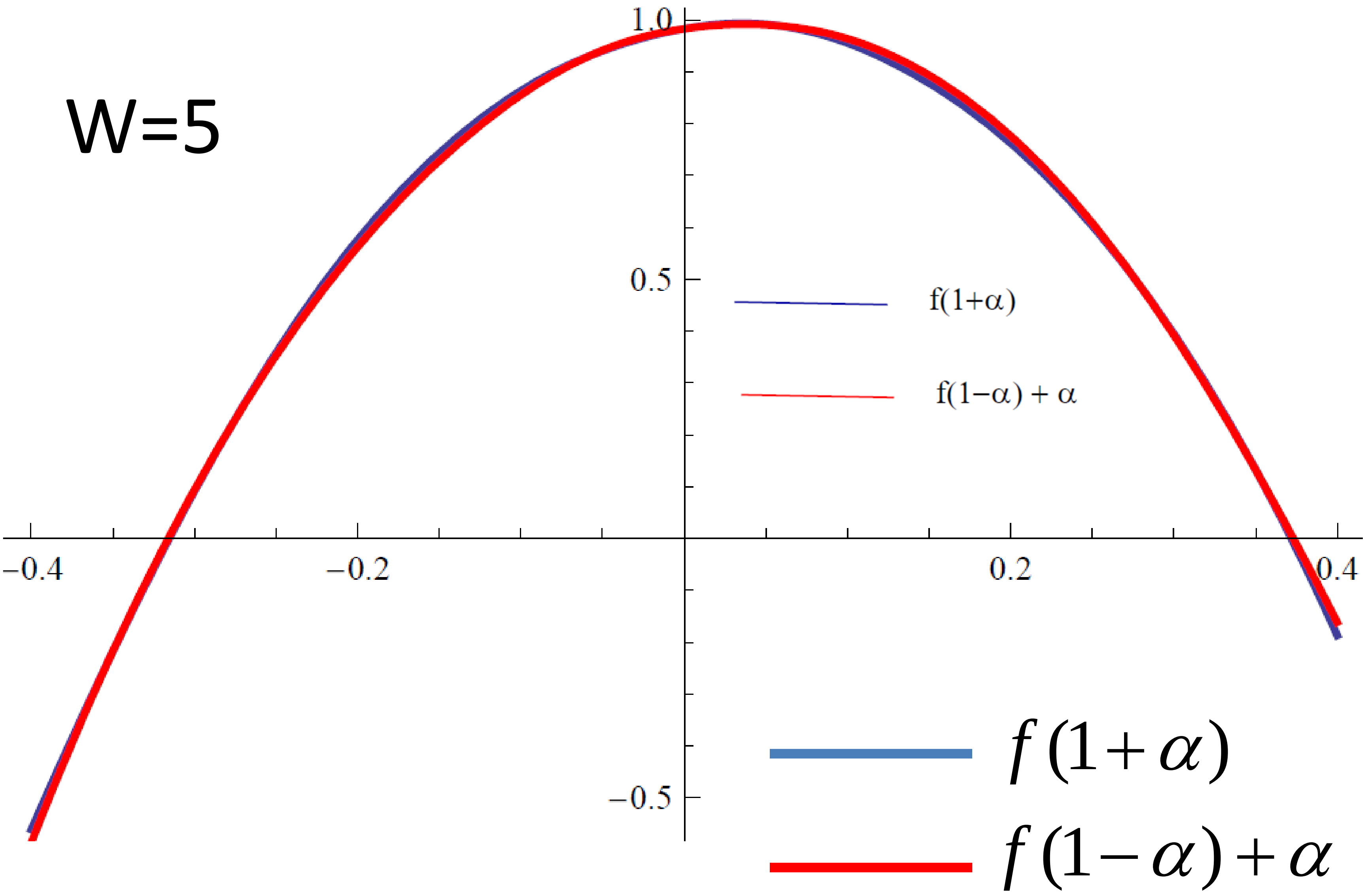


We trust this extrapolation:

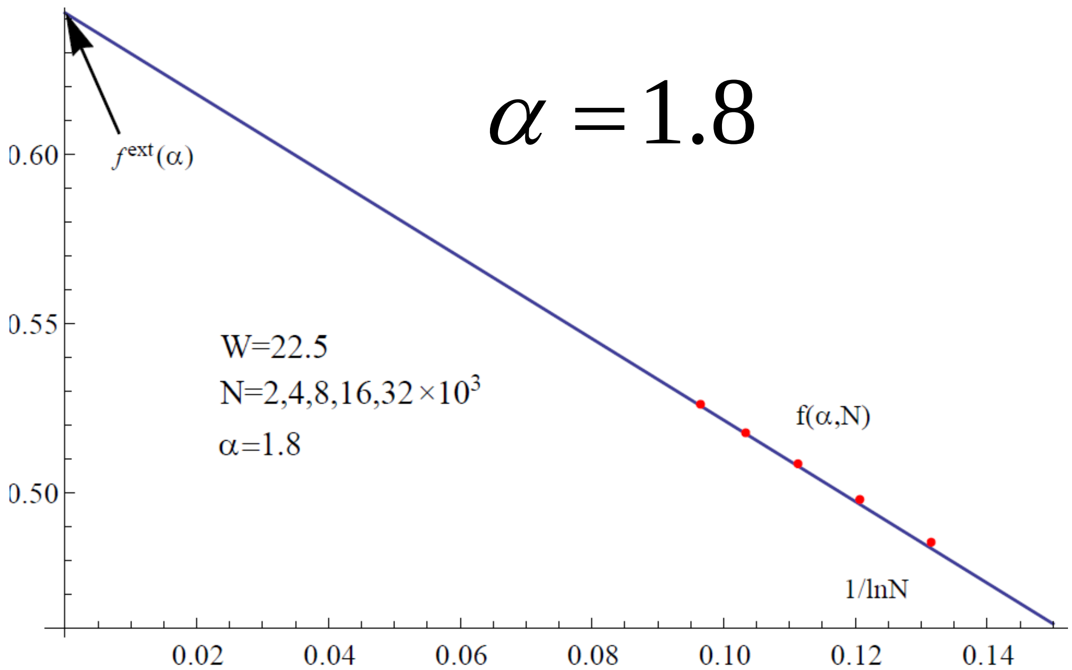
1) $\max\left\{ f^{ext}(\alpha)\right\}$ is very close to **1**, much closer than $\max\left\{ f(\alpha, N)\right\}$ are

2) $f^{ext}(\alpha)$ satisfies the symmetry relation

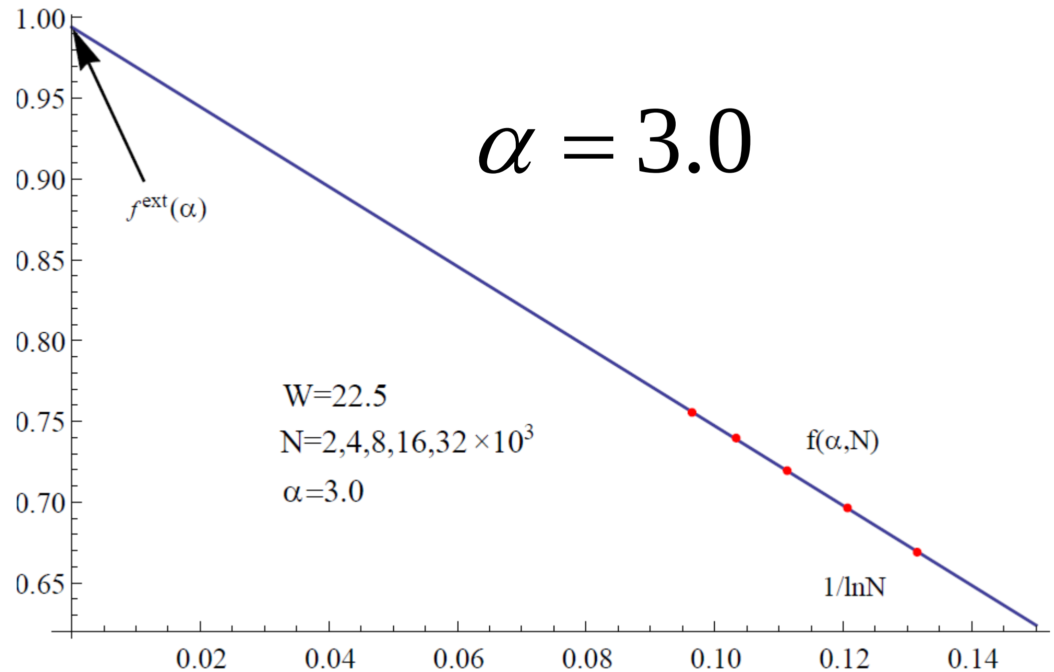
Symmetry relation: $f(1+\alpha) = f(1-\alpha) + \alpha$



$W = 22.5$

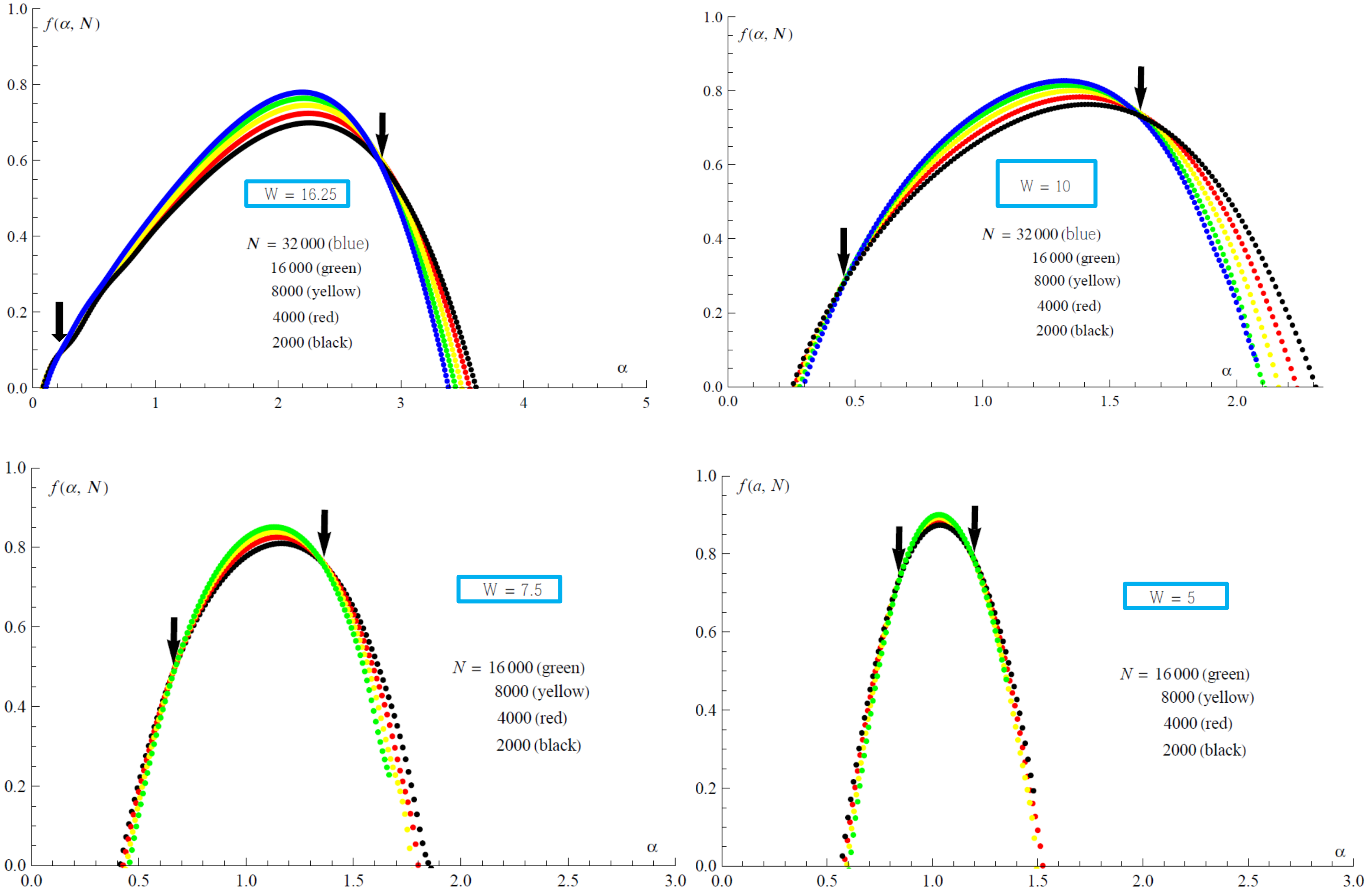


α = 1.8
0.60
0.55
0.50
$f^{ext}(\alpha)$
W=22.5
N=2,4,8,16,32 ×10³
α=1.8
f(α,N)
1/lnN
0.02
0.04
0.06
0.08
0.10
0.12
0.14



α = 3.0
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
$f^{ext}(\alpha)$
W=22.5
N=2,4,8,16,32 ×10³
α=3.0
f(α,N)
1/lnN
0.02
0.04
0.06
0.08
0.10
0.12
0.14

Extended – non-ergodic regime, *W*<17,5:



Extended – non-ergodic regime, W<17,5:

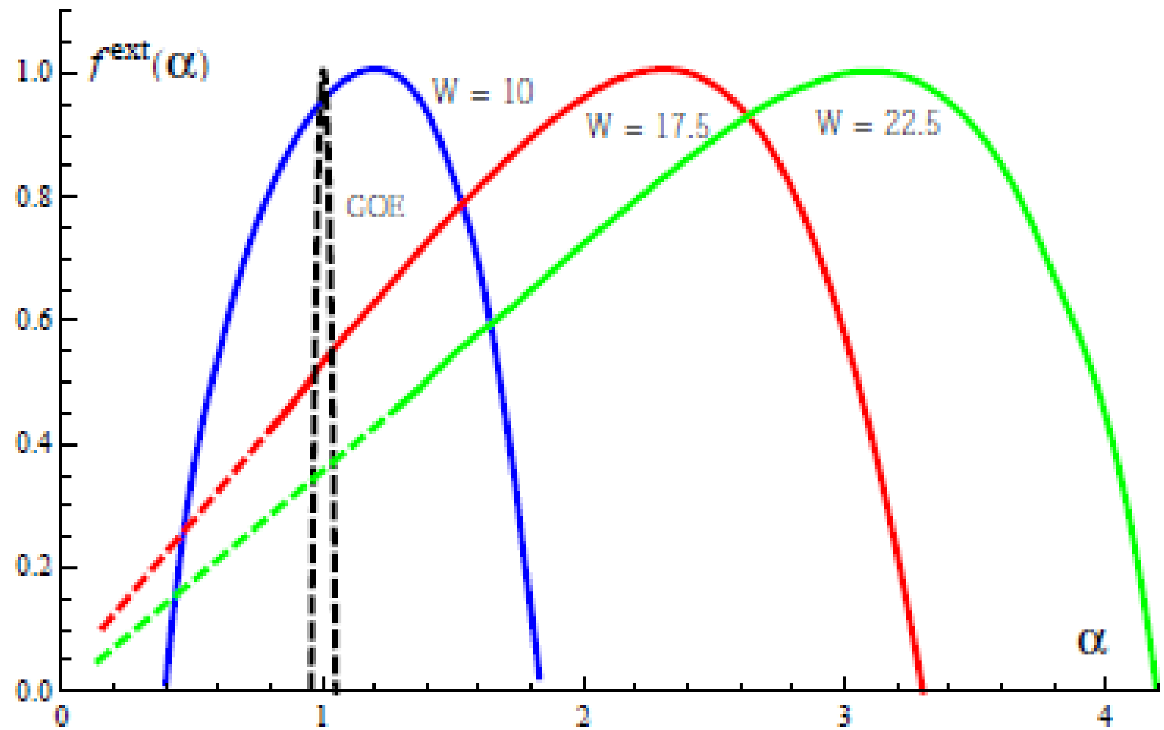
The spectrum of the fractal dimensions $f(\alpha)$ is gradually evolving with the strength of disorder W, but does not collapse to the ergodic limit: $f(1)=1$ and $f(\alpha \neq 1) = -\infty$

This is not a finite size effect:

1) Two fixed points
2) This is not a critical behavior: $f(\alpha, N, W)$ depends on both N and W.

Q: Can extended eigenstates of the Anderson model on the Bethe-Lattice be non-ergodic outside the critical region?

A: YES



Localized states – triangular shape of $f(\alpha)$

Extended states – gradually approach the ergodic limit, but reach it only at $W = 0$

Extended – non-ergodic regime, $W<W_c=17{,}5$:

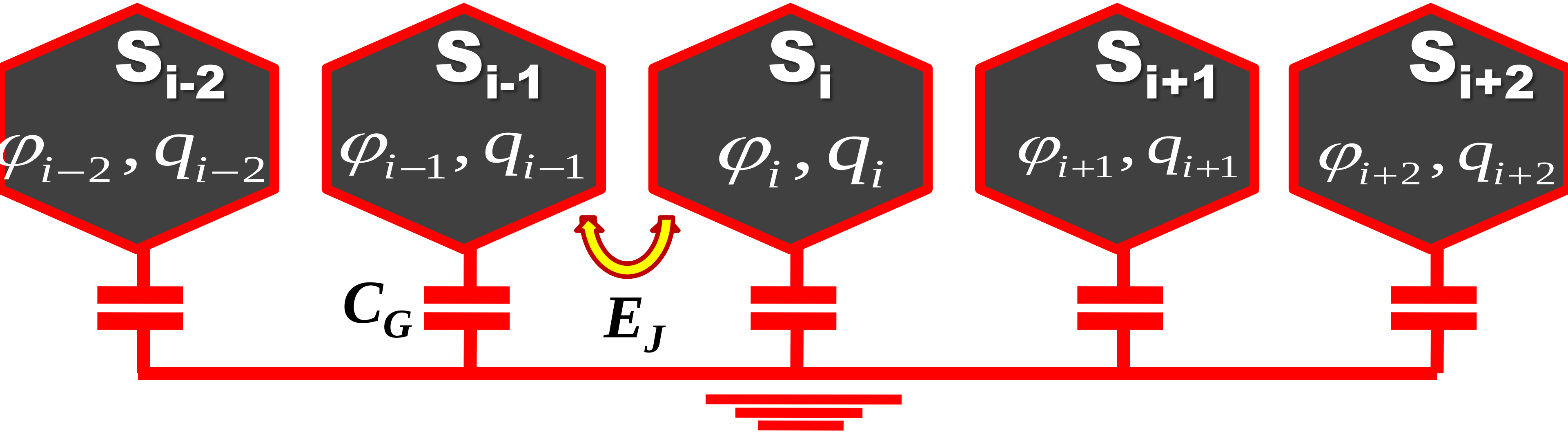
The spectrum of the fractal dimensions $f(\alpha)$ is **gradually** evolving with the strength of disorder W , but does not collapse to the ergodic limit, which is

$$f(1)=1 \qquad f(\alpha\neq 1)=-\infty$$

It is unlikely that this is a finite size effect:

1) Two fixed points
2) This is not a critical behavior: $f(\alpha,N,W)$ **depends on both** N and W.

Ideal (no disorder) 1D Josephson array



$$\hat{H} = \sum_i \left\{ E_J \left[1 - \cos\left(\varphi_{i+1} - \varphi_i \right) \right] + E_c \frac{q_i^2}{2} \right\}$$

$$\left[\varphi_j, q_k \right] = i\delta_{jk}$$

Canonically conjugated variables

Classical Limit $\frac{E_c}{E_J} \to 0$

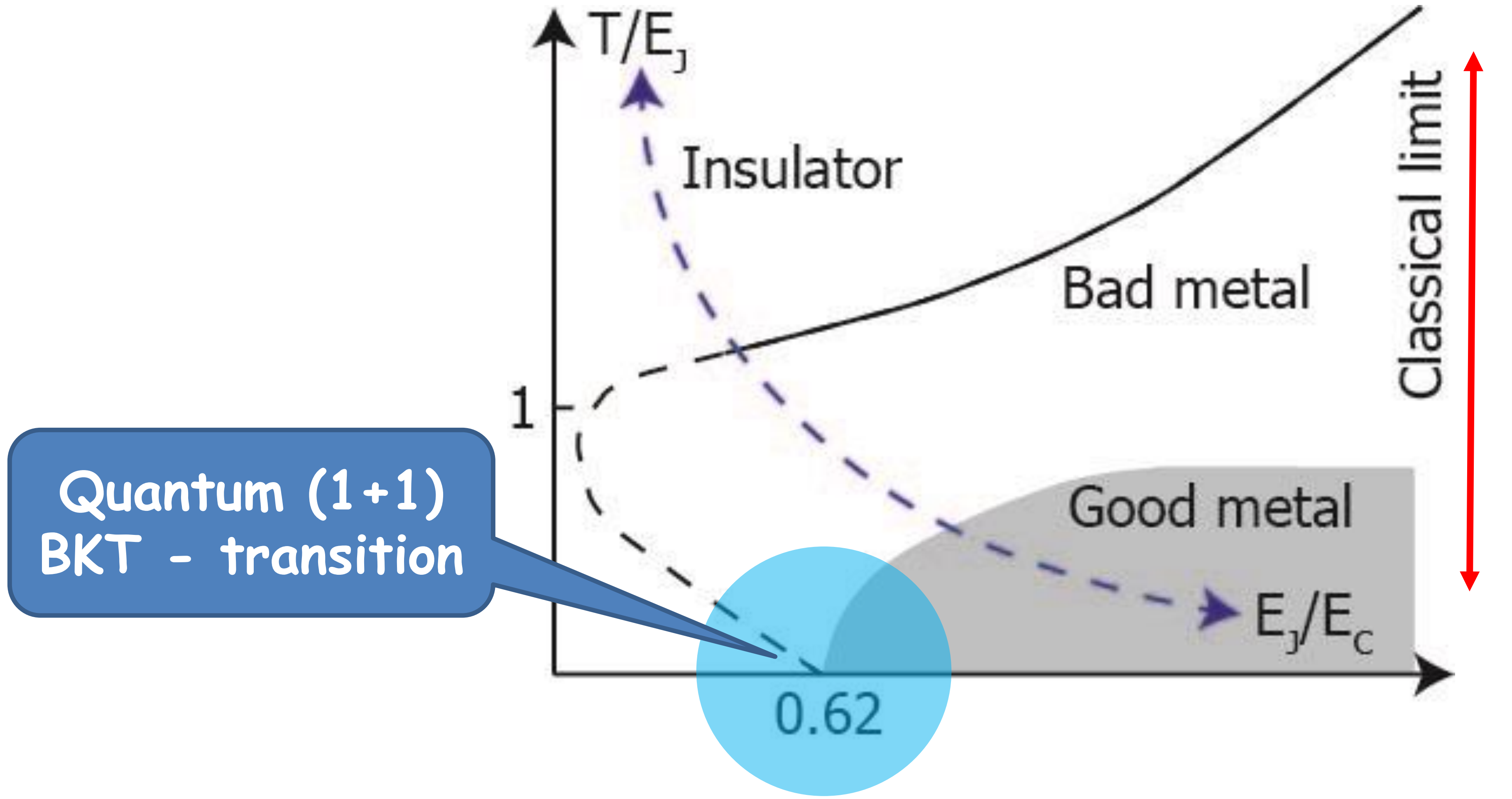
Non-ergodic classical and quantum dynamics

Small entropy at infinite temperature.

M. G. Pino, L.B. Ioffe, BA, to be completed

Ideal (no disorder) 1D Josephson array

$$\hat{H} = \sum_i \left\{ E_J \left[1 - \cos\left(\varphi_{i+1} - \varphi_i\right) \right] + E_c \frac{q_i^2}{2} \right\}$$



Quantum Transition:

$$\hat{H} = \sum_i \left\{ E_J \left[1-\cos\left(\varphi_{i+1}-\varphi_i\right)\right] + E_c \frac{q_i^2}{2} \right\} = \sum_i \left\{ \frac{E_J}{2} \left[\hat{b}_i^\dagger \hat{b}_{i+1} + \hat{b}_i \hat{b}_{i+1}^\dagger\right] + \frac{E_c}{2} q_i^2 \right\}$$

Matrix element of the $\left(q_i, q_{i+1}\right) \Rightarrow \left(q_i+1, q_{i+1}-1\right)$ transition is $E_J/2$

Energy difference of the two states $E_c\left(q_i - q_{i+1} + 1\right)$

$E_c q_i^2 \sim T \;\Rightarrow\; |q_i| \sim \sqrt{\frac{T}{E_c}}$ ⇨ Ratio $\sim \sqrt{\frac{T_c}{T}}$ $\quad T_c = \frac{E_J^2}{E_c}$

Therefore

$T \ll T_c$ ⇨ metal

$T \gg T_c$ ⇨ insulator

Localized phase at *high* temperatures!

Freezing with cooling!

Classical limit: $\frac{E_J}{E_c} \to \infty$ ⇨ $T_c \to \infty$

Classical limit:

equations of motion:

$$\frac{\partial^2 \varphi_i}{\partial \tau^2} = \sin\left(\varphi_{i+1} - \varphi_i\right) + \sin\left(\varphi_{i-1} - \varphi_i\right)$$

$$\tau \equiv t\sqrt{E_J E_c}$$

$$T \leftarrow u \equiv \frac{U}{L} E_J^{-1}$$

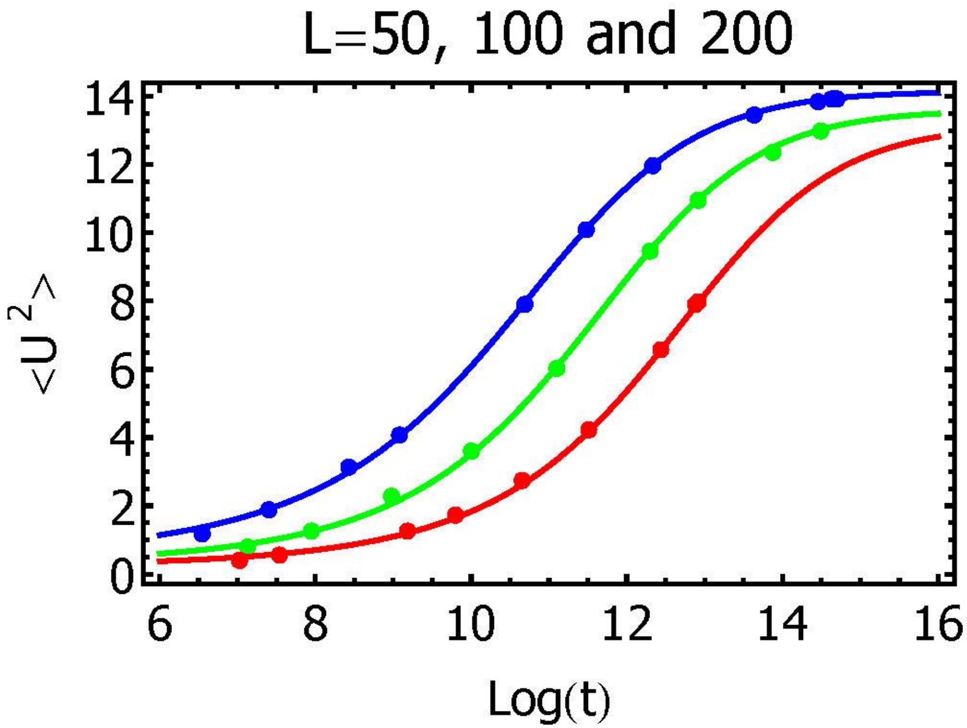
U Total energy

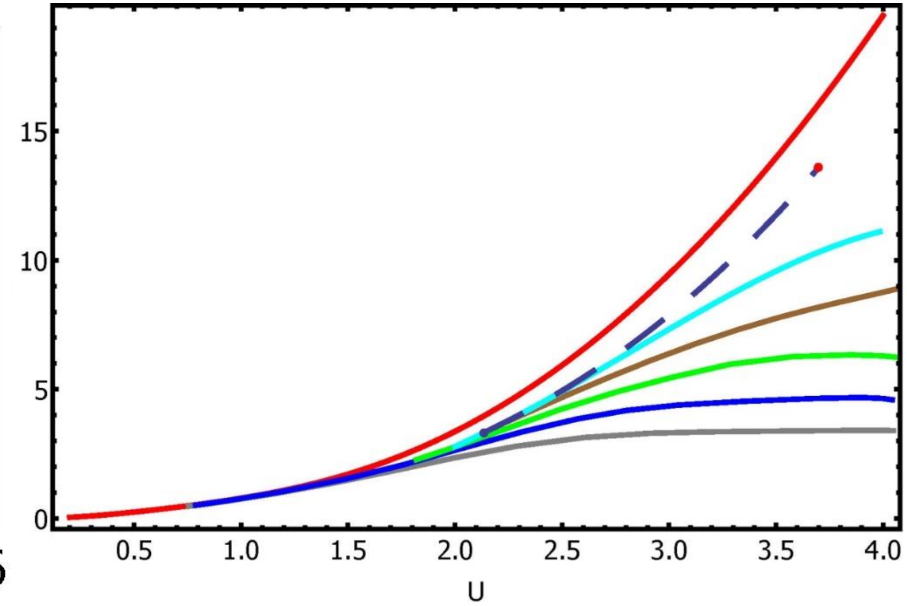
L Length = # of islands

$$u = \frac{1}{L}\sum_i \left\{ \frac{1}{2}\left(\frac{\partial \varphi_i}{\partial \tau}\right)^2 - \cos\left(\varphi_i - \varphi_{i-1}\right) \right\}$$

Dimensionless energy per island.

Slow relaxation in the classical limit





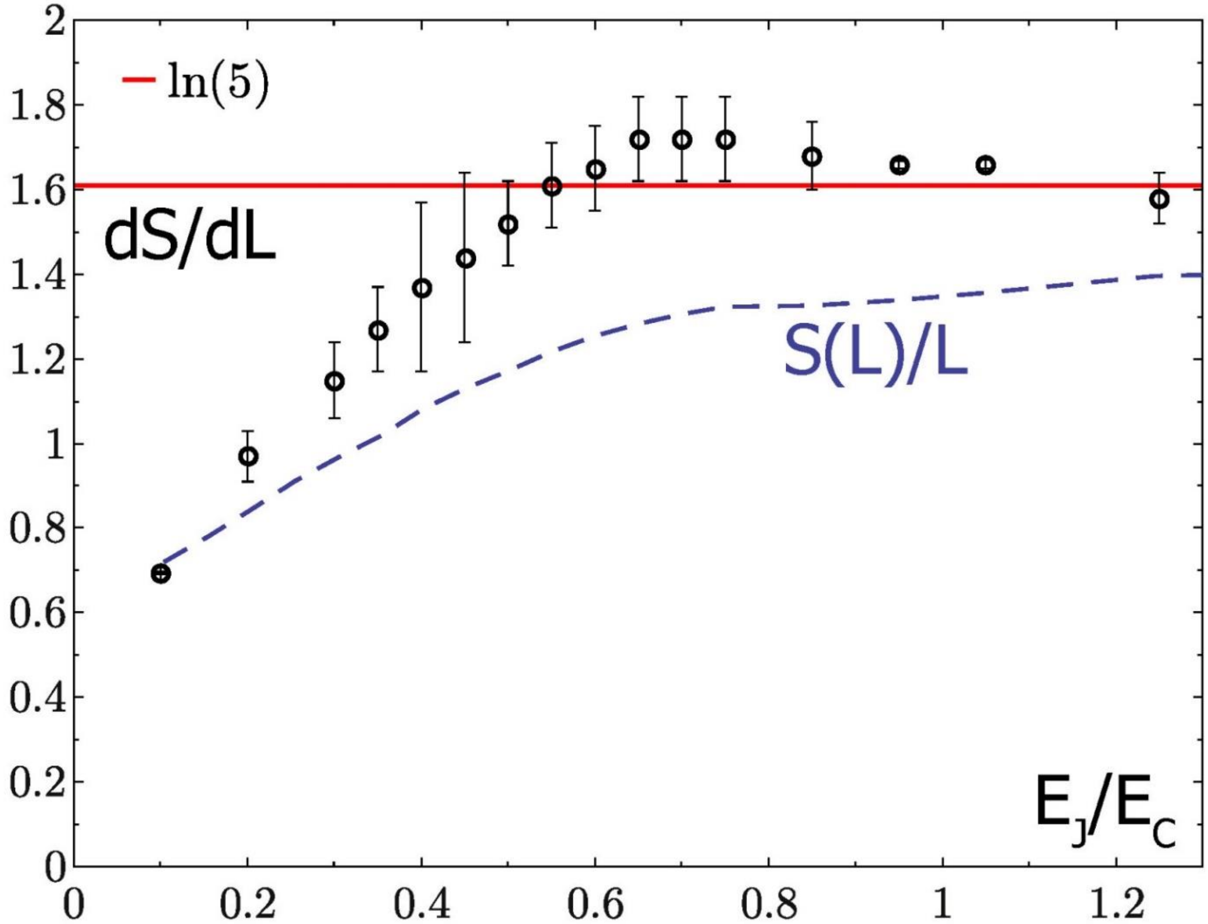
$$\left\langle u^2 \right\rangle_\tau = \frac{\left\langle u^2 \right\rangle_\infty}{1+\beta \exp\left[-\alpha \ln^2\left(\tau/\tau_0\right)\right]}$$

$$\left\langle u^2 \right\rangle_\infty \neq -T^2 \frac{du}{dT}$$!

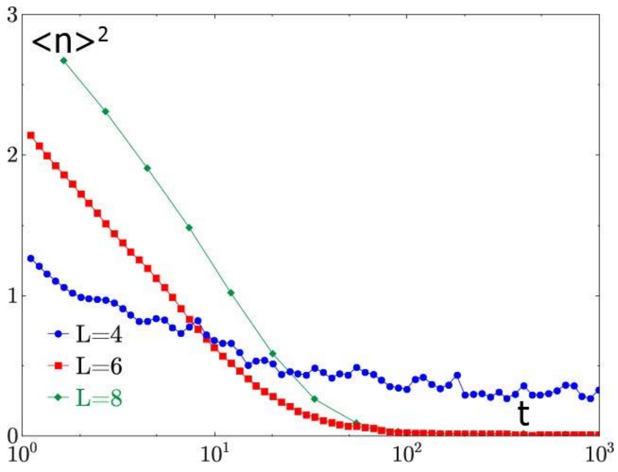
Quantum problem

Limit the number of charge states $q = 0, \pm 1, \pm 2$

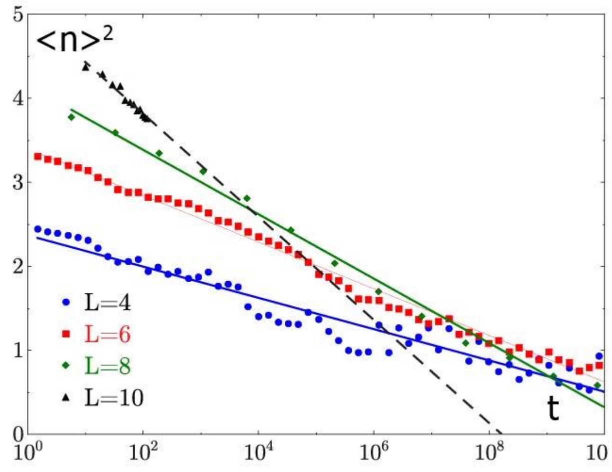
Consider only $T = \infty$ (random initial conditions)



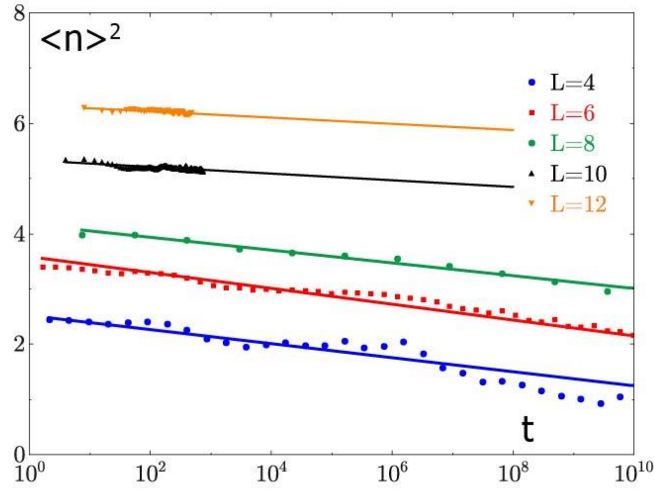
Charge relaxation in Good and Bad metals



Good metal



Bad metal

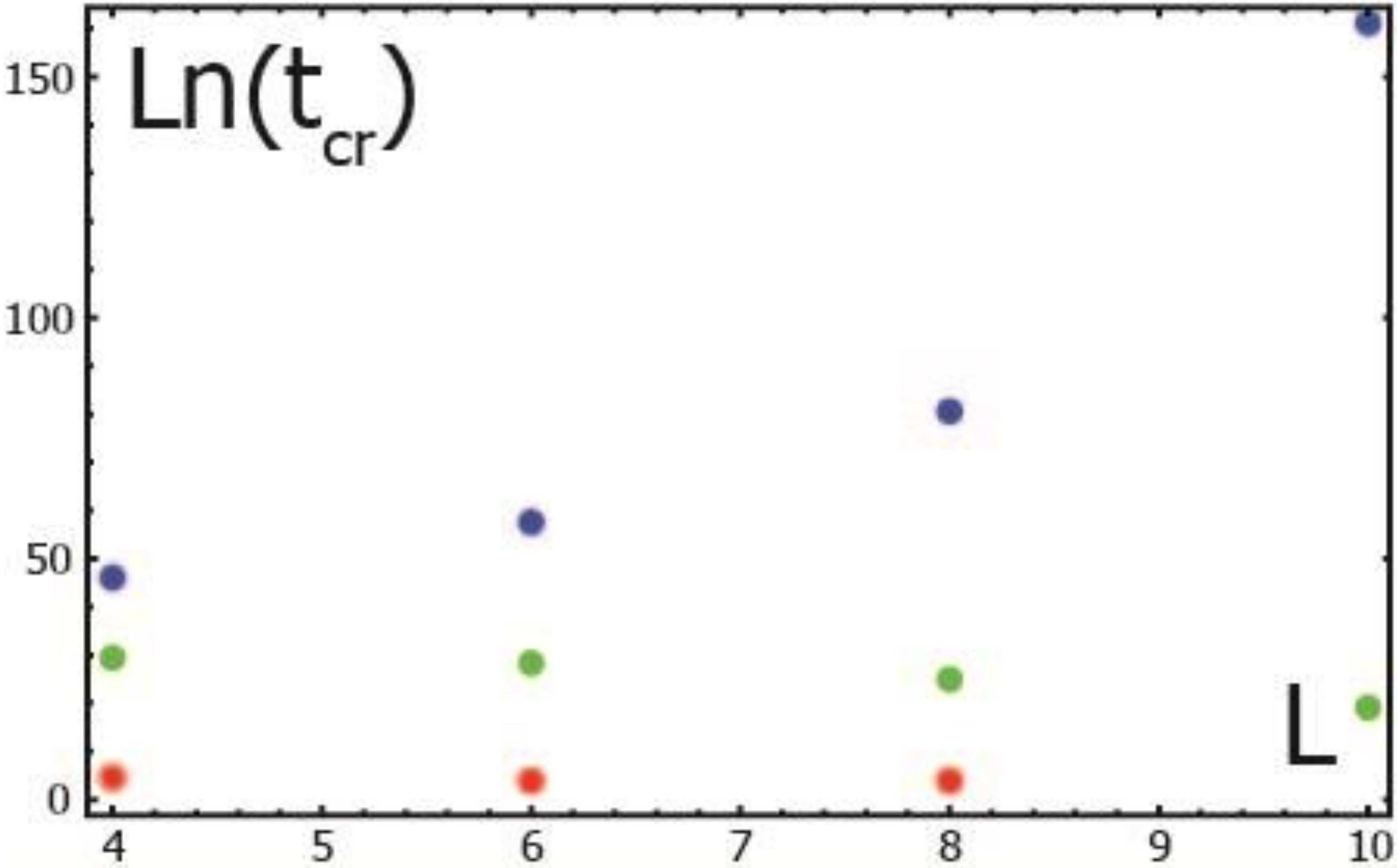


Insulator

As the size increases the characteristic time of the charge relaxation stays roughly constant on log scale

As the size increases the characteristic time of the charge relaxation in the insulator grows exponentially.

Charge relaxation in Good and Bad metals



characteristic times in good and bad metals vs. insulator.

Conclusion:

Multifractality of the eigen-functions of the Anderson Model on the Bethe lattice (random regular graph) persists in a broad interval of the disorder strengths.

This suggests that many-body systems should demonstrate non-ergodic behavior even outside the critical regime of the Many-Body Localization.

Conventional Statistical mechanics might be not fully applicable

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This suggests that many-body systems should demonstrate non-ergodic behavior even outside the critical regime of the Many-Body Localization.

Conventional Statistical mechanics might be not fully applicable

Open problems:

Ergodic – Non-ergodic: crossover of phase transition?

Analytical description of the deviation from the ergodicity (Weak Many-Body Localization)

Non-ergodic time evolution

Driven systems with dissipation