Functional order parameters in disordered systems: the Legendre structure

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Firstly we will give a short introduction on some aspects of the Legendre structure in statistical mechanics.

We outline the analogy and the contrast between ordered and disordered systems.

In particular we will show the route to the establishment of the functional order parameters for the physical description of disordered systems. A complete scheme of dual Legendre variational principles is involved.

These considerations apply, in particular, to the Sherrington-Kirkpatrick model for spin glasses and its p-spin generalizations, to the Hopfield model for neural networks, to generalised multi-species models.

Our itinerary includes the following seven steps.

- The general Legendre structure for statistical mechanics systems: states and energy functionals, entropy and free energy

- Dual variational principles for ordered mean field systems

- The entropy principle for disordered systems. The simple case of the Derrida Random Energy Model (REM) and its generalization (GREM)

- two big surprises: in the ground-breaking treatment, originally given by Giorgio Parisi for the mean field spin glass model, a functional order parameter is involved, moreover the optimization reverses the sign (an *inf* is involved instead of the expected *sup*)

- the reversed variational principle in the simple case of the REM (and the GREM)

- the reversed variational principle in the p-spin models and their generalizations, the functional order parameter, the route to the general Legendre structure, dual functional parameters, dual functionals

- outlook and perspectives

The general Legendre structure for statistical mechanics systems

This is at the basis of everything. It encodes the essence of the second principle of thermodynamics.

Let us describe the thermodynamic system in the frame of the *ensemble* theory.

In an appropriate *coarse graining* the microscopic configurations of the systems are described by a large number of cells $(1, 2, \ldots, K) \ni i$.

A **thermodynamic state** (not necessarily at equilibrium) is described by a probability distribution on cells

$$\mathcal{V} \ni p : (1, 2, \ldots, K) \ni i \to p_i, 0 \leq p_i \leq 1, \sum_i p_i = 1.$$

We call $\mathcal{V}$ the simplex of all states.

The entropy of the state p is defined as

$$p \to S(p) = -\sum_i p_i \log p_i,$$

and turns out to be concave on $\mathcal{V}$.

In fact, let the state p be the mixture of two states $p^{(1)}$ and $p^{(2)}$, in the sense that

$$p_i = \alpha p_i^{(1)} + (1-\alpha) p_i^{(2)},$$

for all i, with the mixing parameter $0 \leq \alpha \leq 1$.

Then

$$S(p) \geq \alpha S(p^{(1)}) + (1-\alpha) S(p^{(2)}).$$

Let us now attribute an energy to each cell

$$E : (1, 2, \ldots, K) \ni i \to E_i,$$

and define the partition function

$$Z(E) = \sum_i e^{-E_i},$$

where the temperature has been incorporated into E.

The free energy is $F(E) = -\log Z(E)$, while the internal energy in a given state p is

$$U(E, p) = \sum_i E_i p_i.$$

Obviously $\log Z(E)$ is convex in E, while $U(E, p)$ is linear in E and affine in p.

The basic Legendre structure (the second principle of thermodymics) is based on the recognition that the state p and the energy E can be assumed as dual variables, while the entropy $S(p)$ and $\log Z(E)$ are dual functionals, involved in the dual Legendre variational principles

$$\log Z(E) = \sup_p (S(p) - \sum_i E_i p_i),$$

$$S(p) = \inf_E (\log Z(E) + \sum_i E_i p_i).$$

$\log Z$ is the Legendre transform of S, and viceversa.

Notice that writing $F(E) = -\log Z(E)$ the variational principle for the free energy involves a *min* over all states, as it is well known from elementary thermodynamics.

At equilibrium, where the optimal values are enforced, the state has the Boltzmann-Gibbs expression in terms of E

$$p_i = -\frac{\partial}{\partial E_i} \log Z(E) = e^{-E_i}/Z,$$

while the E_i are determined up to a constant. This is due to the constraint $\sum_i p_i = 1$.

Notice that the term $\sum_i E_i p_i$ can be also interpreted as a Lagrangian multiplier for the entropy principle. As a matter of fact the optimal state realizes the maximum for the entropy under the constraint of fixed internal energy, in a pure Boltzmann spirit.

This general Legendre structure assumes various shapes in different models in the infinite volume limit.

The case of ordered mean field models, and of REM and GREM, will be considered in the following.

Ordered mean field models

Now we take Ising configurations on N sites

$$\sigma : (1, 2, \ldots, N) \ni i \to \sigma_i = \pm 1.$$

Each σ denotes a *cell* of the system.

The interaction is specified by a function u : $[-1,1] \ni M \to u(M)$, where u is assumed bounded and Lipshitz, *i.e.*

$$u(M) \leq \bar{u},\ |u(M) - u(M')| \leq C|M - M'|,$$

for given constants $\bar{u}$ and C.

Now the partition function is

$$Z_N(u) = \sum_{\sigma} e^{Nu(m)},$$

where $m = \sum_i \sigma_i / N$.

Notice the high degeneracy of the energy functional. Many cells σ have the same energy if they have the same value of the magnetization m.

The term N in the *Boltzmannfaktor* has been introduced in order to have good thermodynamic behavior in the infinite volume limit. In fact, now $\log Z_N(u)$ behaves proportional to N for large N, and we will be interested in the normalized value $N^{-1} \log Z_N(u)$. Notice that $\log Z_N(u)$ is convex in u.

For finite N the Legendre structure is a particular case of that introduced before. But in the infinite volume limit there is a strong degeneracy, due to the mean field character of the interaction.

First of all let me introduce the entropy for a single spin configuration with magnetization M

$$s(M) = -\frac{1+M}{2}\log\frac{1+M}{2} - \frac{1-M}{2}\log\frac{1-M}{2}.$$

Notice that s is symmetric under $M \to -M$, and concave in M.

We can easily establish the following. Let u be bounded and Lipshitz, then the infinite volume limit of $N^{-1} \log Z_N(u)$ does exist and has the following value

$$A(u) = \lim_{N \to \infty} \frac{1}{N} \log Z_N(u) = \sup_M \Big(s(M) + u(M) \Big).$$

In general many different values for M realize the *sup*. There are multiple phases. In each phase the state is factorized as a product state on the corresponding value of M.

Due to the high degeneracy in the infinite volume limit, now we have that M must be

considered as the dual variable to the interaction u. Moreover the convex function $A(u)$ of u and the concave function $s(M)$ of M participate to the dual variational principles

$$A(u) = \sup_M \Big(s(M) + u(M) \Big),$$

$$s(M) = \inf_u \Big(A(u) - u(M) \Big).$$

We see that duality is fully preserved, while the Legendre structure is defaced by degeneracy. In fact the Lagrange multiplier assumes the form $u(M)$ (the energy read at the

value M) which has the right linear structure in u, but not in M.

Recall the in the finite volume case the Lagrange multiplier has the form $\sum_i E_i p_i$, linear in the energy, and affine over the states.

Here the states are product states, for which affinity does not hold.

The entropic principle in the Random Energy Model

As it is very well known, this is a disordered model characterized by the random partition function

$$Z_N(\beta, J) = \sum_{\sigma} e^{\beta\sqrt{\frac{N}{2}}J_\sigma},$$

where β is the inverse temperature, and the J's are a family of centered normalized independent Gaussian random variables, with averages $\mathbb{E}J_\sigma = 0$, $\mathbb{E}(J_\sigma J_{\sigma'}) = \delta_{\sigma\sigma'}$.

Obviously, $\log Z_N(\beta, J)$ is convex in β for each sample of the J's.

The infinite volume limit can be controlled by standard arguments. Therefore, we have superadditivity for the quenched averages ($N = N_1 + N_2$)

$$\mathbb{E} \log Z_N(\beta, J) \geq \mathbb{E} \log Z_{N_1}(\beta, J) + \mathbb{E} \log Z_{N_2}(\beta, J),$$

which implies convergence for the quenched densities

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E} \log Z_N(\beta, J) = \sup_N \frac{1}{N} \mathbb{E} \log Z_N(\beta, J) = A(\beta).$$

Moreover, by a concentration of measure argument and Borel-Cantelli lemma, we have

also the convergence of the densities

$$\lim_{N\to\infty} \frac{1}{N} \log Z_N(\beta, J) = A(\beta),$$

with probability one.

The entropy can be even explicitely calculated in the infinite volume limit, as Derrida has shown in his original paper.

For a parameter ϵ (with the meaning of minus the internal energy density), let us define the random microcanonical entropy density

$$\frac{1}{N} \log \sum_{\sigma} \chi(J_\sigma \geq \epsilon\sqrt{2N}),$$

where $\chi(\mathcal{A})$ is the truth function for the event $\mathcal{A}$ appearing.

We must resist the temptation of taking the quenched average, which is $-\infty$ because the χ's turn out to be all zero on a set of finite measure.

On the other hand, the annealed average is well defined. An easy calculation shows

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \sum_{\sigma} \chi(J_\sigma \geq \epsilon\sqrt{2N}) = \log 2 - \epsilon^2,$$

for any $\epsilon \geq 0$, while the limit equals $\log 2$ for negative values of ϵ.

With the help of the Borel-Cantelli lemma, we can easily establish the convergence of the random entropy density with probability one in the form

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} \sum_{\sigma} \chi(J_\sigma \geq \epsilon\sqrt{2N}) = s(\epsilon),$$

where: $s(\epsilon) = \log 2$ for $\epsilon \leq 0$,
$s(\epsilon) = \log 2 - \epsilon^2$ for $0 \leq \epsilon \leq \sqrt{\log 2}$
(as in the annealed case),
$s(\epsilon) = -\infty$, for $\epsilon > \sqrt{\log 2}$.

Notice that $s(\epsilon)$ is decreasing and concave in the parameter ϵ.

The usual chain of arguments connecting the microcanonical ensemble with the canonical one, leads us to the dual variational principles

$$A(\beta) = \sup_{\epsilon}\Big(s(\epsilon) + \beta\epsilon\Big),$$

where the *sup* is reached in the interval $0 \leq \epsilon \leq \sqrt{\log 2}$, and

$$s(\epsilon) = \inf_{\beta}\Big(A(\beta) - \beta\epsilon\Big).$$

Since $s(\epsilon)$ is completely known, the first variational principle gives us the explicit form of $A(\beta)$ as

$$A(\beta) = \log 2 + \frac{\beta^2}{4},$$

for $0 \leq \beta \leq \beta_c = 2\sqrt{\log 2}$, and

$$A(\beta) = \beta\sqrt{\log 2},$$

for $\beta \geq \beta_c$.

In conclusion we see that there is a full Legendre structure, where β and ϵ are conjugated parameters, while the convex $A(\beta)$ and

the concave $s(\epsilon)$ are conjugated functions related by the dual variational principles.

By the way, the recognition of the Legendre structure is useful, because we derive the free energy from the known entropy.

These considerations can be easily extended to the Generalized Random Energy Model.

Functional order parameter and the inverted variational principle

It is very well known that Giorgio Parisi introduced a functional order parameter, connected to the overlap distribution, for the description of disordered mean field models, as for example the Sherrington-Kirkpatrick model for a spin glass.

Moreover, in this scheme, the free energy is given by a *sup* on the functional order parameter for an appropriately chosen trial

functional. The Parisi variational principle for the free energy appears in a form inverted with respect to the usual entropic principle.

It is our purpose to interpret these features in the frame of a generalized Legendre structure.

First of all we give a simple explanation of the inversion, by exploiting the simple case of the REM, where everything can be explicitly calculated.

It turns out that the inversion is due to the fact that the free energy, with the usual minus sign, $\log Z$, convex in the interaction, is also concave in the **covariance** of the random interaction.

The Parisi functional order parameter turns out to be dual, in the Legendre sense, to the covariance of the random interaction. The two variational principles, for the dual functionals, one convex in the functional order parameter, the other concave in the covariance, can be also established in general.

The inverted principle in the REM

First of all let us establish a general fact concerning the (-)free energy density in the infinite volume limit

$$A(\beta) = \lim_{N\to\infty} \frac{1}{N} \log \sum_{\sigma} e^{\beta\sqrt{\frac{N}{2}} J_\sigma}.$$

The covariance of the interaction is $\beta^2 \frac{N}{2} \delta_{\sigma\sigma'}$. Therefore it is linear in β^2. For our purpose, it is convenient to make the change of variables $\beta^2 = t$, $\beta = \sqrt{t}$, and interpret $A(\beta)$ as a function of t, by introducing $\tilde{A}(t) = A(\beta)$.

Even without knowing the explicit form of $A(\beta)$, it is easy to establish that $\tilde{A}(t)$ must be **concave** in t, through a simple interpolation argument.

Now we state the following *a priori bound* on $\tilde{A}(t)$:

$$\tilde{A}(t) \leq \frac{\log 2}{m} + \frac{1}{4}tm,$$

holding for any $0 < m \leq 1$.

The proof is simple. For any $0 < m \leq 1$, we have for the quenched free energy per site

the inequality

$$N^{-1}\mathbb{E}\log\sum_{\sigma}\exp(\sqrt{t}\sqrt{\frac{N}{2}}J_{\sigma})\leq$$

$$m^{-1}N^{-1}\mathbb{E}\log\sum_{\sigma}\exp(m\sqrt{t}\sqrt{\frac{N}{2}}J_{\sigma}),$$

which holds for any spin system for purely thermodynamic reasons (positivity of the entropy).

Now we apply the annealed inequality, coming from convexity, $\mathbb{E}\log..\leq\log\mathbb{E}...$ The $\mathbb{E}$

of the *Boltzmannfaktor* is immediately calculated.

$$\mathbb{E}\exp(\sqrt{t}\sqrt{\frac{N}{2}}J_\sigma) = \exp(\frac{1}{4}tN).$$

By taking into account that $\sum_\sigma = 2^N$, we end with the inequality, uniform in N,

$$N^{-1}\mathbb{E}\log\sum_\sigma \exp(\sqrt{t}\sqrt{\frac{N}{2}}J_\sigma) \leq \frac{\log 2}{m} + \frac{1}{4}tm.$$

By taking the limit $N \to \infty$ we find the stated inequality for $\tilde{A}(t)$.

Of course, the optimal value for m is found if we take the *inf* (inverted principle). Now

$$\inf_m (\frac{\log 2}{m} + \frac{1}{4} tm),$$

is precisely in the form of a good Lagrange variational principle. In fact, $\frac{\log 2}{m}$ is convex in m, and the second term is linear in both t and m separately.

It is easy to find the optimal m. It turns out that $m = 1$ for $0 \leq t \leq t_c = 4 \log 2$, while $m = \sqrt{t_c/t}$ for $t \geq t_c$. Moreover, by an

explicit calculation the optimal value of the trial function turns out to be exactly $\tilde{A}(t)$.

Therefore, we see that the concavity of $\tilde{A}(t)$ allows us to introduce the following clean Legendre structure. t and m are conjugated parameters, $t \geq 0$, $0 < m \leq 1$. The concave $\tilde{A}(t)$ and the convex $\psi(m) = \log 2/m$ are the conjugated functions involved in the dual Legendre variational principles

$$\tilde{A}(t) = \inf_m (\psi(m) + \frac{1}{4} tm),$$

$$\psi(m) = \sup_t(\tilde{A}(t) - \frac{1}{4}tm).$$

At the optimal value for $m(t)$ we have

$$\tilde{A}(t) = \psi(m(t)) + \frac{1}{4}tm(t).$$

If we take the t derivative, only the term where t appears explicitly must be taken into account

$$\frac{d}{dt}\tilde{A}(t) = \frac{1}{4}m(t).$$

It is easy to give the physical interpretation of the optimal $m(t)$. In fact, by a well known

direct calculation we have

$$\frac{d}{dt}\tilde{A}(t) = \frac{1}{4}(1 - < \delta_{\sigma^{(1)}\sigma^{(2)}} >_t),$$

where $< >_t$ denotes the quenched average on the replicated state with variables $\sigma^{(1)}, \sigma^{(2)}$. By a comparison with the previous one we get

$$m(t) = 1 - < \delta_{\sigma^{(1)}\sigma^{(2)}} >_t .$$

Since the replica "overlap" δ can take only the values $0, 1$, the meaning of the order parameter $m(t)$ is obvious. It is the probability that the two replicas are different, *i.e.* $\delta = 0$.

Therefore, in this Legendre structure, with the inverted variational principle, the dual parameter m is connected with the overlap distribution at the optimal value.

As a matter of fact, we can introduce the Legendre structure according to the following simple canonical procedure. Start from $\tilde{A}(t)$ and its derivative

$$\frac{d}{dt}\tilde{A}(t) = \frac{1}{4}(1 - < \delta_{\sigma^{(1)}\sigma^{(2)}} >_t) = \frac{1}{4}m(t),$$

where the last expression is only a definition of $m(t)$.

However, from the very expression of the derivative it is immediately recognized that we must introduce the parameter m dual to t, and that the Lagrange term should be of the form

$$\frac{1}{4}tm.$$

The dual function $\psi(m)$, convex in m, is obtained through the variational principle

$$\psi(m) = \sup_t(\tilde{A}(t) - \frac{1}{4}tm).$$

The inverse Legendre transfom $\underline{A}(t)$, con-

cave in t, is defined as

$$\underline{A}(t) = \inf_m(\psi(m) + \frac{1}{4}tm).$$

Since we know that $\tilde{A}(t)$ is concave in t, we conclude with the identification $\underline{A}(t) = \tilde{A}(t)$, and the Legendre structure is fully established. The bound

$$\tilde{A}(t) \leq \frac{\log 2}{m} + \frac{1}{4}tm$$

allows immediately to recognize the explicit expression of $\psi(m) = \log 2/m$.

It is also possible to connect the entropy s

and the function ψ in the frame of a Legendre scheme, by an appropriate change of the conjugated variables.

We will see in the following how this generalizes to more general models.

The Legendre structure in mean field spin glass models

We consider models where the (-) free energy per site is defined as

$$A_N(g) = \frac{1}{N}\mathbb{E}\log\sum_{\sigma} e^{\sqrt{\frac{N}{2}}\mathcal{K}(\sigma)},$$

where $\sigma \to \mathcal{K}(\sigma)$ is a family of centered Gaussian random variables with variances given by

$$\mathbb{E}\Big(\mathcal{K}(\sigma)\mathcal{K}(\sigma')\Big) = g\Big(q(\sigma,\sigma')\Big).$$

Here q is the overlap between two configurations

$$q(\sigma,\sigma') = \frac{1}{N}\sum_i \sigma_i\sigma_i' \ .$$

Of course g must be positive definite as a function of the two configurations. This can be easily obtained by taking $\mathcal{K}$ as a sum over p-spin glass interactions ($p \geq 2$). For the

sake of notational simplification we neglect any external field.

There is no problem with the infinite volume limit of the (-) free energy per site $\lim_{N\to\infty} A_N(g) = A(g)$.

To proceed toward the Legendre structure we must recognize the order parameter and the associated Lagrange multiplier.

In order to evaluate how $A(g)$ changes with g, let us rescale $\mathcal{K}$ to the form $\sqrt{t}\mathcal{K}$. Then a

standard calculation gives

$$\frac{d}{dt}A_N(g) = \frac{1}{4} < \Big(g(1) - g(q(\sigma,\sigma')\Big) >,$$

where $< >$ as usual is the quenched average on the two replica Boltzmann-Gibbs state.

According to a very general principle (Talagrand positivity arising from the Ghirlanda-Guerra identities), to the effect of the free energy evaluation, only the region where the overlaps are non-negative give a contribution. Therefore, by calling $\rho(q)$ the positive

overlap distribution, we have

$$\frac{1}{4} < \Big(g(1) - g(q(\sigma,\sigma'))\Big) > = \frac{1}{4}\int_0^1 \Big(g(1) - g(q)\Big)\rho(q)\; dq$$

$$= \frac{1}{4}\int_0^1 g'(q)x(q)\; dq,$$

where we have written $\Big(g(1) - g(q)\Big) = \int_q^1 g'(q')\; dq'$, ($g'$ is the derivative of $g(q)$), have defined $x(q) = \int_0^q \rho(q')\; dq'$, and have exchanged the integrations on q and q'.

From the obtained expression, we can immediately recognize the two conjugated vari-

ables which will be involved in the dual variational principles. The first is $g' : q \to g'(q)$, connected with the interaction, the second is a general $x : q \to x(q)$, which only at the optimal value is connected with the overlap distribution.

Of course g' is sufficient to characterize the interaction, because in any case we can take $g(0) = 0$, without loss of generality. In conclusion, we see that the Lagrange multiplier must have the form

$$\frac{1}{4} \int_0^1 g'(q) x(q) \; dq,$$

for a generic x.

The Legendre transform of $A(g)$ is defined as

$$\psi(x) = \sup_g \Big(A(g) - \frac{1}{4} \int_0^1 g'(q) x(q) \; dq \Big),$$

and turns out to be automatically convex in x.

The inverse Legendre transform is

$$\underline{A}(g) = \inf_x \Big(\psi(x) + \frac{1}{4} \int_0^1 g'(q) x(q) \; dq \Big).$$

Of course, $\underline{A}(g)$ turns out be be concave in g, and the full Legendre structure is established only if $A(g)$ is also concave in g, in which case $A(g) = \underline{A}(g)$, and effectively

$$A(g) = \inf_x \Big(\psi(x) + \frac{1}{4} \int_0^1 g'(q) x(q) \; dq \Big).$$

Therefore, the concavity of $A(g)$ in g plays an important role here.

Up to this point we have seen that the whole structure is a simple generalization of that found for the Random Energy Model. In this

simple case however, it was possible to prove directly the *a priori* concavity, and an upper bound in the Legendre form was easily found. The present scheme reproduces that found in the REM, by taking $g'(q) = \delta(1-q)$.

Let us see what can be said about the upper bound for the general model.

First of all let us recall that by using a simple interpolation argument (Guerra CMP2003)

it was possible to obtain the following uniform upper bound

$$A_N(g) \leq \phi(g, x),$$

where the trial functional has the form

$$\phi(g, x) = \log 2 + f(0, 0; x, g) - \frac{1}{4} \int_0^1 q g''(q) x(q)\, dq.$$

Here g'' is the second derivative of the covariance g with respect to q, and $f(0, 0; x, g)$ is the value at $q = 0, y = 0$ of the function $[0, 1] \times \mathbb{R} \ni (q, y) \to f(q, y; x, g)$ defined as solution of the differential equation

$$\partial_q f + \frac{g''}{4} \left(\partial_{yy}^2 + x(q) (\partial_y f)^2 \right) = 0,$$

with final condition

$$f(1, y; x, g) = \log \cosh y.$$

The trial functional, which comes for free from the interpolation method, is identical to that found by Parisi in the frame of the replica trick, and then subject to optimization in the form of $\inf_x$, perfectly consistent with the nature of the *upper* bound. Of course the bound holds also for the infinite volume limit

$$A(g) \leq \phi(g, x).$$

We call

$$A_P(g) = \inf_x \phi(g, x).$$

Then Talagrand (Annals of Mathematics 2006) was able to establish that $A(g) = A_P(g)$.

The latest result is due to Auffinger and Chen who have established (arXiv 2014) that $f(0, 0; x, g)$ as a functional of x is stricly convex. Since the other term in $\phi(g, x)$ is affine in x, we immediately have that the functional order parameter x coming from Parisi optimization is uniquely defined. A truly remarkable result!

In particular we notice that a phase transition is NOT characterized by multiple values of the order parameter, as in the ordered models.

Of course, the form of the trial functional $\phi(g, x)$ is not consistent with a Legendre form. The term

$$-\frac{1}{4}\int_0^1 qg''(q)x(q)\ dq$$

has the right properties of being affine in x and linear in g, but it is not in the expected

form

$$+\frac{1}{4}\int_0^1 g'(q)x(q)\ dq\Big).$$

Moreover, $f(0,0;x,g)$ depends heavily on both x and g. Therefore the variational principle does not give $A(g)$ as the Legendre transform of some $\psi(x)$, in the expected form

$$A(g) = \inf_x\Big(\psi(x) + \frac{1}{4}\int_0^1 g'(q)x(q)\ dq\Big).$$

However, there is a way out. Let us start from the bound

$$A(g) \leq \phi(g,x),$$

written in the identical form

$$A(g) \leq \phi(g,x) - \frac{1}{4}\int_0^1 g'(q)x(q)\,dq + \frac{1}{4}\int_0^1 g'(q)x(q)\,dq,$$

where we have subtracted and added the expected Lagrange multiplier. Now let us *define*

$$\psi(x) = \sup_g \Big(\phi(g,x) - \frac{1}{4}\int_0^1 g'(q)x(q)\;dq\Big).$$

Then the bound becomes

$$A(g) \leq \psi(x) + \frac{1}{4}\int_0^1 g'(q)x(q)\;dq,$$

which is in the Legendre form. It can be easily seen that the optimization gives the

same value as in the original bound. Therefore, the Legendre structure can be enforced also in the general case.

outlook and perspectives

Let us start from a simple observation.

If we consider the interpolating bound

$$A_N(g) \leq \phi(g, x),$$

as explained before, and let apply it to a sequence of p-spin models with $p \to \infty$. It is

very well known that the REM will be obtained in the limit. Consider an order parameter of the constant shape $x(q) = m$. Through a long cumbersome calculation it can be proven that if we take $g(q) = tq^p$ and a constant order parameter$x(q) = m$, then we have

$$\lim_{p\to\infty} \phi(g, x) = \log 2 + \frac{1}{4}mt,$$

which is the bound for the REM easily obtained through simple thermodynamic arguments, as shown before.

Therefore, the broken replica symmetry bounds give also the simple Legendre bound in REM through a cumbersome limiting procedure.

Research is under way on how to modify the broken replica symmetry bound procedure in order to get bounds directly in the Legendre form.

Another perspective for future developments is to establish the Legendre structure in the case of multi-species spin glass models, as the bipartite ones. This would have interesting applications for neural networks.