Steps to go beyond the Bethe approximation in disordered models: large deviations of critical correlations and loop corrections

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Motivations

- Strongly disordered systems (e.g. spin glasses, RFIM) are in part well understood at the mean field level.

  Still many open questions at the MF level, e.g.:
  - dynamics
  - state-following
  - fRSB solution at the Bethe level

- Non-perturbative phenomena make even more difficult to extend results at finite dimensions

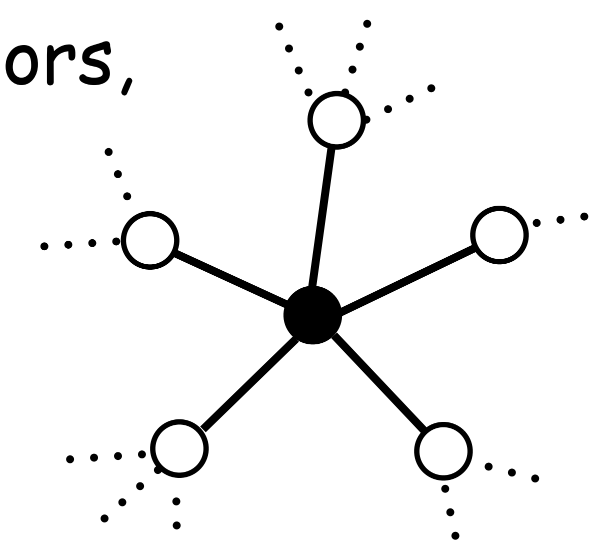
  -> forced us to resort to numerical simulations
  (see talks by Young, Martin-Mayor, Ruiz-Lorenzo)

Fluctuations around mean-field

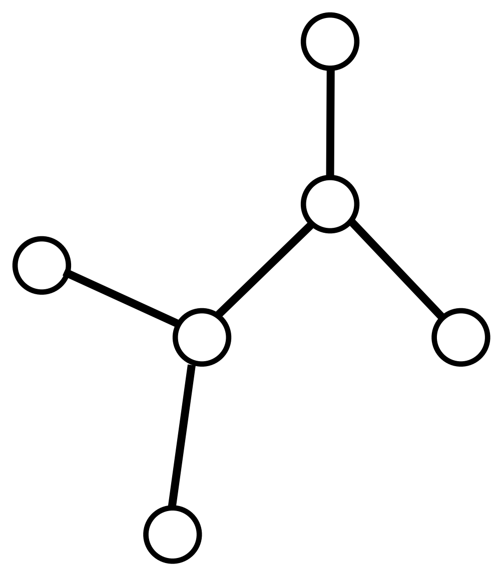
- At the mean-field level the Parisi solution is very likely to be correct one, but...
- the epsilon expansion (De Dominicis, Kondor, Temesvari) is extremely difficult (for T<Tc no full 1-loop exp)
- predictions on propagators decay do not match numerical results, e.g.
  -> Janus equilibrium data on $q=0$ and $q>0$ sectors ($d=3$)
  -> relation between exponents at and below Tc in $d=4$
  (see Nicolao, Parisi, FRT 2014)
- Need to find a better expansion around mean-field

Mean-field approximations

- naive Mean-Field assumes weakly interacting variables
- TAP approximation = weakly interacting variables + Onsager reaction term
- nMF and TAP are the first terms in the weak couplings Plefka expansion
  -> they fail in the strong coupling regime
- Bethe approximation (i.e. cavity method) assumes conditional independence of neighbors, but <u>no</u> weak couplings assumptions
  -> exact on trees (loopless graphs)



Models on trees



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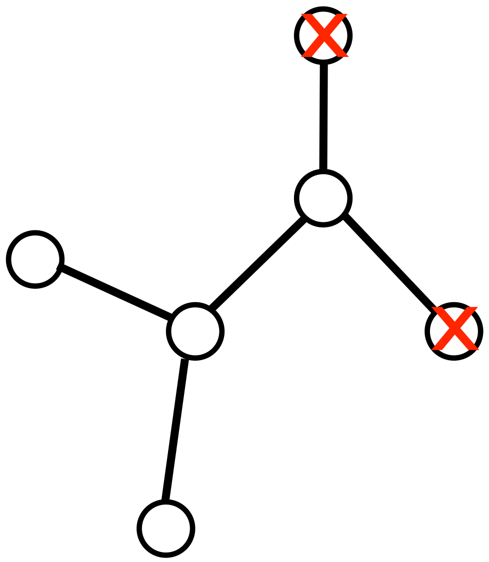

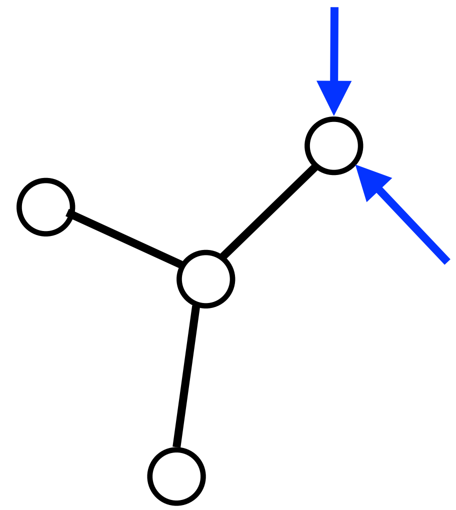

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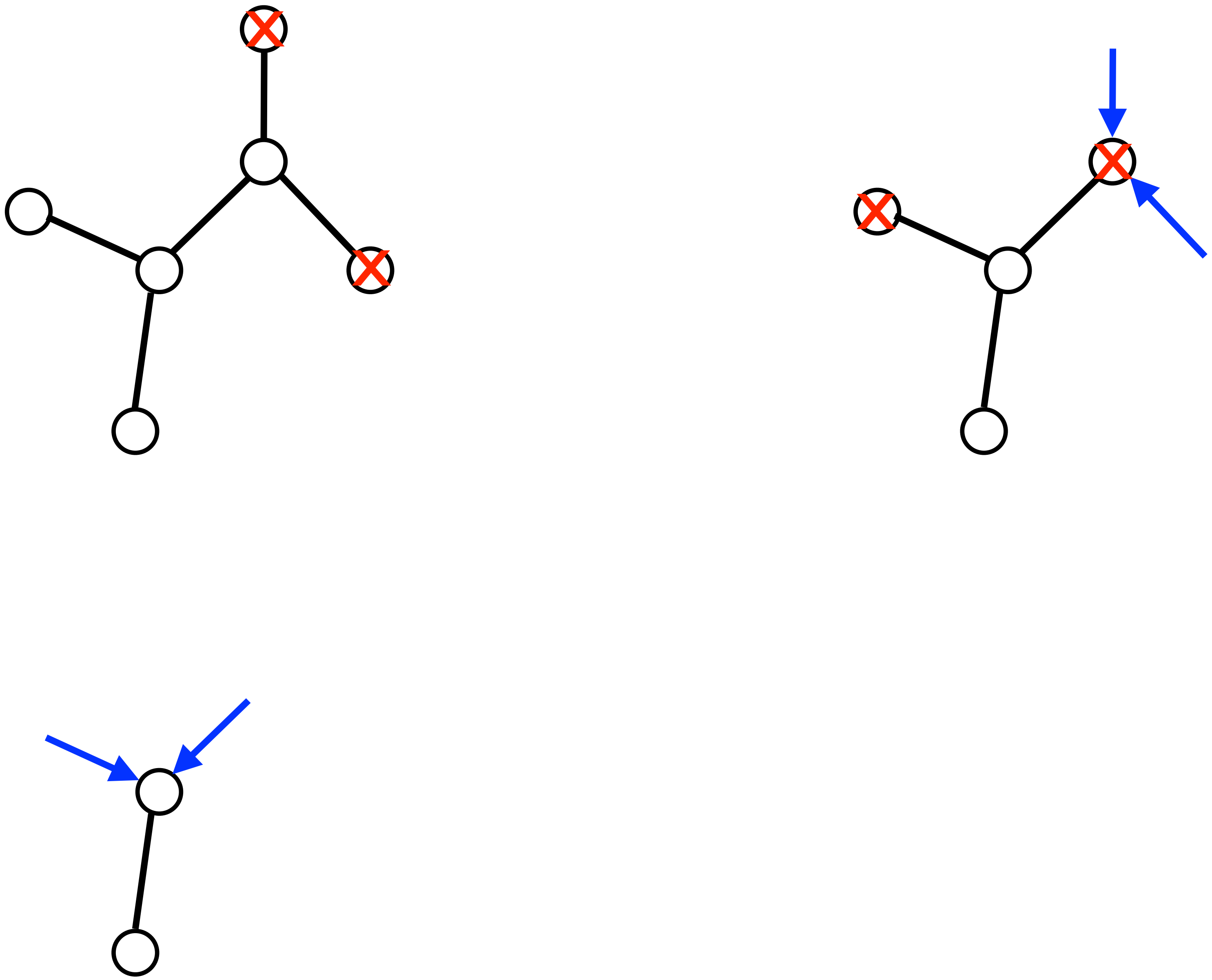

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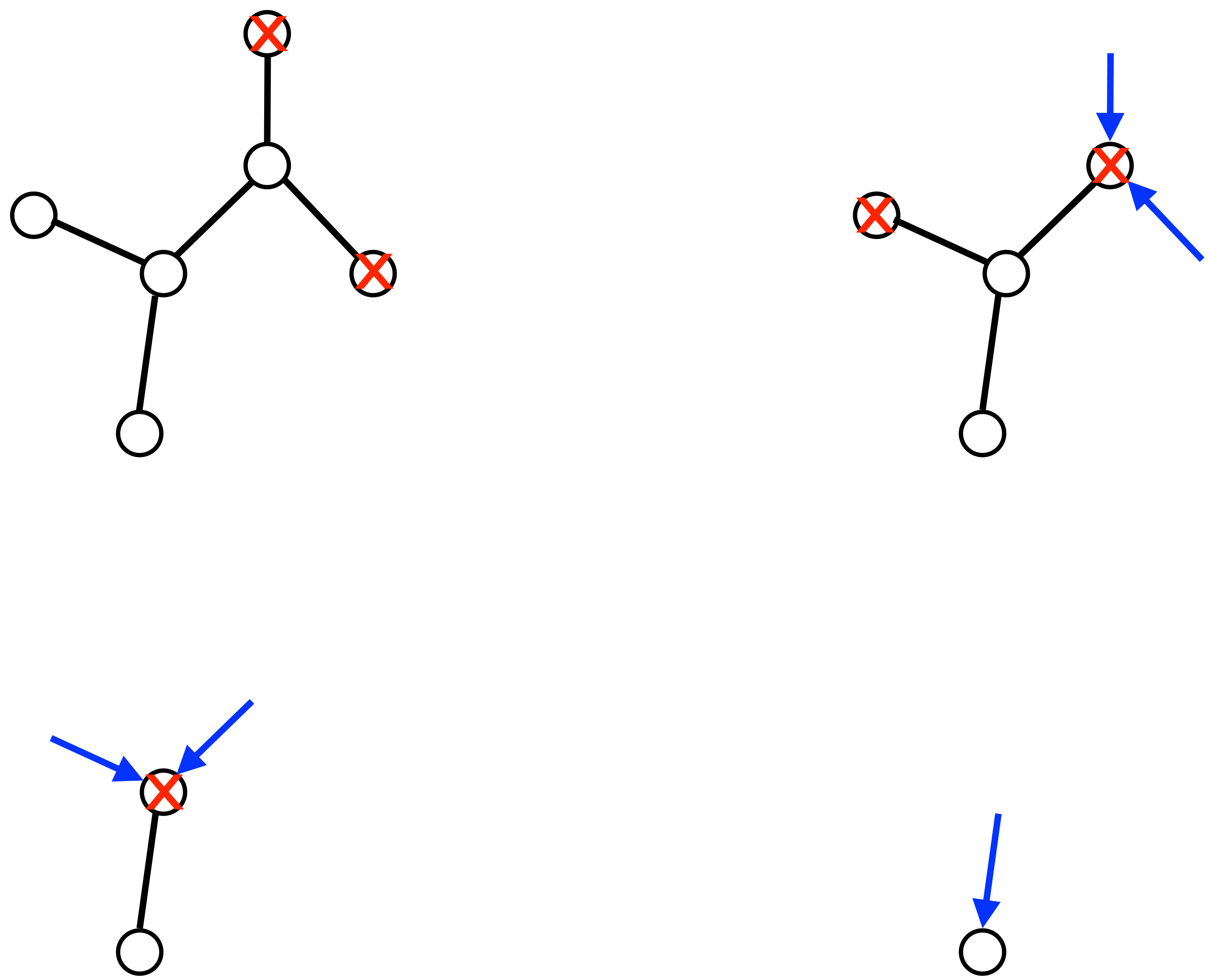
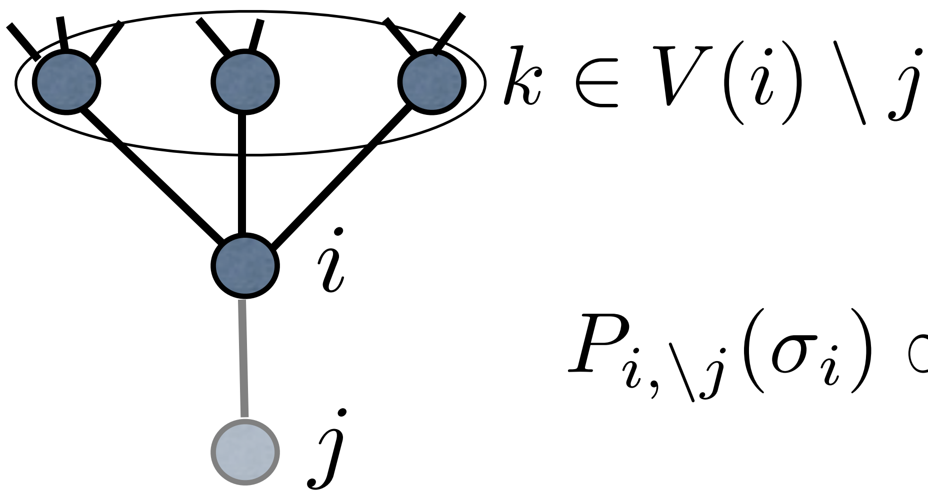


Bethe approximation

- Factorization approximation

$$P(\boldsymbol{s}) = \prod_i P_i(s_i) \prod_{ij} \frac{P_{ij}(s_i,s_j)}{P_i(s_i)P_j(s_j)}$$

- Self-consistent equations for cavity marginals $P_{i,\setminus j}(\sigma_i)$



Belief Propagation: iterative solution to these equations

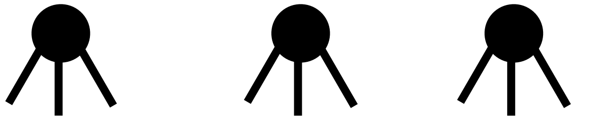
$$P_{i,\setminus j}(\sigma_i) \propto \sum_{\{\sigma_k\}} \prod_k P_{k,\setminus i}(\sigma_k)\psi(\sigma_i,\sigma_k)$$

- From cavity marginals to full marginals $\{P_i(\sigma_i), P_{ij}(\sigma_i,\sigma_j)\}$

$$P_i(\sigma_i) \propto \sum_{\{\sigma_k\}_{k\in V(i)}} \prod_{k\in V(i)} P_{k,\setminus i}(\sigma_k)\psi(\sigma_i,\sigma_k)$$

Random c-regular graphs

- Random graphs with fixed degree c for each node
- Construction:
  - N vertices, with c “legs” each



  - Connected pairs of legs at random (avoiding self-loops and double-links)
- Bethe approximation is correct, but there are multiple solutions depending on the boundary conditions (i.e. initial conditions for the BP iterative algorithm)
- Highly non-trivial low-temperature phases...

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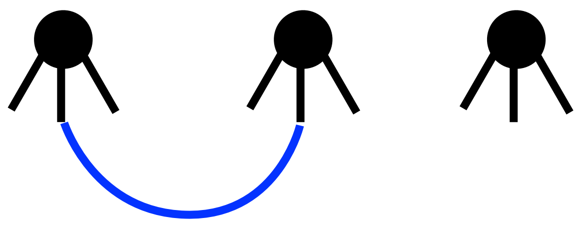

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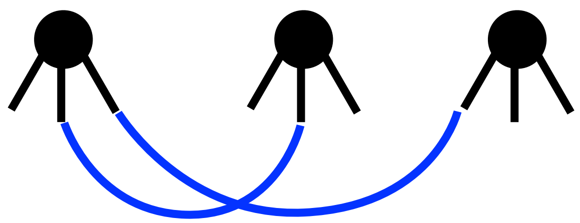

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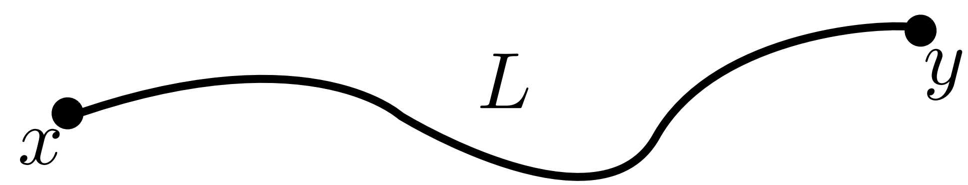
Previous attempts to add loops to the Bethe approximation

- Montanari, Rizzo (2005): BP + loop corrections
- Chertkov, Chernyak (2006): loop calculus = sum over generalized loops
- Parisi, Slanina (2006): effective field theory for lattice models, whose zero-order is Bethe approximation
- None is really effective nor conclusive...

Fat diagrams

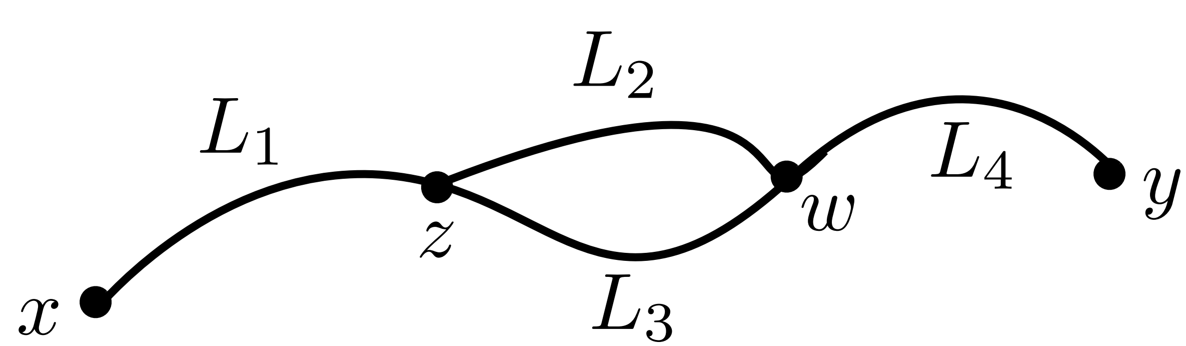
Efetov (1990)   Parisi (2006)



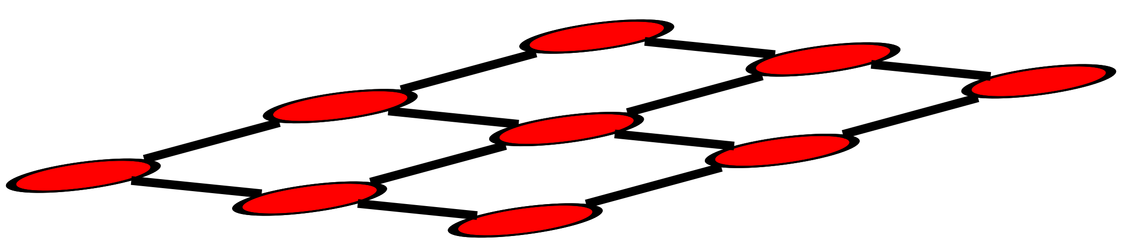
$$P[\phi(x),\phi(y)] = \sum_L \mathcal{N}(x-y,L)\; p[\phi(x),\phi(y),L] +$$

$$\sum_{L_1,L_2,L_3,L_4} \int \mathcal{N}(x-z,L_1)\mathcal{N}(z-w,L_2)\mathcal{N}(z-w,L_3)\mathcal{N}(w-y,L_4)$$

$$p[\phi(x),\phi(y),\phi(z),\phi(w),\{L_i\}]\; + \ldots$$



M-layer model



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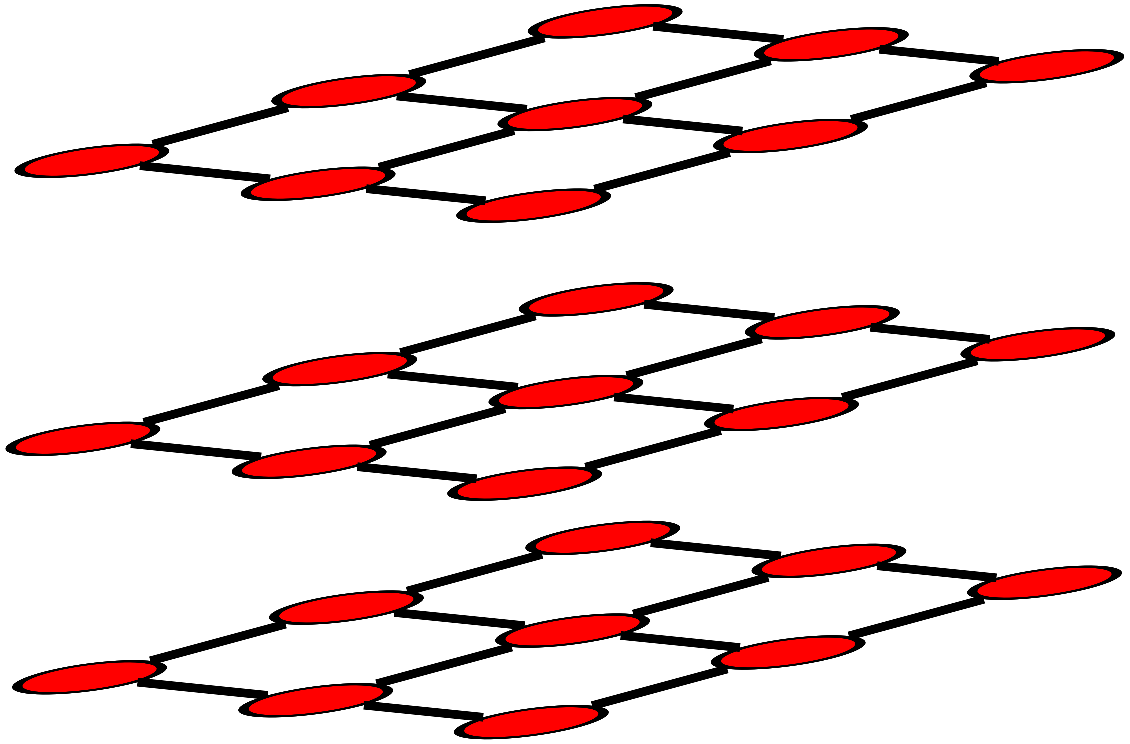

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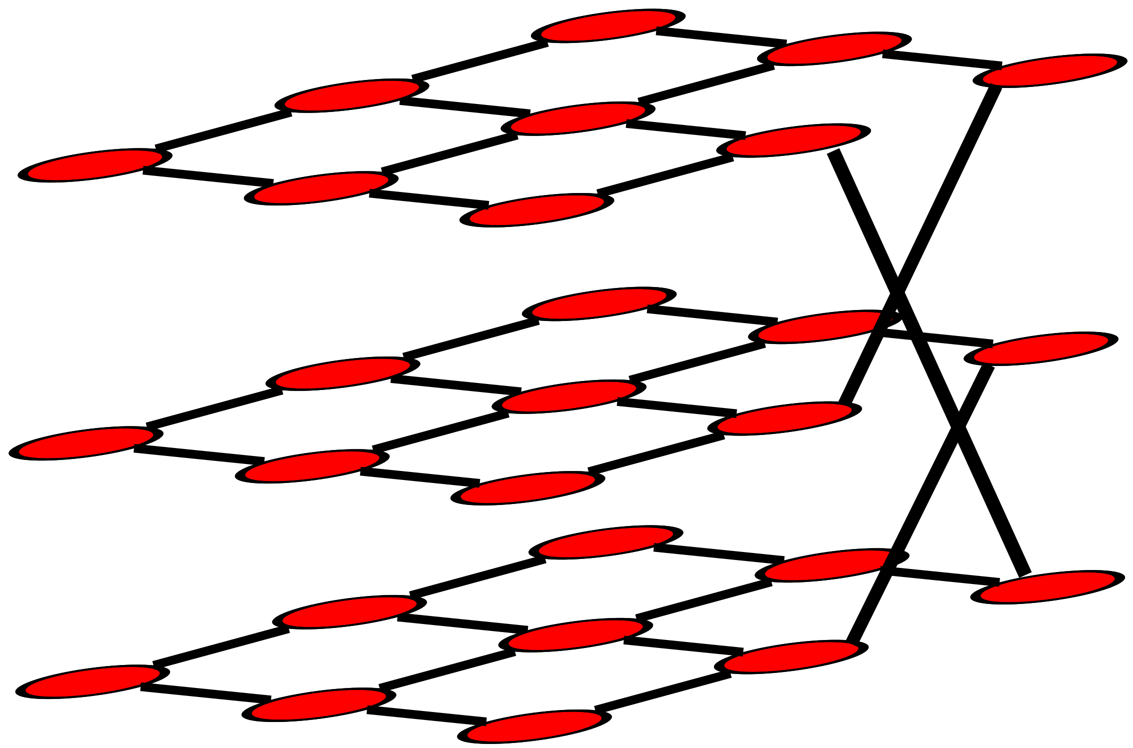

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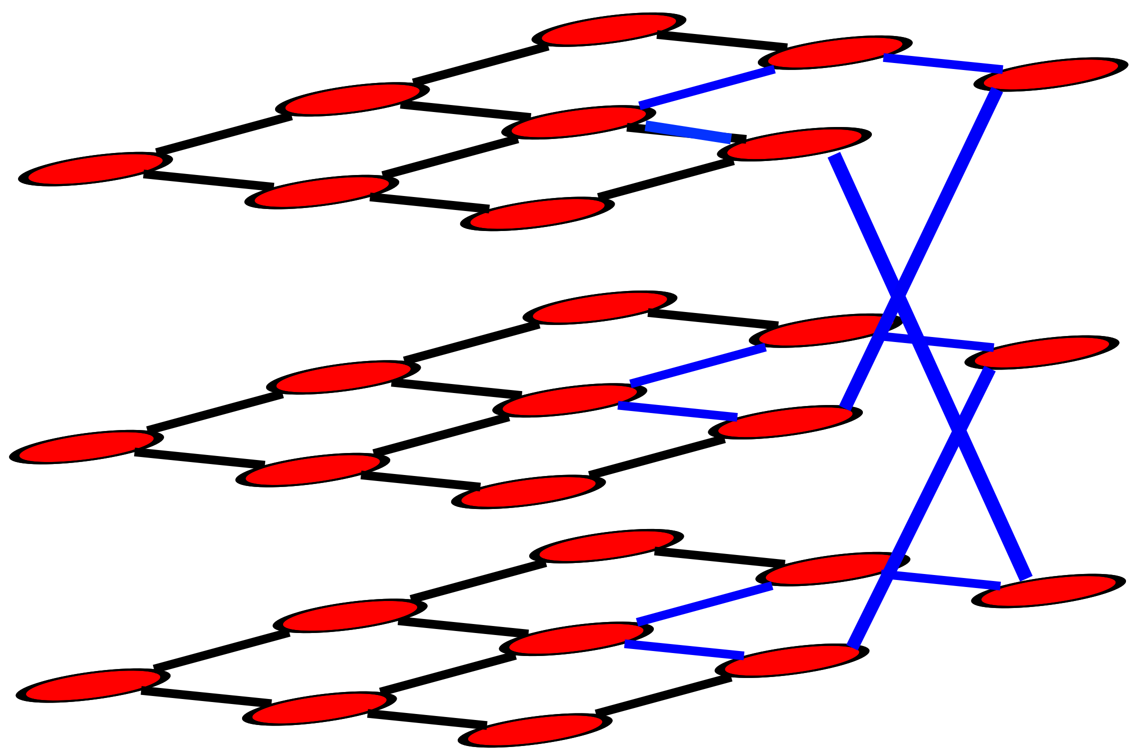
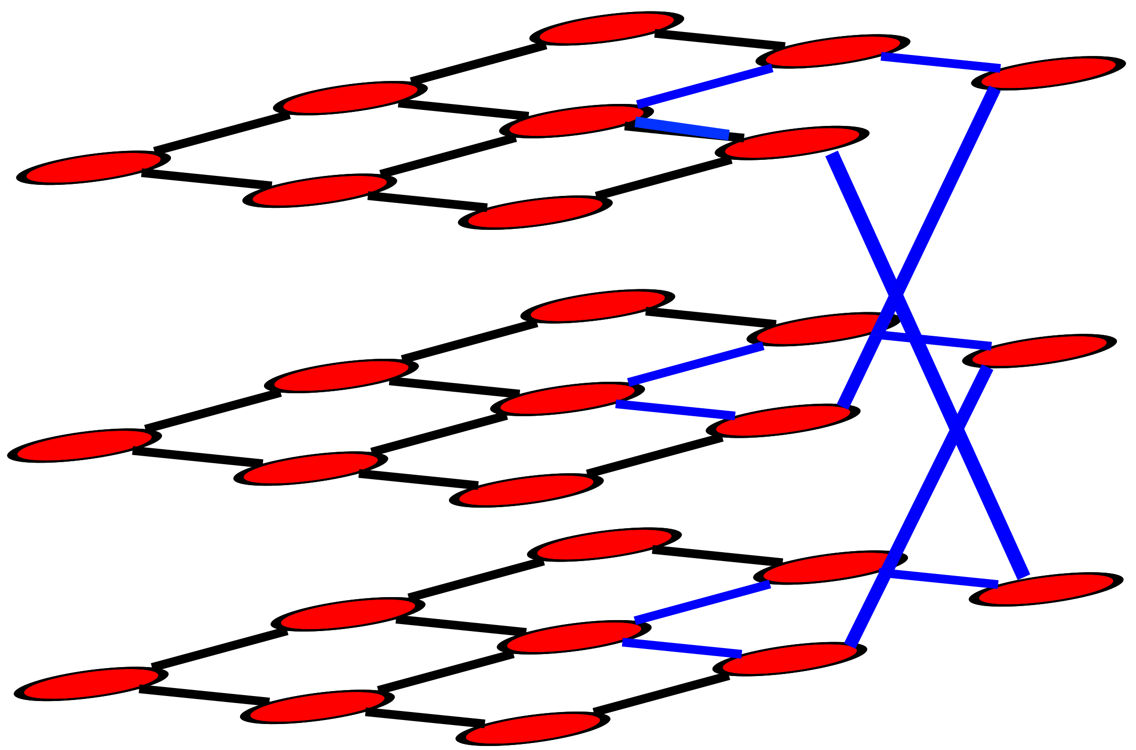


M-layer model

- In the limit $M \to \infty$ we get a random graph
  Bethe approximation is exact (Vontobel, 2012)
  d-dimensional regular lattice -> 2d-regular random graph
- Typical loops are O(M)
- M=1 -> original model
- 1/M expansion =
  loop expansion
- First order in 1/M =
  Bethe + single loops



Rest of the talk

- Computing critical properties of strongly disordered models on random regular graphs
  (SG in a field and RFIM)

- Computing the effect of adding loops to a random regular graph

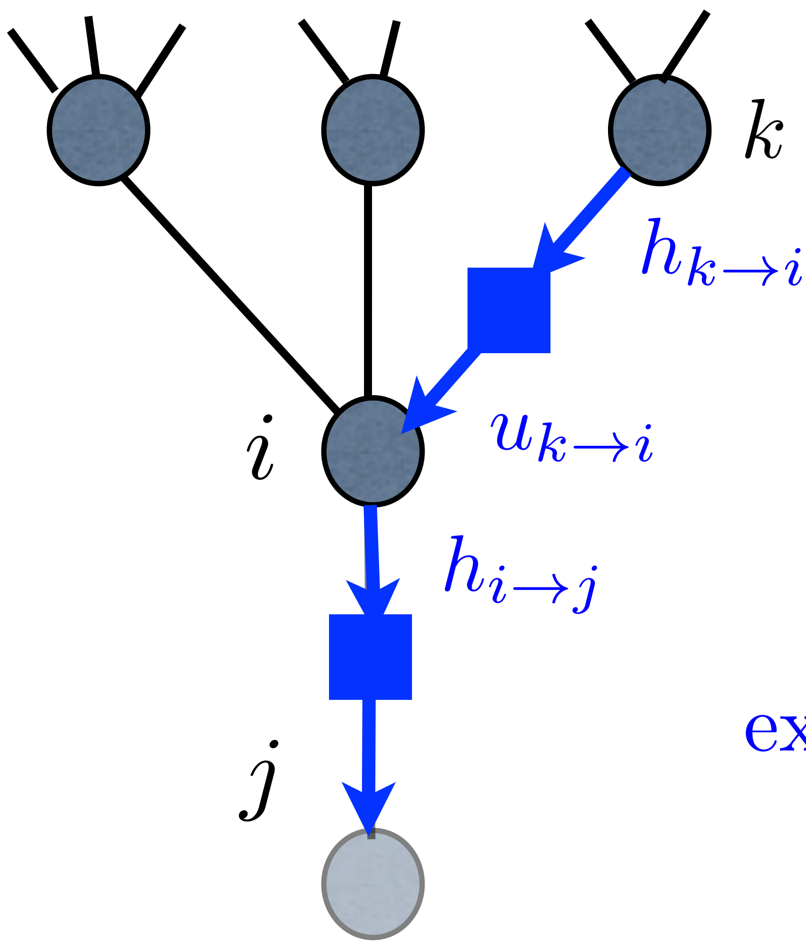
  -> finite size corrections to models defined on random regular graphs

Computing correlations in Bethe approx.

$$H = -\sum_{ij} J_{ij} s_i s_j - \sum_i h_i s_i$$

- Ising spins
- (ij) on a random c-regular graph with c=z+1
- SG in a field: $J_{ij} = \pm 1 \quad h_i = h$
- RFIM: $J_{ij} = 1 \quad h_i \sim N(0, \sigma^2)$

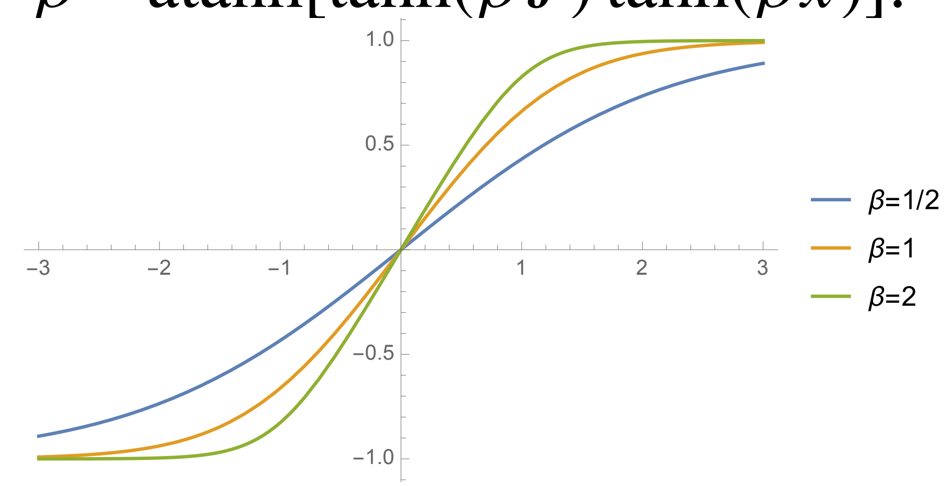
Self-consistency equations as a message passing algorithm (Belief Propagation)



$$P_{i,\setminus j}(\sigma_i) \propto \sum_{\{\sigma_k\}} \prod_k P_{k,\setminus i}(\sigma_k)\psi(\sigma_i,\sigma_k)$$

$\exp[\beta h_{i\to j} s_i]$ $\exp[\beta u_{k\to i} s_i]$

$$\hat{u}(\beta, J, x) = \beta^{-1}\mathrm{atanh}[\tanh(\beta J)\tanh(\beta x)].$$



$$u_{k\to i} = \hat{u}(\beta, J_{ik}, h_{k\to i})$$

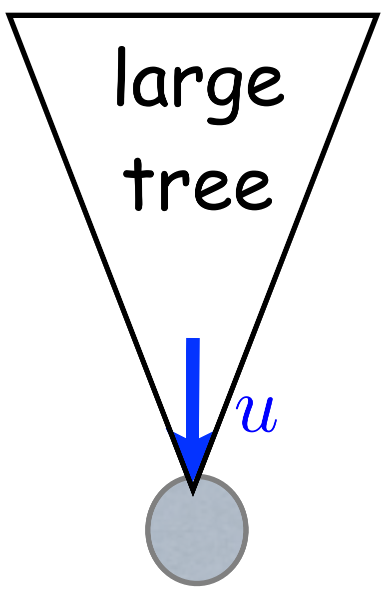
$$h_{i\to j} = h_i + \sum_{k\in V(i)\setminus j} u_{k\to i}$$

Averaging over the random graphs ensemble

$$u_{k\to i} = \hat{u}(\beta, J_{ik}, h_{k\to i})$$

$$h_{i\to j} = h_i + \sum_{k\in V(i)\backslash j} u_{k\to i}$$

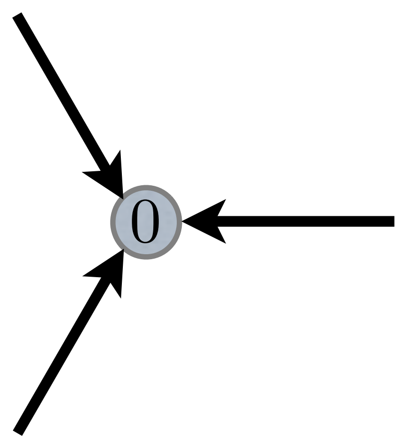
$\{h_{i\to j}, u_{i\to j}\}$ are random variables



$$P(h) = \int \left[\prod_{i=1}^{z} dQ(u_i)\right] \overline{\delta\left(h - h_R - \sum_{i=1}^{z} u_i\right)}^{h_R}$$

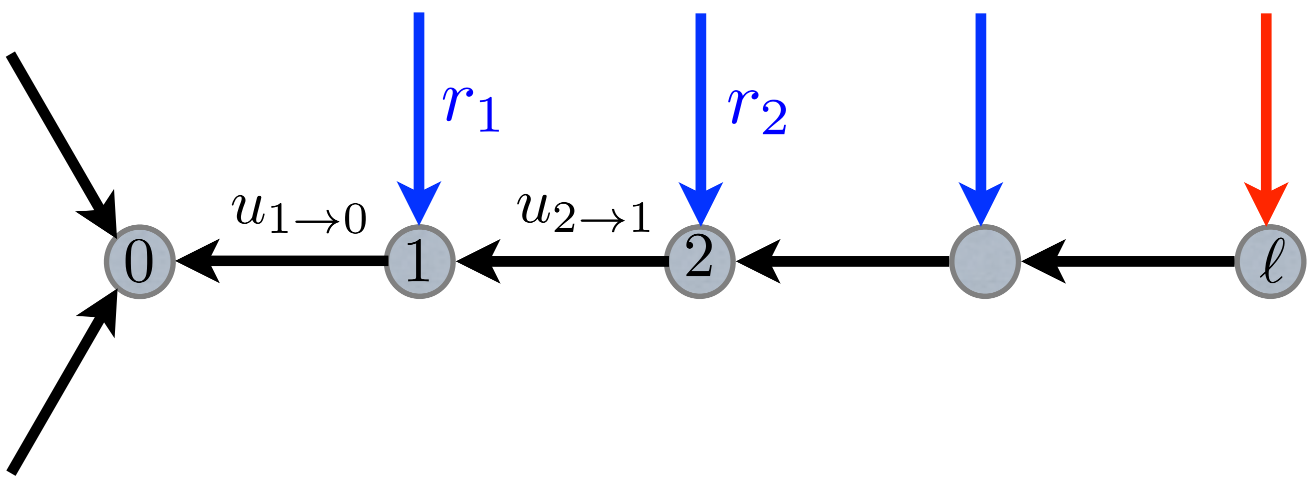
$$Q(u) = \int dP(h)\, \overline{\delta[u - \hat{u}(\beta, J, h)]},$$

Computing correlations in Bethe approx.



$$\langle s_0 \rangle = \tanh\left[\beta(h_R + \sum u_i)\right]$$

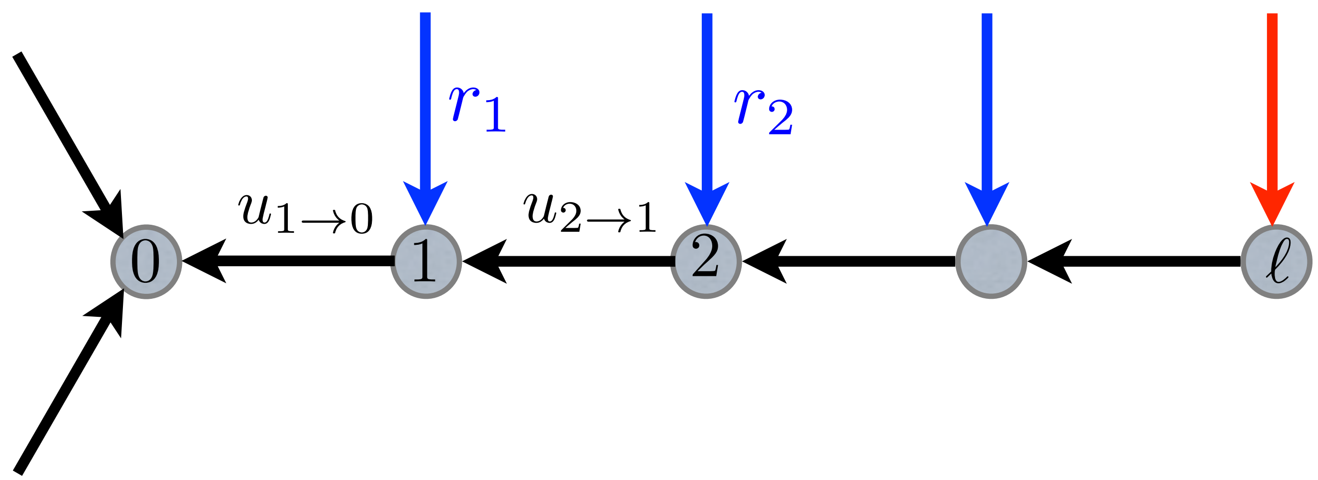
Computing correlations in Bethe approx.



$$\langle s_0 \rangle = \tanh\left[\beta(h_R + \sum u_i)\right]$$

$$\langle s_0 s_\ell \rangle_c = \beta^{-1} \frac{\partial \langle s_0 \rangle}{\partial h_\ell} = (1 - \langle s_0 \rangle^2) \prod_{k=1}^{\ell} \frac{\partial u_{k\to k-1}}{\partial u_{k+1\to k}}$$

Computing correlations in Bethe approx.



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$\mathcal{C}(\ell)$

$\mathcal{C}(\ell)$ is a random variable depending on $(u_{1\to 0}, u_{2\to 1}, \ldots)$

Decay rates and physical interpretation

- on a chain correlations always decay $\mathcal{C}(\ell) \propto e^{-\gamma\ell}$

- decay rate of typical chains $\gamma_0 = -\lim_{\ell\to\infty} \frac{\overline{\log \mathcal{C}(\ell)}}{\ell}$

- ferromagnetic susceptibility $\gamma_1 = -\lim_{\ell\to\infty} \frac{\log \overline{\mathcal{C}(\ell)}}{\ell}$

$$\chi_F \propto \sum_{\ell=1}^{\infty} z^\ell \, \overline{\mathcal{C}(\ell)} \propto \frac{1}{1 - z\exp(-\gamma_1)}$$

- spin glass susceptibility $\gamma_2 = -\lim_{\ell\to\infty} \frac{\log \overline{\mathcal{C}(\ell)^2}}{\ell}$

$$\chi_{SG} \propto \sum_{\ell=1}^{\infty} z^\ell \, \overline{\mathcal{C}(\ell)^2} \propto \frac{1}{1 - z\exp(-\gamma_2)}$$

al criticality $\gamma_{1,2} = \log z$

Large deviation function for the decay rate

$$P_\ell(\gamma) \approx e^{-\ell \Sigma(\gamma)} \quad \text{for} \quad \ell \to \infty \qquad \Sigma(\gamma) \geq 0 \quad \Sigma(\gamma_0) = 0$$

$$\chi_F \propto \sum_\ell z^\ell \int d\gamma \, e^{-\ell[\Sigma(\gamma)+\gamma]} \simeq \sum_\ell z^\ell e^{-\ell[\Sigma(\gamma^*)+\gamma^*]}$$

with $\left.\frac{\partial \Sigma(\gamma)}{\partial \gamma}\right|_{\gamma^*} = -1 \qquad \Longrightarrow \; \gamma^* < \gamma_0 \quad \Sigma(\gamma^*) > 0$

$$\boxed{\chi_F \approx \sum_\ell \mathcal{N}_\ell(\gamma^*) e^{-\ell \gamma^*}}$$

number of chains $\ll z^\ell$

decaying in a much more slower way

Large deviation function for the decay rate

- We actually compute $$\lambda(q) = -\lim_{\ell\to\infty} \frac{\log \overline{\mathcal{C}(\ell)^q}}{\ell}$$

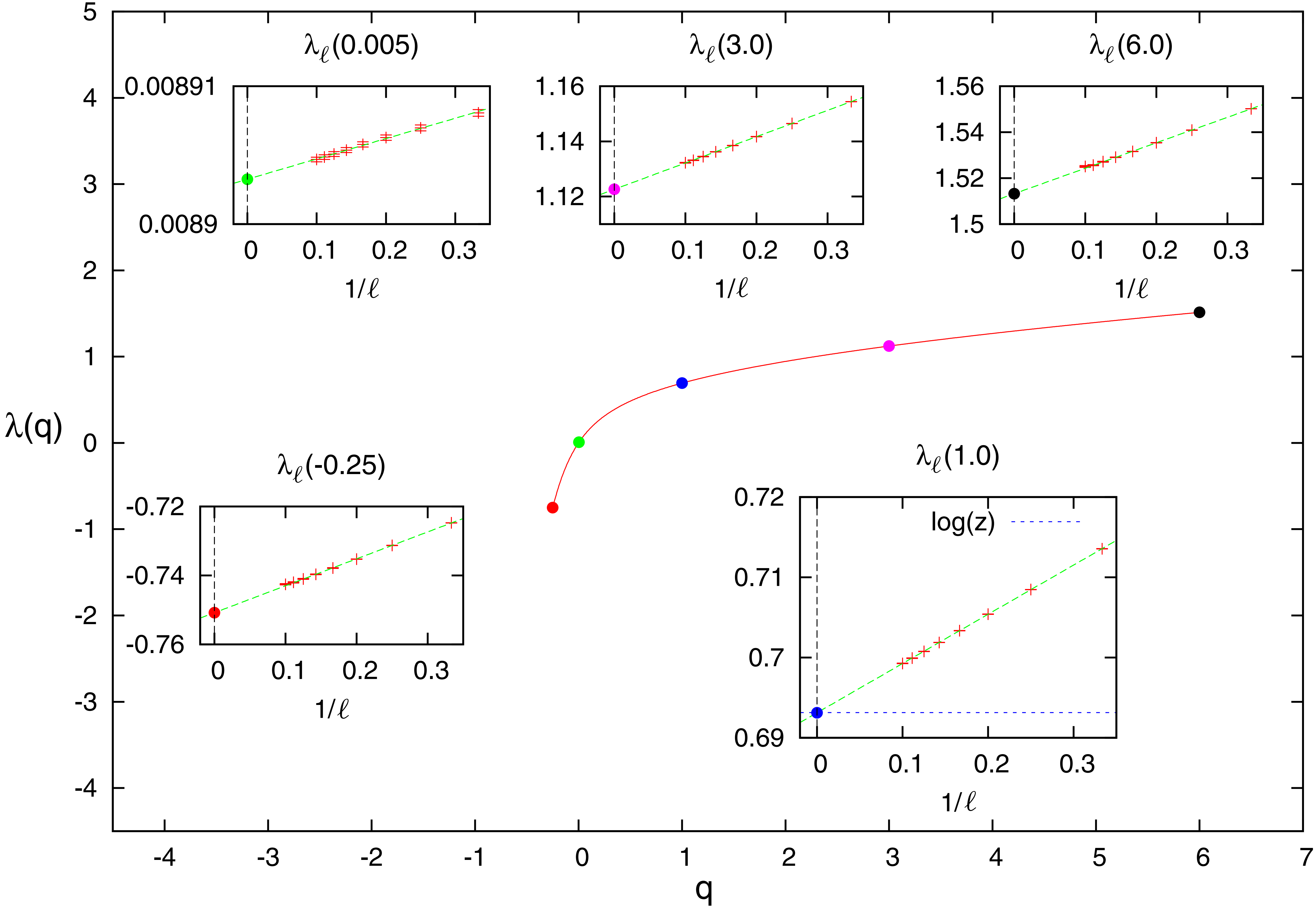
  from which we get $\Sigma(\gamma) = \sup_{q\in\mathbb{R}}[\lambda(q) - q\gamma].$

- We use 2 methods:

  1) "brute force" -> average over a huge number of chains of finite length $$\lambda_\ell(q) = -\frac{\log \overline{\mathcal{C}(\ell)^q}}{\ell}$$

     extrapolate to large distances $$\lambda_\ell(q) = \lambda(q) + \frac{A(q)}{\ell}$$

  2) solve by population dynamics an integral equation providing the result directly in the thermodynamic limit



λ(q)
q
λℓ(0.005)
λℓ(3.0)
λℓ(6.0)
λℓ(-0.25)
λℓ(1.0)
log(z)
1/ℓ
0.00891
0.0089
1.16
1.14
1.12
1.56
1.54
1.52
1.5
-0.72
-0.74
-0.76
0.72
0.71
0.7
0.69
5
4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4 5 6 7
0 0.1 0.2 0.3

Analytic expression for the large deviation function in the thermodynamic limit

$\lambda(q)$ is the largest eigenvalue of the following equation

$$\mathbb{E}_r \int du' g(u',q)\delta[u - \hat{u}(\beta,J,u'+r)] \left(\frac{\partial \hat{u}}{\partial u'}\right)^q = e^{-\lambda(q)} g(u,q)$$

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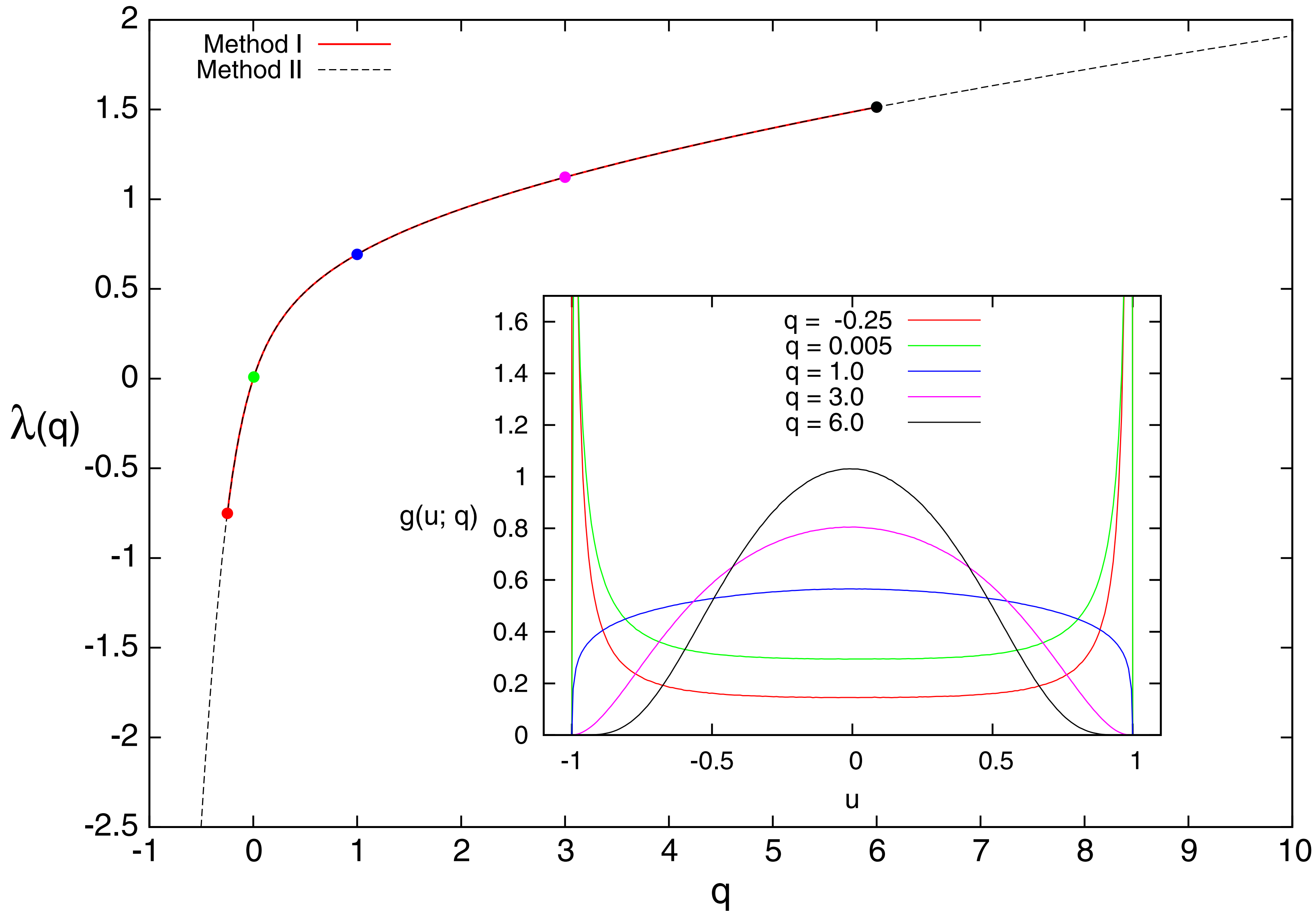


Analytic expression for the large deviation function in the thermodynamic limit

$$\mathcal{C}(\ell) = \prod_{k=1}^{\ell} \frac{\partial u_{k\to k-1}}{\partial u_{k+1\to k}} \qquad u_{k\to k-1} = \hat{u}(\beta, J, r + u_{k+1\to k})$$

$$\mathcal{C}(\ell+1) = \frac{\partial u_{1\to 0}}{\partial u_{2\to 1}}\mathcal{C}(\ell), \quad u_{1\to 0} = \hat{u}(\beta, J, r + u_{2\to 1}).$$

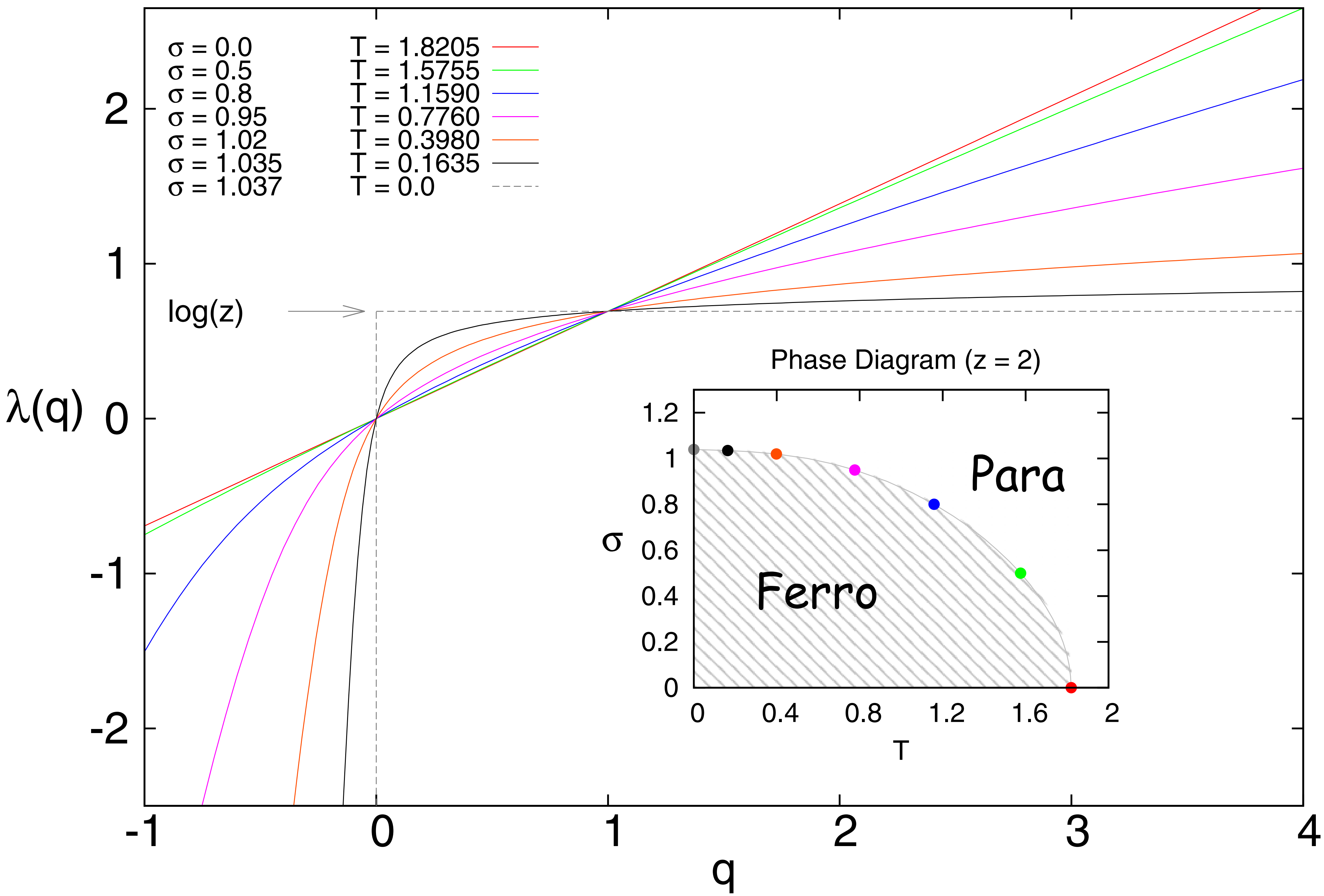
$$P_{\ell+1}(\mathcal{C}, u) = \mathbb{E}_r \int d\mathcal{C}'\, du'\, P_\ell(\mathcal{C}', u')\delta\left[\mathcal{C} - \frac{\partial \hat{u}(\beta, J, r + u')}{\partial u'}\mathcal{C}'\right]\delta[u - \hat{u}(\beta, J, r + u')].$$

$$\psi_\ell(u,q) = \int d\mathcal{C}\, P_\ell(\mathcal{C}, u)\mathcal{C}^q, \qquad \psi_{\ell+1}(u,q) = \mathbb{E}_r \int du'\, \psi_\ell(u',q)\delta[u - \hat{u}(\beta, J, r + u')]\left(\frac{\partial \hat{u}}{\partial u'}\right)^q$$

$$\int du\, \psi_\ell(u,q) = \overline{\mathcal{C}(\ell)^q}, \qquad g_\ell(u,q) = \psi_\ell(u,q)e^{\ell\lambda(q)},$$

$$\boxed{\mathbb{E}_r \int du' g(u',q)\delta[u - \hat{u}(\beta, J, u' + r)]\left(\frac{\partial \hat{u}}{\partial u'}\right)^q = e^{-\lambda(q)}g(u,q)}$$

Gaussian RFIM on random 3-regular graphs



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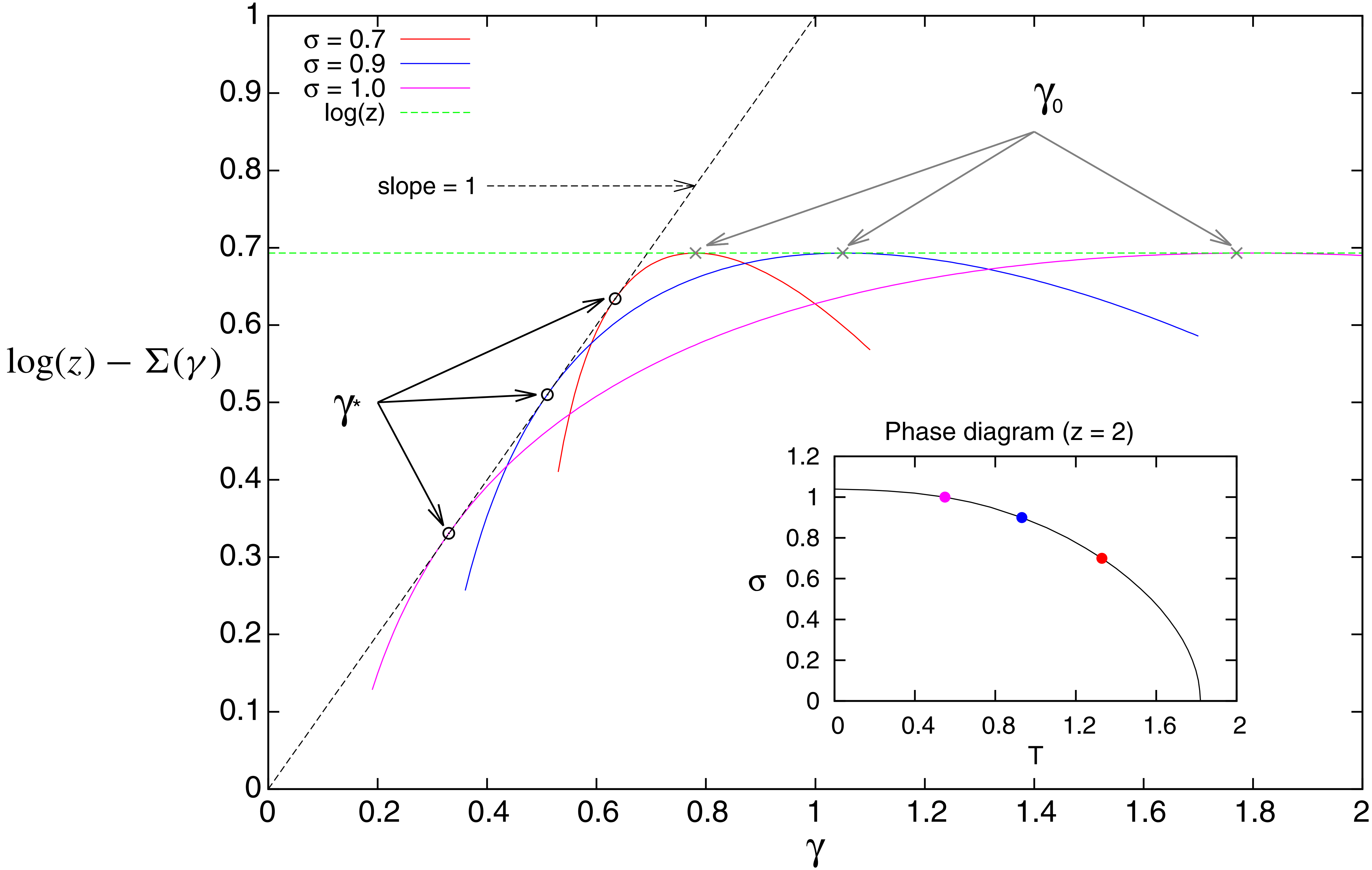

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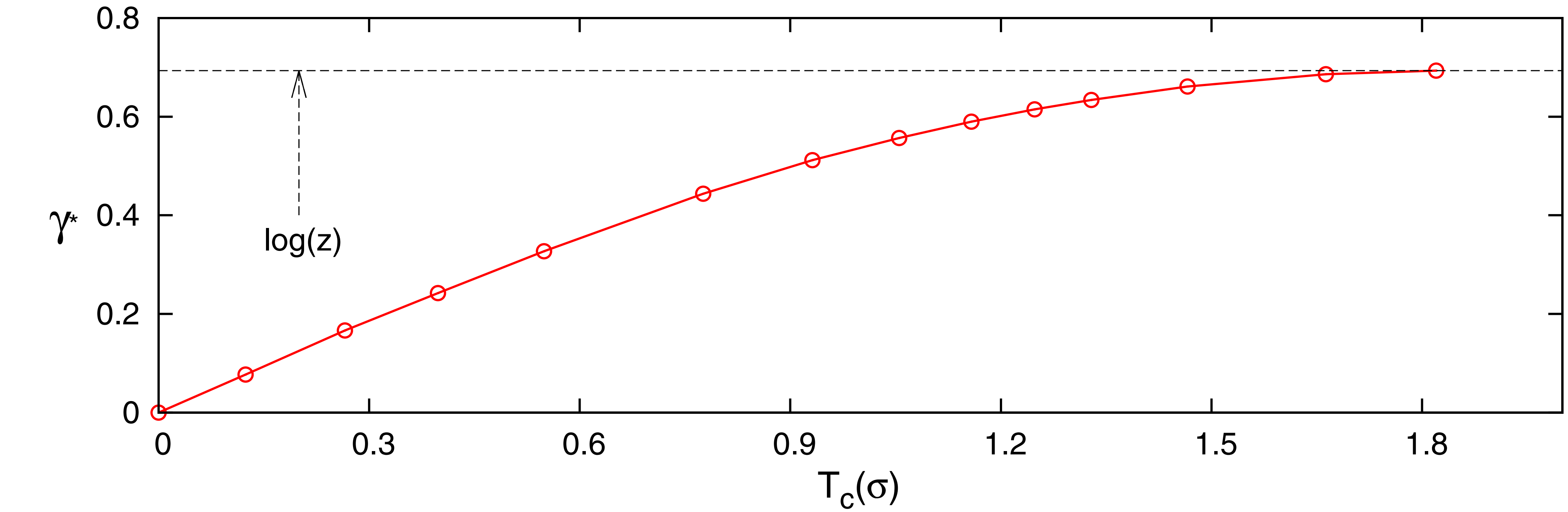

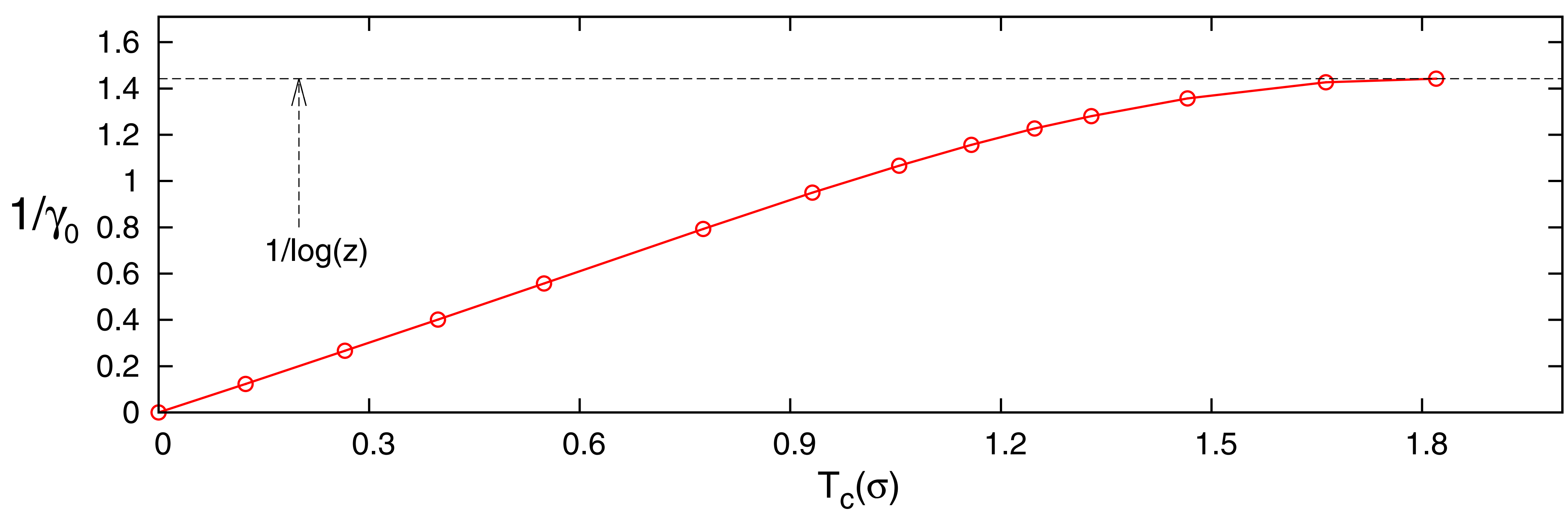

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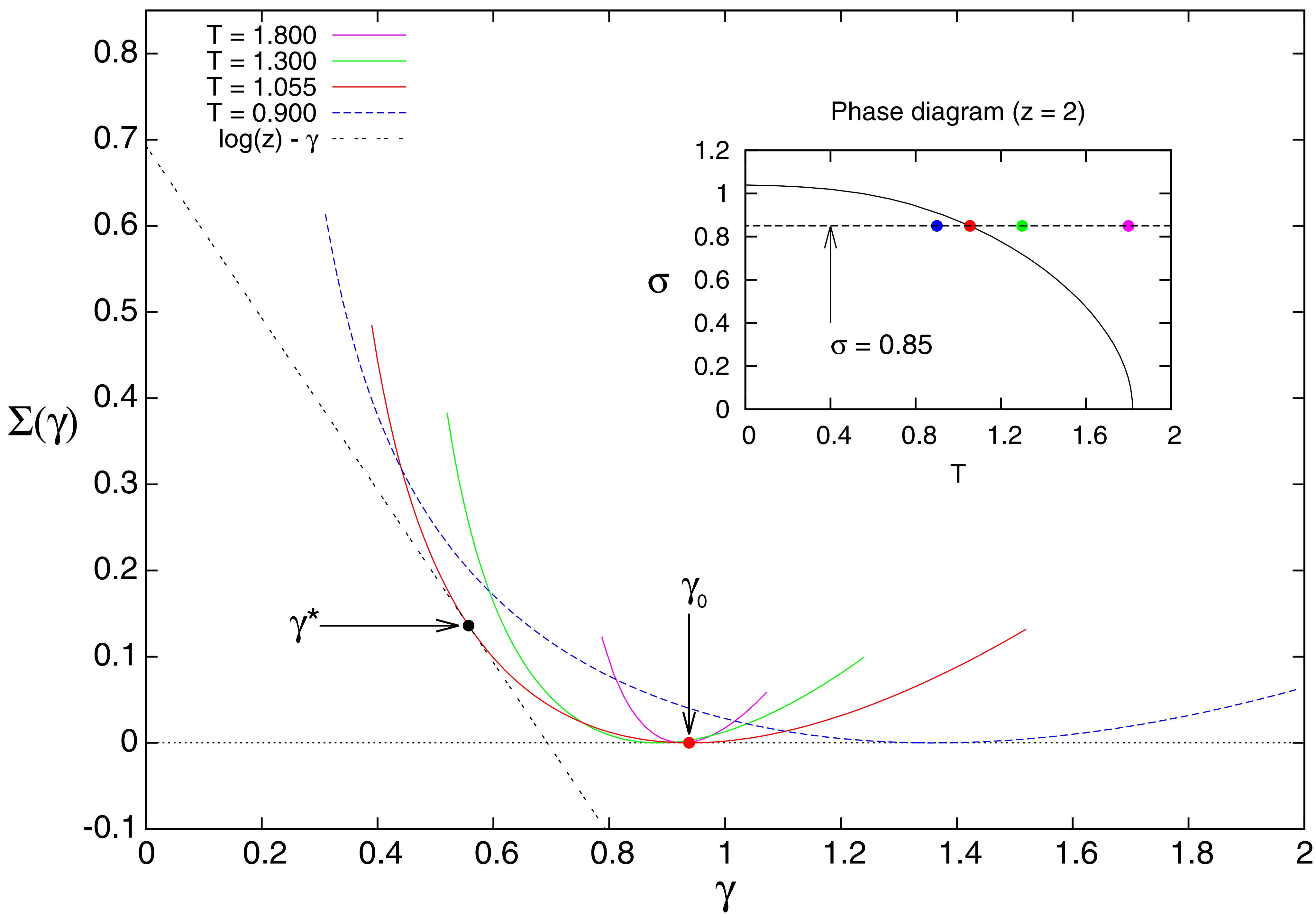

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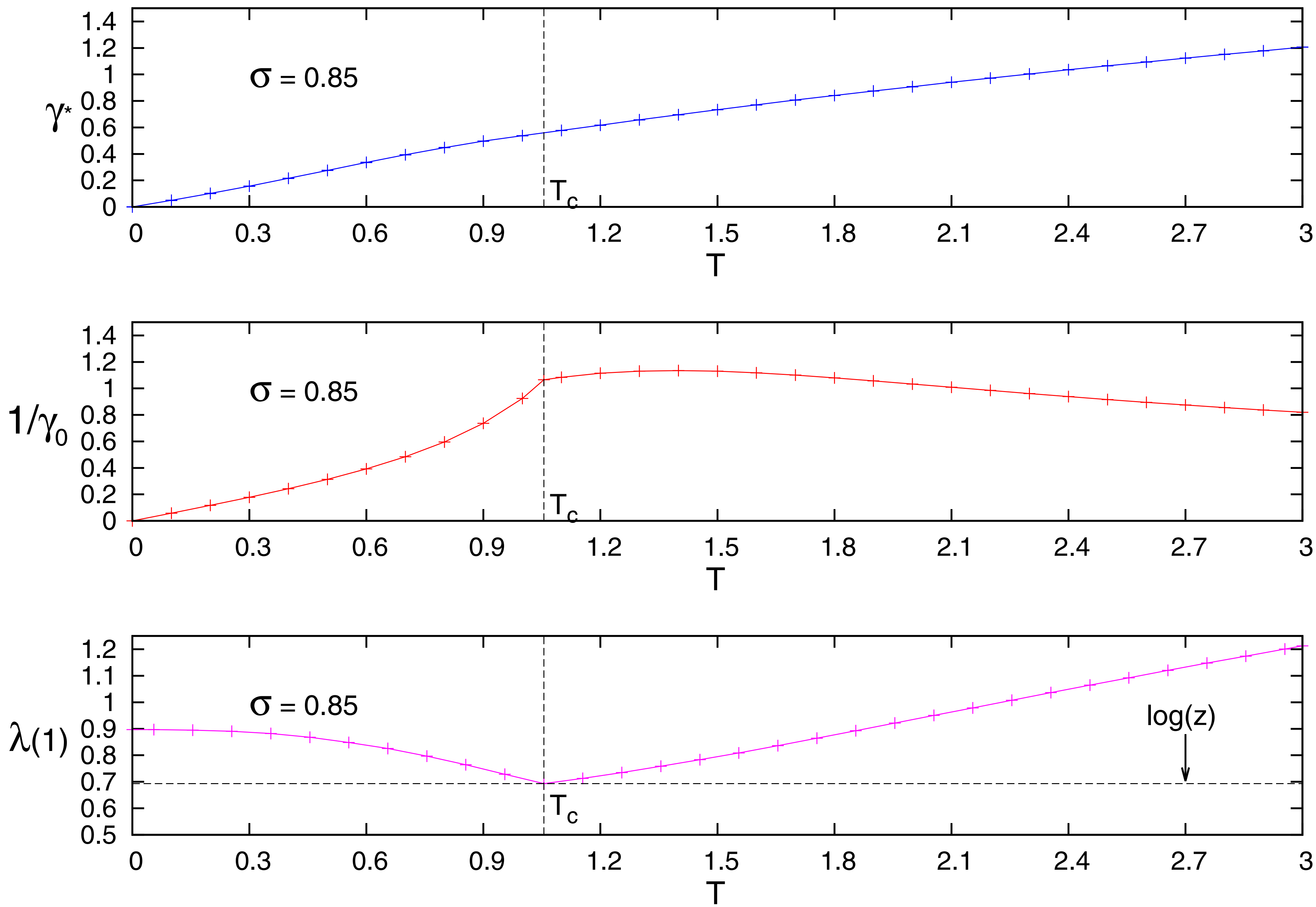

$$\Sigma(\gamma^*) + \gamma^* = \lambda(1).$$

In the zero temperature limit

- phase transition induced by an infinitesimal fraction of highly correlated chains chains

- responses/couplings at large distances from ground state recursion relation $E_\ell(s_0,s_\ell) = -h_0^{(\ell)} s_0 - h_\ell s_\ell - J_\ell s_0 s_\ell + \mathcal{E}_\ell$

$$E_{\ell+1}(s_0, s_{\ell+1}) = \min_{s_\ell} E_\ell(s_0, s_\ell) + s_\ell s_{\ell+1} + h_{\ell+1} s_{\ell+1}$$

$$\boxed{P_\ell(J) \;=\; \rho\lambda^{\ell-1}\ell(\ell-1)(1-\rho J)^{\ell-2} + (1-\ell\lambda^{\ell-1})\delta(J)}$$

$$\lambda = \int_{-1}^{1} dh\, Q_z^{\mathrm{cav}}(h),$$

$$\rho = 2Q_z^{\mathrm{cav}}(1)/\lambda,$$

cavity fields pdf on chains with $J_\ell \neq 0$

- $J_\ell \sim 1/\ell\,, \qquad \overline{J_\ell} = \lambda^{\ell-1}$

  $\mathcal{N}_\ell = \ell(z\lambda)^\ell \quad \text{for} \quad \ell \gg 1$

- $\overline{\langle s_0 s_\ell \rangle_c} \sim \lambda^\ell \qquad \overline{\langle s_0\rangle\langle s_\ell\rangle} \sim \ell\,\lambda^\ell$

Loops in a random regular graph

- A random c-regular graph has $\frac{(c-1)^\ell}{2\ell}$ loops of length ℓ
- Density of loops is $O(1/N)$
- Can we approximate a random graph of <u>finite size</u> as a tree + $O(1/N)$ corrections due to the loops ?

1. Compute analytically physical observables (e.g. energy, free-energy) on a tree with few loops
2. Compute numerically the same observables on a random regular graph of finite size

Finite size corrections by the replica method
(i.e. Gaussian fluctuations around the saddle point)

$$f(N) = f_0 + \frac{f_1}{N} \qquad f_1 = \sum_{\ell=3}^{\infty} \frac{(c-1)^\ell}{2\ell} \Delta\phi_\ell$$

$$\Delta\phi_\ell = \phi_\ell^c - \ell\,\phi$$

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mean number of loops of length ℓ

free-energy shift for adding a loop of length ℓ to an infinite tree

$$\Delta\phi_\ell = \phi_\ell^c - \ell\,\phi$$

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mean number of loops of length ℓ

free-energy shift for adding a loop of length ℓ to an infinite tree → $\Delta\phi_\ell = \phi_\ell^c - \ell\,\phi$

$$\phi \equiv \lim_{\ell\to\infty} \frac{\phi_\ell^c}{\ell}$$

mean free-energy of a loop of length ℓ

mean free-energy per link on the infinite tree

Finite size corrections by the replica method

(i.e. Gaussian fluctuations around the saddle point)

$$f(N) = f_0 + \frac{f_1}{N} \qquad f_1 = \sum_{\ell=3}^{\infty} \frac{(c-1)^\ell}{2\ell} \Delta\phi_\ell$$

mean number of loops of length ℓ

free-energy shift for adding a loop of length ℓ to an infinite tree → $\Delta\phi_\ell = \phi_\ell^c - \ell\,\phi$

$$\phi \equiv \lim_{\ell\to\infty} \frac{\phi_\ell^c}{\ell}$$

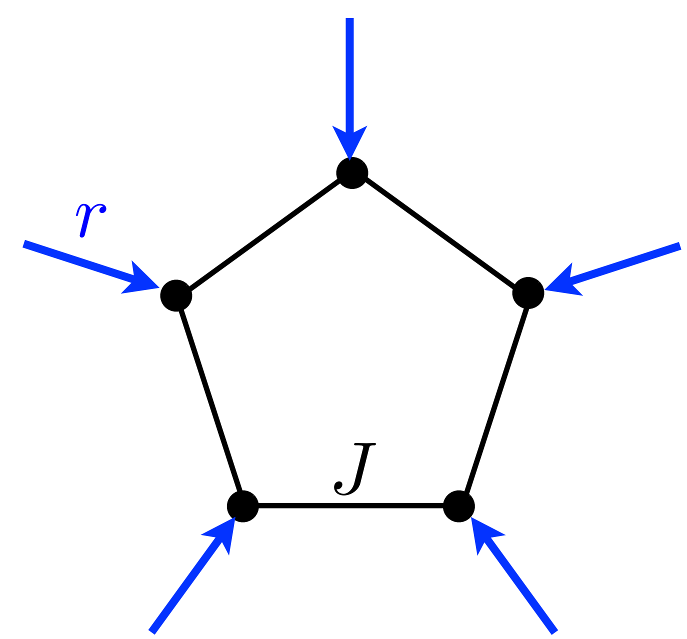
mean free-energy of a loop of length ℓ

mean free-energy per link on the infinite tree

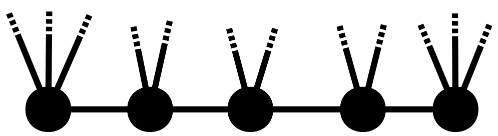
$$\phi_\ell^c \equiv -\frac{1}{\beta}\left[\log Z_\ell^c\right]_{\text{av}}$$

$$Z_\ell^c \equiv \sum_{\sigma_1,\dots,\sigma_\ell} e^{\beta(r_1\sigma_1 + J_1\sigma_1\sigma_2 + \cdots + r_\ell\sigma_\ell + J_\ell\sigma_\ell\sigma_1)}$$

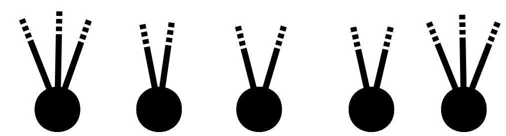
$$R(r) = \mathbb{E}_{J,H} \int \prod_{k=1}^{c-2} \mathrm{d}h_k\, P(h_k)\, \delta\left[r - H - \sum_{k=1}^{c-2} \hat{u}(\beta, J, h_k)\right]$$



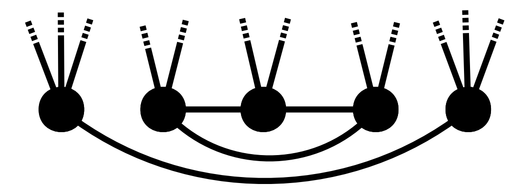
Probabilistic/cavity derivation



randomly chosen path
$(\ell + 1\ edges, \ell + 2\ vertices)$
in an infinite regular tree



remove edges
-> cavity tree



infinite regular tree
with a loop of length ℓ

$$Z_{cav}(\sigma_0, \ldots, \sigma_{\ell+1}) = \tilde{Z} e^{\beta(h_0\sigma_0 + r_1\sigma_1 + \ldots + r_\ell\sigma_\ell + h_{\ell+1}\sigma_{\ell+1})}$$

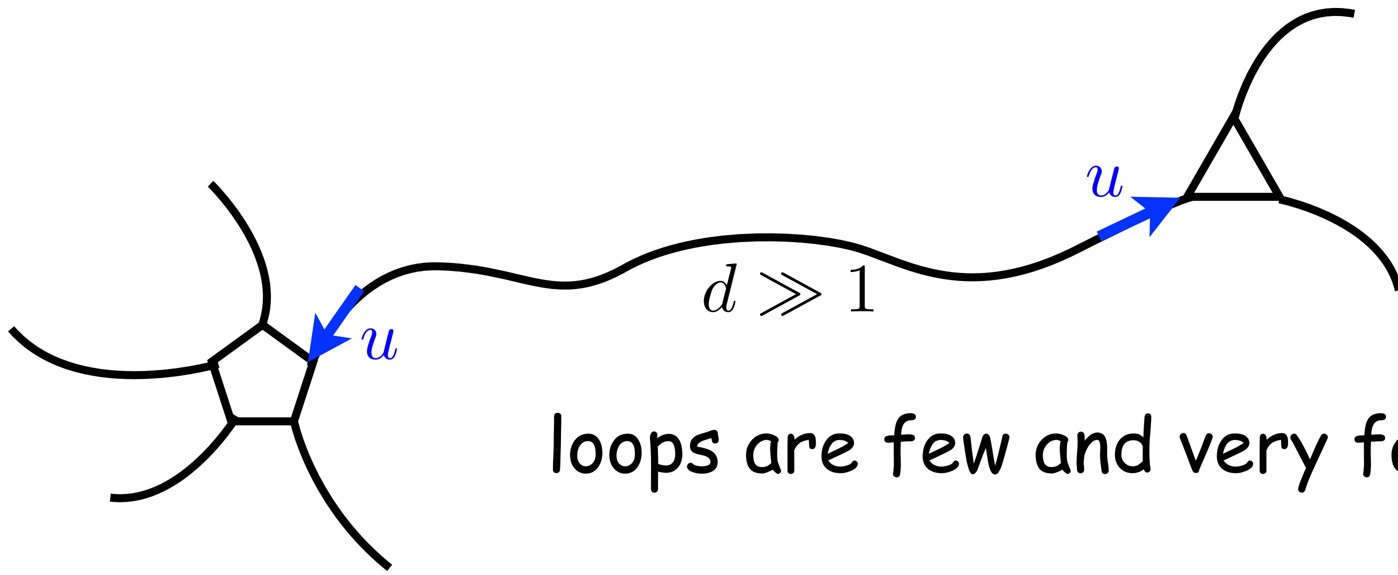
$$Z_T = \tilde{Z} \times Z^o_{\ell+1} \qquad Z_G = \tilde{Z} \times Z^o_1 \times Z^c_\ell$$

$$\Delta\phi_\ell = -\frac{1}{\beta} \lim_{\mathrm{N}\to\infty} [\log Z_G - \log Z_T]_{\mathrm{av}}$$

$$\Delta\phi_\ell = \phi^c_\ell + \phi^o_1 - \phi^o_{\ell+1} = \boxed{\phi^c_\ell - \ell\phi}$$

since on a random regular graph holds $\phi^o_L = L\,\phi + \phi_s$

Probabilistic/cavity derivation



loops are few and very far from each other

messages u arriving on loops are like on the infinite tree

Wormald (1981): numbers of "short" loops of lengths $\ell \geq 3$ are independent Poisson variables with means $\frac{(c-1)^\ell}{2\ell}$

$$f_1 = \sum_{\ell=3}^{\infty} \frac{(c-1)^\ell}{2\ell} \Delta\phi_\ell$$

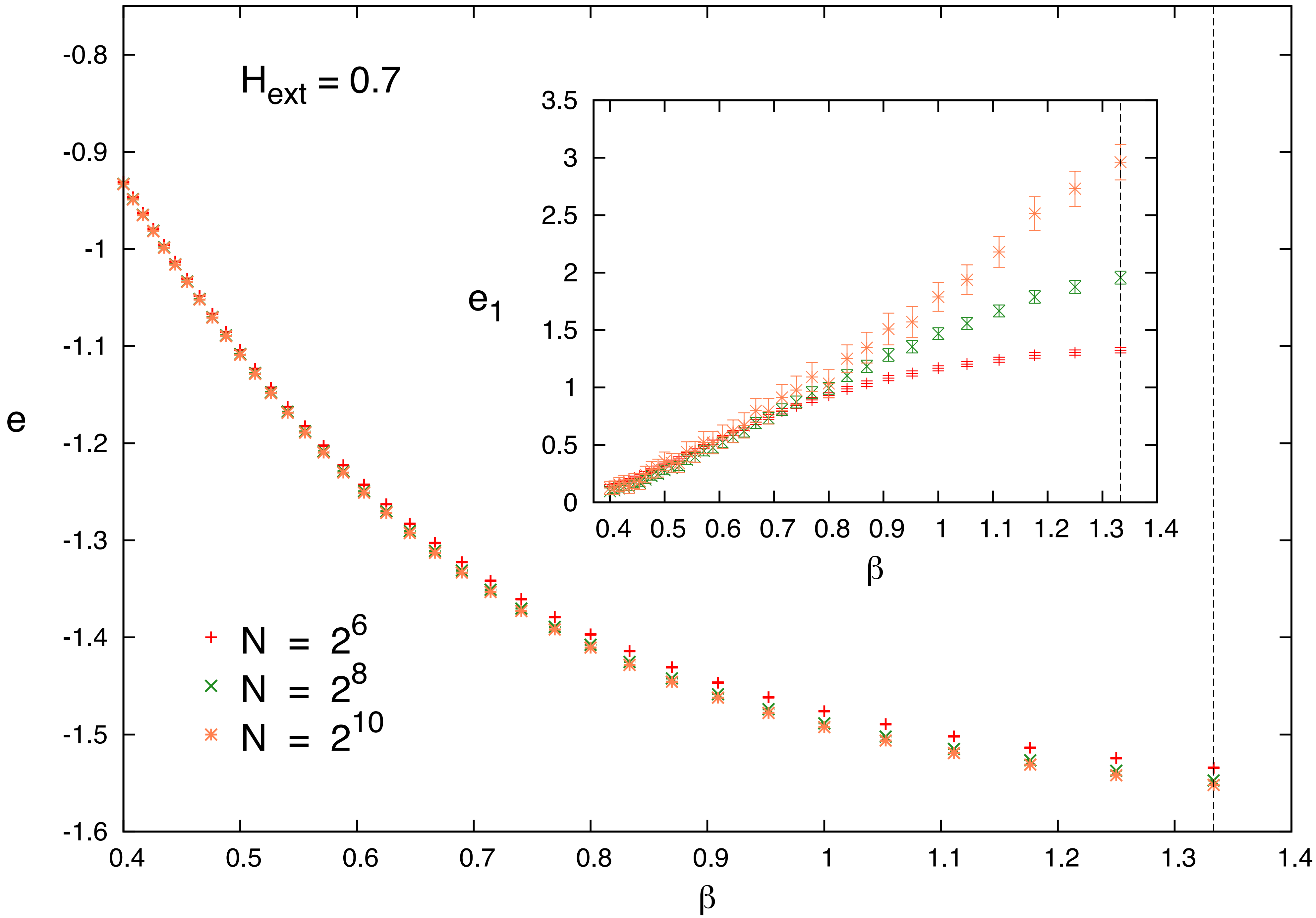
Finite size corrections for spin glass models with magnetic external field on Bethe lattices

- Numerical check of the $O(1/N)$ corrections to the energy, computed analytically through

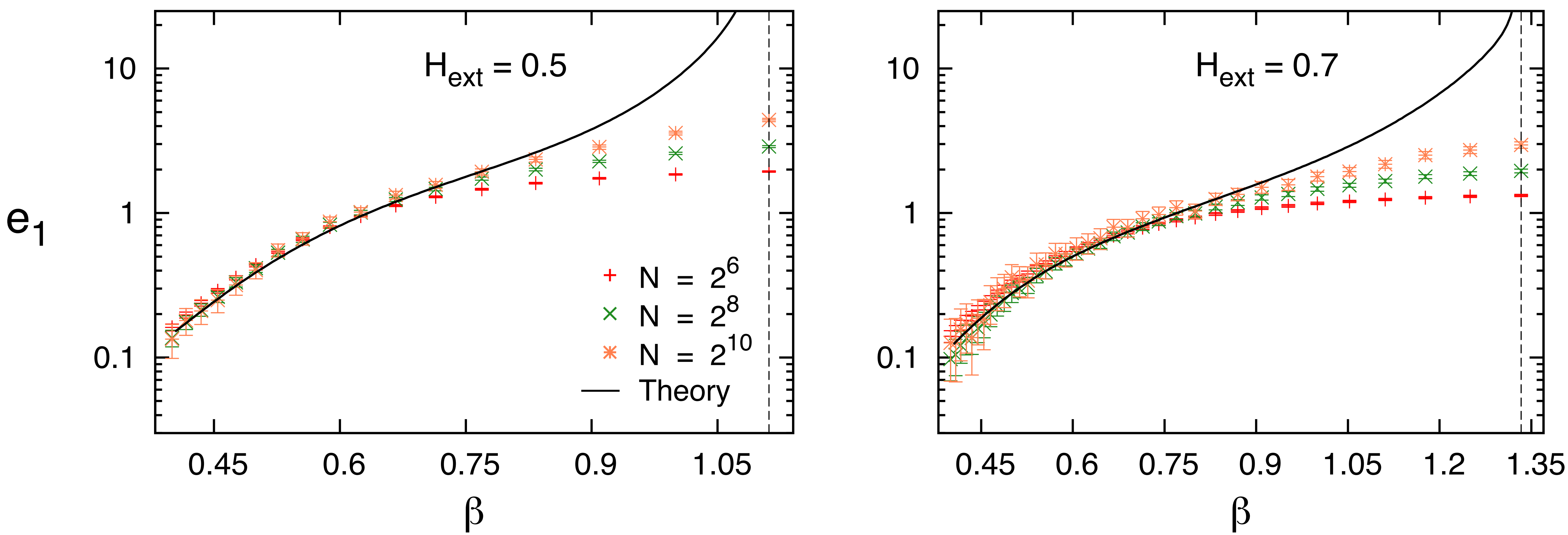
$$e_1 = f_1 + \beta \frac{\partial f_1}{\partial \beta} \qquad f_1 = \sum_{\ell=3}^{\infty} \frac{(c-1)^\ell}{2\ell} \Delta\phi_\ell$$

- Terms in the series are computed explicitly up to $\ell = 7$ and then resumed using the asymptotic $\Delta\phi_\ell \sim A\lambda^\ell$
- Spin glass models (J=+/-1) in a constant field H
- On random 4-regular graphs of sizes from 64 to 1024

Finite size corrections for SG in a field



Finite size corrections for SG in a field



Summary and outlook

- Bethe approximation for strongly disordered systems is quite well under control (at least at the RS level)
- We know how to compute:
  - Full probability distributions of critical correlations (and higher cumulants)
  - Energy and free-energy shifts due to short loops (i.e. finite size corrections to models on random graphs)
- What to do next?
  - Compute fat diagrams to study renormalized propagators
  - Derive a better loop expansion -> algorithm better than BP

Some recent references

- Large deviations of correlation functions in random magnets
  Phys. Rev. E **89**, 214202 (2014)
  F. Morone, G. Parisi, and F. Ricci-Tersenghi

- Finite-size corrections to disordered Ising models on random regular graphs
  Phys. Rev. E **90**, 012146 (2014)
  C. Lucibello, F. Morone, G. Parisi, F. Ricci-Tersenghi, and T. Rizzo

- One-dimensional disordered Ising models by replica and cavity methods
  Phys. Rev. E **90**, 012140 (2014)
  C. Lucibello, F. Morone, and T. Rizzo

Thank you !