

Of course. This is a fundamental concept in quantum field theory, statistical field theory, and many other areas of theoretical physics. Let's break it down from simple Gaussian integrals to the powerful and abstract "functional" version.

1. The Foundation: Ordinary Gaussian Integrals

It all starts with the standard one-dimensional Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2} dx = \sqrt{\frac{2\pi}{a}}, \quad \text{for } a > 0$$

A more useful generalization includes a linear term (Jx), which is completed by shifting the variable.

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + Jx} dx$$

We complete the square:

$$-\frac{1}{2}ax^2 + Jx = -\frac{1}{2}a \left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a}$$

The integral becomes:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}a\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a}} dx = e^{\frac{J^2}{2a}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}ay^2} dy = e^{\frac{J^2}{2a}} \sqrt{\frac{2\pi}{a}}$$

So, the key result is:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}ax^2 + Jx} dx = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}$$

The **source** J is a mathematical tool. Notice that if we differentiate with respect to J and then set $J=0$, we can compute moments (expectation values of x^n):

$$\langle x^n \rangle = \frac{\int x^n e^{-\frac{1}{2}ax^2} dx}{\int e^{-\frac{1}{2}ax^2} dx} = \frac{1}{Z(0)} \frac{\partial^n}{\partial J^n} Z(J) \Big|_{J=0}$$

where $Z(J) = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}$.

2. Multi-Dimensional Gaussian Integrals

Now, let's generalize to N variables. We have a vector $\mathbf{x} = (x_1, \dots, x_n)$ and a symmetric, positive-definite $N \times N$ matrix A . The integral is:

$$Z_0 = \int d^N x \exp\left(-\frac{1}{2} \sum_{i,j=1}^N x_i A_{ij} x_j\right) = \int d^N x e^{-\frac{1}{2} \mathbf{x}^T A \mathbf{x}}$$

The result is a generalization of the one-dimensional case:

$$Z_0 = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

Again, we add a source term, now a vector $\mathbf{J} = (J_1, \dots, J_n)$:

$$Z(\mathbf{J}) = \int d^N x \exp\left(-\frac{1}{2} \mathbf{x}^T A \mathbf{x} + \mathbf{J}^T \mathbf{x}\right)$$

The result is:

$$Z(\mathbf{J}) = \frac{(2\pi)^{N/2}}{\sqrt{\det A}} \exp\left(\frac{1}{2} \mathbf{J}^T A^{-1} \mathbf{J}\right)$$

This is the central formula. The two-point correlation function (or propagator) is given by the inverse of the matrix A :

$$\langle x_i x_j \rangle = \frac{1}{Z(0)} \frac{\partial^2 Z(\mathbf{J})}{\partial J_i \partial J_j} \Big|_{\mathbf{J}=0} = (A^{-1})_{ij}$$

3. The Leap to Gaussian Functional Integrals

A **functional integral** (or path integral) is an integral over a space of *functions*, $\phi(x)$, instead of a finite set of variables. Think of the function $\phi(x)$ as a vector where the

index i is the continuous variable x . The sum over i, j becomes an integral over x, y .

The dictionary is:

Finite Dimension (N variables)	Infinite Dimension (Field Theory)
Vector x_i	Field $\phi(x)$
Matrix A_{ij}	Differential Operator $K(x, y)$
Sum $\sum_{ij} x_i A_{ij} x_j$	Double Integral $\int dx dy \phi(x) K(x, y) \phi(y)$
Inverse Matrix $(A^{-1})_{ij}$	Green's Function / Propagator $G(x, y)$
Determinant $\det A$	Functional Determinant $\text{Det } K$
Source J_i	Source Field $J(x)$

A typical Gaussian action in field theory looks like:

$$S[\phi] = \frac{1}{2} \int d^d x \phi(x) \hat{O} \phi(x)$$

where \hat{O} is a differential operator, e.g., $\hat{O} = -\nabla^2 + m^2$ for a free scalar field.

The corresponding Gaussian functional integral with a source $J(x)$ is:

$$Z[J] = \int \mathcal{D}\phi \exp \left(-\frac{1}{2} \int d^d x \phi(x) \hat{O} \phi(x) + \int d^d x J(x) \phi(x) \right)$$

By direct analogy with the multi-dimensional case, the result is:

$$Z[J] = \frac{\mathcal{N}}{\sqrt{\text{Det } \hat{O}}} \exp \left(\frac{1}{2} \int d^d x d^d y J(x) G(x, y) J(y) \right)$$

where:

- \mathcal{N} is a formal normalization constant (often absorbed into the definition).

- $\text{Det } \hat{O}$ is the **functional determinant**, the product of all eigenvalues of the operator \hat{O} .
- $G(x, y)$ is the **Green's function** (or propagator), which is the inverse of the operator \hat{O} , satisfying:

$$\hat{O}_x G(x, y) = \delta^{(d)}(x - y)$$

The two-point correlation function is the propagator:

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{Z[0]} \left. \frac{\delta^2 Z[J]}{\delta J(x)\delta J(y)} \right|_{J=0} = G(x, y)$$

Key Points and Physical Significance

1. **Definition of the Measure** $\mathcal{D}\phi$: Formally,

$\mathcal{D}\phi = \prod_x d\phi(x)$, an infinite product over all spacetime points. This is mathematically problematic, but in practice, we define it by taking a continuum limit of a discretized (lattice) version. The answers we get are well-defined and physically correct.

2. **The Source** $J(x)$: This is a powerful computational trick. All correlation functions (n-point functions) can be generated by taking functional derivatives with respect to $J(x)$ and then setting $J=0$.

3. **Why is it so Important?**

- **Free Field Theories:** The exact solution for any free (non-interacting) quantum field theory is a Gaussian functional integral. For example, the quantum theory of a free scalar, fermion, or photon field is defined by a Gaussian integral.
- **Perturbation Theory:** For interacting theories, the action is $S = S_{\text{free}} + S_{\text{int}}$. We treat S_{int} as a perturbation and expand the exponential $e^{-S_{\text{int}}}$. The calculation then reduces to evaluating Gaussian integrals of polynomials in $\phi(x)$, which are just moments of the Gaussian distribution. This is the foundation of Feynman diagrams.
- **Semiclassical Approximation:** The saddle-point approximation (or loop expansion) around a classical solution often leads to a Gaussian integral at the lowest order, describing small quantum fluctuations.

A Simple Example: Free Scalar Field

For a free scalar field $\phi(x)$ in Euclidean space, the operator is

$$\hat{H} = -\nabla^2 + m^2.$$

- The propagator is

$$G(x, y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2}.$$

- The partition function is $Z[\theta] \propto 1 / \sqrt{\text{Det}(-\nabla^2 + m^2)}$.
- The two-point function $\langle \phi(x)\phi(y) \rangle$ is exactly $G(x, y)$.

In summary, the Gaussian functional integral is the infinite-dimensional generalization of the standard Gaussian integral. It provides the mathematical foundation for defining and solving free quantum field theories and serves as the starting point for perturbative calculations in interacting theories.

good. can you explain the basis of the first order expansion in g of the Landau Ginzburg ϕ to the fourth theory? How do we compute the integrals.

Excellent question. This is the canonical example of moving from a Gaussian (soluble) theory to an interacting one using perturbation theory. Let's go through it step-by-step.

We start with the **Landau-Ginzburg Hamiltonian (or Euclidean Action)** for a scalar field $\phi(x)$:

$$S[\phi] = \int d^d x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4 \right]$$

We can separate this into the **free (Gaussian) part** and the **interaction part**:

$$S_0[\phi] = \int d^d x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right] = \frac{1}{2} \int d^d x \phi(x)(-\nabla^2 + m^2)\phi(x)$$

$$S_{\text{int}}[\phi] = \int d^d x \frac{g}{4!}\phi^4(x)$$

The full partition function is:

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \int \mathcal{D}\phi e^{-S_0[\phi] - S_{\text{int}}[\phi]}$$

1. The Perturbation Expansion

The coupling constant g is assumed to be small. We expand the exponential of the interaction term:

$$Z = \int \mathcal{D}\phi e^{-S_0[\phi]} \left(1 - S_{\text{int}}[\phi] + \frac{1}{2!} S_{\text{int}}[\phi]^2 + \dots \right)$$

Substituting S_{int} :

$$Z = \int \mathcal{D}\phi e^{-S_0[\phi]} \left(1 - \frac{g}{4!} \int d^d x \phi^4(x) + \frac{1}{2!} \left(\frac{g}{4!} \right)^2 \int d^d x d^d y \phi^4(x) \phi^4(y) + \mathcal{O}(g^3) \right)$$

We can write this as:

$$Z = Z_0 \left(1 - \frac{g}{4!} \int d^d x \langle \phi^4(x) \rangle_0 + \frac{1}{2!} \left(\frac{g}{4!} \right)^2 \int d^d x d^d y \langle \phi^4(x) \phi^4(y) \rangle_0 + \mathcal{O}(g^3) \right)$$

Where:

- $Z_0 = \int \mathcal{D}\phi e^{-S_0[\phi]}$ is the Gaussian partition function.
- $\langle \mathcal{O} \rangle_0 = \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0[\phi]} \mathcal{O}[\phi]$ is the expectation value in the free theory.

2. Computing the Integrals: Wick's Theorem

The expectation values $\langle \phi^4(x) \rangle_0$, etc., are integrals over all field configurations weighted by the Gaussian e^{-S_0} . For a Gaussian theory, these are given by **Wick's Theorem**.

Wick's Theorem states that the expectation value of a product of fields is the sum of all possible full contractions (pairings). For two fields, this is just the propagator:

$$\langle \phi(x)\phi(y) \rangle_0 = G(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{ip \cdot (x-y)}}{p^2 + m^2}$$

Let's compute the first-order term: $\langle \phi^4(x) \rangle_0$.

$\phi^4(x)$ means $\phi(x)\phi(x)\phi(x)\phi(x)$ at the same point x . According to Wick's theorem, we must contract them in all possible ways.

1. Type 1: Disconnected Contraction

We can pair the first with the second and the third with the fourth. There are 3 such distinct pairings (this is the combinatoric factor $4!/(2! \cdot 2! \cdot 2) = 3$).

- Contraction 1-2 and 3-4: $\langle \phi(x)\phi(x) \rangle_0 \langle \phi(x)\phi(x) \rangle_0 = [G(0)]^2$
- Contraction 1-3 and 2-4: $[G(0)]^2$
- Contraction 1-4 and 2-3: $[G(0)]^2$

Total from these contractions: $3[G(0)]^2$

So, we have:

$$\langle \phi^4(x) \rangle_0 = 3[G(0)]^2$$

What is $G(0)$?

$$G(0) = \langle \phi(x)\phi(x) \rangle_0 = \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2}$$

This integral is divergent for $d \geq 2$! This is our first encounter with a **divergence** in QFT. For now, we can regulate it (e.g., with a momentum cutoff Λ) and write $G(0)$ as a formal, potentially large, number.

3. The First-Order Correction to the Partition Function

Plugging this back in:

$$Z = Z_0 \left(1 - \frac{g}{4!} \int d^d x (3[G(0)]^2) + \mathcal{O}(g^2) \right)$$

$$Z = Z_0 \left(1 - \frac{g}{8} [G(0)]^2 \int d^d x + \mathcal{O}(g^2) \right)$$

The $\int d^d x$ factor is just the volume of space, V . So the first-order correction is proportional to the volume and the square of the propagator at zero separation.

4. A More Interesting Calculation: The Two-Point Function

The partition function correction is often not the most physically relevant. Let's look at the **two-point correlation function** $\langle \phi(y)\phi(z) \rangle$, which tells us how fluctuations at point y are correlated with those at point z .

$$\langle \phi(y)\phi(z) \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]} \phi(y)\phi(z)$$

Expanding to first order in g :

$$\langle \phi(y)\phi(z) \rangle \approx \frac{1}{Z_0(1 - \frac{g}{8}[G(0)]^2 V)} \int \mathcal{D}\phi e^{-S_0[\phi]} \left(1 - \frac{g}{4!} \int d^d x \phi^4(x) \right) \phi(y)\phi(z)$$

The denominator can be expanded as $(1 + \frac{g}{8}[G(0)]^2 V + \dots)$. However, the V -term will cancel against disconnected pieces in the numerator. Let's focus on the connected part, which is the numerator directly:

$$\langle \phi(y)\phi(z) \rangle \approx \frac{1}{Z_0} \int \mathcal{D}\phi e^{-S_0[\phi]} \phi(y)\phi(z) - \frac{g}{4!} \frac{1}{Z_0} \int d^d x \int \mathcal{D}\phi e^{-S_0[\phi]} \phi^4(x)\phi(y)\phi(z)$$

The first term is just the free propagator:

$$\langle \phi(y)\phi(z) \rangle_0 = G(y - z)$$

The second term is more interesting: $\langle \phi^4(x)\phi(y)\phi(z) \rangle_0$. We now have 6 fields. We need to contract them all using Wick's theorem. The full contraction must connect the external points y and z to the "interaction vertex" at x .

1. **First, contract $\phi(y)$ and $\phi(z)$ to the vertex at x .** There are 4 choices for which $\phi(x)$ contracts with $\phi(y)$, and then 3 remaining choices for which $\phi(x)$ contracts

with $\phi(z)$. So there are $4 \times 3 = 12$ such contractions.

- The result of one such contraction is: $G(x - y)G(x - z)$
- The combinatorial factor for this structure is 12.

2. **Now, contract the two remaining $\phi(x)$ fields together.** This gives a factor of $\langle \phi(x)\phi(x) \rangle_0 = G(0)$.

So, the value of this expectation value is:

$$\langle \phi^4(x)\phi(y)\phi(z) \rangle_0 = 12 \cdot G(0) \cdot G(x - y) \cdot G(x - z) + (\text{disconnected terms})$$

The disconnected terms will cancel with the denominator expansion.

Putting it all together, the **connected** two-point function to first order in g is:

$$\langle \phi(y)\phi(z) \rangle_{\text{connected}} \approx G(y - z) - \frac{g}{4!} \int d^d x (12 \cdot G(0) \cdot G(x - y) \cdot G(x - z))$$

$$\langle \phi(y)\phi(z) \rangle \approx G(y - z) - \frac{g}{2} G(0) \int d^d x G(x - y)G(x - z) + \mathcal{O}(g^2)$$

5. The Feynman Diagram Picture

This result is perfectly captured by **Feynman diagrams**:

- **Zeroth Order:** $G(y - z)$ is a single line from y to z .

[y]----[z]

- **First Order:** The correction term $-\frac{g}{2}G(0) \int d^d x G(x - y)G(x - z)$ corresponds to the diagram:

[y]----[x]----[z] with a loop on the vertex [x].

Let's break it down:

- There is an **interaction vertex** at point x (this is the ϕ^4 term, which has 4 "legs").
- Two of its legs are connected to the external points y and z (the factors $G(x - y)$ and $G(x - z)$).
- The two remaining legs are connected to each other, forming a **loop**. This loop is $G(0)$, the propagator starting and ending at the same point.

- The factor $-\frac{g}{2}$ is the **symmetry factor** for this diagram. The $\frac{1}{4!}$ from the interaction, combined with the combinatorial factor of 12 we calculated, gives $12/4! = 1/2$. The minus sign comes from the $-S_{int}$ in the expansion.

Summary

The basis of the first-order expansion is:

1. **Expand** the exponential $e^{-S_{int}}$ in powers of the coupling g .
2. **Compute** the resulting expectation values in the free Gaussian theory using **Wick's Theorem**.
3. **Interpret** the results, which involve products of free propagators $G(x - y)$ and divergent terms like $G(0)$.
4. **Handle Divergences:** The divergences (like $G(0)$) signal that a more careful treatment (renormalization) is needed, where physical parameters like m and g are defined at a certain scale and the divergences are absorbed into their renormalized values.

This first-order calculation is the foundation upon which the entire framework of Feynman diagrams and perturbative renormalization is built.

wick's theorem

Of course. **Wick's Theorem** is a fundamental result in quantum field theory and many-body physics that provides a systematic way to compute expectation values of products of operators in a *Gaussian* (or free) theory.

It's the machinery that makes perturbative calculations possible.

1. The Core Idea in Simple Terms

In a Gaussian theory, the probability distribution is like a bell curve. Wick's Theorem tells us that the average (expectation value) of a product of many variables in such a theory is given by:

The sum of all possible ways to pair the variables, where the value of each pair is its covariance (the two-point function).

If you have an odd number of variables, the expectation value is zero.

2. The Formal Statement

Let $A_1, A_2, A_3, \dots, A_{2n}$ be linear operators (or fields) that are quadratic in the creation/annihilation operators (e.g., free fields). The **time-ordered** vacuum expectation value in a free (Gaussian) theory is:

$$\langle T\{A_1 A_2 A_3 \dots A_{2n}\} \rangle_0 = \sum_{\text{all pairings}} \langle T\{A_{i_1} A_{j_1}\} \rangle_0 \langle T\{A_{i_2} A_{j_2}\} \rangle_0 \dots \langle T\{A_{i_n} A_{j_n}\} \rangle_0$$

Where:

- $\langle \dots \rangle_0$ denotes the expectation value in the free vacuum.
- $T\{\}$ denotes time-ordering.
- The sum is over all distinct ways to partition the set $\{1, 2, \dots, 2n\}$ into n pairs. For example, $\{(1, 2), (3, 4)\}$ is one pairing, $\{(1, 3), (2, 4)\}$ is another, etc.
- The expectation value for an odd number of operators is zero:
 $\langle T\{A_1 A_2 \dots A_{2n+1}\} \rangle_0 = 0.$

3. Step-by-Step Examples

Let's use a common notation for the two-point function (the propagator):

$$\langle T\{\phi(x)\phi(y)\} \rangle_0 = G(x, y)$$

Example 1: $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_0$ (where $\phi_i = \phi(x_i)$)

We have 4 operators, so $2n = 4 \Rightarrow n = 2$. We need to find all pairings.

1. Pair (1,2) and (3,4): $G(x_1, x_2) \cdot G(x_3, x_4)$

2. Pair (1,3) and (2,4): $G(x_1, x_3) \cdot G(x_2, x_4)$

3. Pair (1,4) and (2,3): $G(x_1, x_4) \cdot G(x_2, x_3)$

There are no other distinct pairings. Therefore:

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle_0 = G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3)$$

Example 2: $\langle \phi_1 \phi_1 \phi_1 \phi_1 \rangle_0$ (All fields at the same point x_1)

This is the case we saw in the ϕ^4 theory. The calculation is the same as above, but now all coordinates are equal: $x_1 = x_2 = x_3 = x_4 = x$.

So, $G(x_i, x_j) = G(x, x) \equiv G(0)$ for all i, j .

Applying the result from Example 1:

$$\langle \phi^4(x) \rangle_0 = G(0)G(0) + G(0)G(0) + G(0)G(0) = 3[G(0)]^2$$

This matches the calculation we did before.

4. Combinatorics and "All Possible Pairings"

The number of pairings for $2n$ operators is given by the **double factorial**:

$$(2n - 1)!! = (2n - 1) \cdot (2n - 3) \cdot \dots \cdot 5 \cdot 3 \cdot 1$$

This counts the number of distinct ways to partition $2n$ distinct items into n unordered pairs.

- For $n = 1$ (2 operators): $1!! = 1$ pairing.
- For $n = 2$ (4 operators): $3!! = 3$ pairings (as we saw).
- For $n = 3$ (6 operators): $5!! = 15$ pairings.

5. Why is Wick's Theorem True?

The deepest reason comes from the nature of Gaussian integrals. Consider a multi-dimensional Gaussian integral with a source J . The generating functional is:

$$Z[J] = \exp\left(\frac{1}{2} J^T A^{-1} J\right)$$

The expectation value $\langle x_i x_j \cdots x_k \rangle$ is obtained by taking functional derivatives with respect to J_i, J_j, \dots, J_k and then setting $J = 0$:

$$\langle x_i x_j x_k x_l \rangle_0 = \left. \frac{\delta}{\delta J_i} \frac{\delta}{\delta J_j} \frac{\delta}{\delta J_k} \frac{\delta}{\delta J_l} Z[J] \right|_{J=0}$$

When you take these derivatives, the product rule ensures that you get a sum over all ways to "hook" the derivatives to the J 's in the exponential. Each derivative $\delta/\delta J_m$ must be paired with another derivative $\delta/\delta J_n$ to "saturate" the J 's and get a non-zero result when $J = 0$. This process exactly generates the sum over all pairings, where each pair (m, n) gives a factor of $(A^{-1})_{mn} = \langle x_m x_n \rangle_0$.

6. Application in Perturbation Theory & Feynman Diagrams

This is where Wick's Theorem becomes indispensable. In an interacting theory (like ϕ^4), we expand $e^{-S_{\text{int}}}$:

$$e^{-S_{\text{int}}} \approx 1 - \frac{g}{4!} \int d^d x \phi^4(x) + \dots$$

To compute a correlation function like $\langle \phi(y)\phi(z) \rangle$, we need to calculate integrals of the form:

$$\int \mathcal{D}\phi e^{-S_0[\phi]} \phi(y)\phi(z)\phi^4(x)$$

Wick's Theorem is the tool that reduces this complex functional integral to a simple combinatorial problem: "How many ways can I contract $\phi(y)$ and $\phi(z)$ with the four $\phi(x)$'s?"

Each distinct contraction scheme corresponds directly to a **Feynman Diagram**:

- Each **pairing** $\langle \phi(a)\phi(b) \rangle_0$ is a **propagator line** from point a to b .
- Each **interaction term** $\phi^4(x)$ is a **vertex** with 4 legs.
- The **combinatorial factor** in front of the integral (like the $-\frac{g}{2}$ we found) is the **symmetry factor** of the diagram, which is determined by the number of equivalent Wick contractions that produce the same diagram.

In summary, **Wick's Theorem is the formal justification for the Feynman rules in perturbation theory**. It's the bridge between the path integral formulation and the diagrammatic technique that makes calculations tractable.

can you give me a clear and extended proof of Wick's thorem?

Of course. Here is a clear and extended proof of Wick's theorem. I'll present it in two main ways: first for finite-dimensional Gaussian integrals (which is rigorous and captures the essence), and then for the operatorial case in Quantum Field Theory (which is the original context).

Part 1: Proof for Finite-Dimensional Gaussian Integrals

This is the most straightforward and mathematically rigorous proof. It generalizes directly to the functional integral case.

1.1 Setup and Generating Functional

Let $\mathbf{x} = (x_1, x_2, \dots, x_N)$ be a set of real variables. Consider a Gaussian probability measure:

$$P(\mathbf{x}) d^N x = \frac{1}{Z_0} \exp\left(-\frac{1}{2} \mathbf{x}^T A \mathbf{x}\right) d^N x, \quad Z_0 = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}$$

where A is a symmetric, positive-definite matrix.

The expectation value of a function $F(\mathbf{x})$ is:

$$\langle F(\mathbf{x}) \rangle_0 = \frac{1}{Z_0} \int F(\mathbf{x}) \exp\left(-\frac{1}{2} \mathbf{x}^T A \mathbf{x}\right) d^N x$$

The crucial object is the **generating functional (or function)** with a source term $\mathbf{J} = (J_1, \dots, J_N)$:

$$Z(\mathbf{J}) = \langle \exp(\mathbf{J}^T \mathbf{x}) \rangle_0 = \frac{1}{Z_0} \int \exp\left(-\frac{1}{2} \mathbf{x}^T A \mathbf{x} + \mathbf{J}^T \mathbf{x}\right) d^N x$$

As we know from Gaussian integration:

$$Z(\mathbf{J}) = \exp\left(\frac{1}{2} \mathbf{J}^T A^{-1} \mathbf{J}\right)$$

Let's define the covariance matrix (the propagator) as $\Delta = A^{-1}$, so $\Delta_{ij} = \langle x_i x_j \rangle_0$.

Then:

$$Z(\mathbf{J}) = \exp\left(\frac{1}{2} \sum_{i,j} J_i \Delta_{ij} J_j\right)$$

1.2 Key Observation: Differentiation and Contractions

The power of the generating functional is that derivatives with respect to the sources \mathbf{J} bring down factors of \mathbf{x} :

$$\frac{\delta}{\delta J_i} Z(\mathbf{J}) = \langle x_i \exp(\mathbf{J}^T \mathbf{x}) \rangle_0$$

More generally:

$$\langle x_{k_1} x_{k_2} \dots x_{k_m} \rangle_0 = \frac{\delta}{\delta J_{k_1}} \frac{\delta}{\delta J_{k_2}} \dots \frac{\delta}{\delta J_{k_m}} Z(\mathbf{J}) \Big|_{\mathbf{J}=0}$$

So, to prove Wick's theorem, we need to compute:

$$\frac{\delta}{\delta J_{k_1}} \frac{\delta}{\delta J_{k_2}} \dots \frac{\delta}{\delta J_{k_m}} \exp\left(\frac{1}{2} \sum_{i,j} J_i \Delta_{ij} J_j\right) \Big|_{\mathbf{J}=0}$$

1.3 The Proof by Induction

Statement of Wick's Theorem (Finite-Dimensional):

$$\langle \mathbf{x}_{k_1} \mathbf{x}_{k_2} \dots \mathbf{x}_{k_m} \rangle_0 = \begin{cases} 0, & \text{if } m \text{ is odd} \\ \sum_{\text{all pairings}} \Delta_{p_1 p_2} \Delta_{p_3 p_4} \dots \Delta_{p_{m-1} p_m}, & \text{if } m \text{ is even} \end{cases}$$

The sum is over all distinct ways to pair the indices k_1, \dots, k_m into $m/2$ pairs.

Base Case (m=2):

$$\langle \mathbf{x}_{k_1} \mathbf{x}_{k_2} \rangle_0 = \frac{\delta}{\delta J_{k_1}} \frac{\delta}{\delta J_{k_2}} \exp\left(\frac{1}{2} \mathbf{J}^T \Delta \mathbf{J}\right) \Big|_{\mathbf{J}=0}$$

Let's compute this carefully.

$$\frac{\delta}{\delta J_{k_2}} Z(\mathbf{J}) = Z(\mathbf{J}) \cdot \left(\sum_j \Delta_{k_2 j} J_j \right)$$

$$\frac{\delta}{\delta J_{k_1}} \frac{\delta}{\delta J_{k_2}} Z(\mathbf{J}) = \frac{\delta}{\delta J_{k_1}} \left[Z(\mathbf{J}) \cdot \left(\sum_j \Delta_{k_2 j} J_j \right) \right]$$

Using the product rule:

$$\begin{aligned} &= \left(\frac{\delta}{\delta J_{k_1}} Z(\mathbf{J}) \right) \cdot \left(\sum_j \Delta_{k_2 j} J_j \right) + Z(\mathbf{J}) \cdot \left(\frac{\delta}{\delta J_{k_1}} \sum_j \Delta_{k_2 j} J_j \right) \\ &= Z(\mathbf{J}) \cdot \left(\sum_i \Delta_{k_1 i} J_i \right) \left(\sum_j \Delta_{k_2 j} J_j \right) + Z(\mathbf{J}) \cdot \Delta_{k_1 k_2} \end{aligned}$$

Now set $\mathbf{J} = 0$. Since $Z(0) = 1$, the first term vanishes (it's proportional to $J_i J_j$), and we are left with:

$$\langle \mathbf{x}_{k_1} \mathbf{x}_{k_2} \rangle_0 = \Delta_{k_1 k_2}$$

This proves the theorem for $m = 2$. The single pairing is just (k_1, k_2) .

Inductive Step:

Assume the theorem holds for all $m' < m$. Consider:

$$F = \frac{\delta}{\delta J_{k_1}} \frac{\delta}{\delta J_{k_2}} \dots \frac{\delta}{\delta J_{k_m}} Z(\mathbf{J}) \Big|_{\mathbf{J}=0}$$

We use the product rule trick from the $m=2$ case, but now on the *first* derivative.

Think of it as:

$$\frac{\delta}{\delta J_{k_1}} \left[\frac{\delta}{\delta J_{k_2}} \cdots \frac{\delta}{\delta J_{k_m}} Z(\mathbf{J}) \right]$$

Let $G(\mathbf{J}) = \frac{\delta}{\delta J_{k_2}} \cdots \frac{\delta}{\delta J_{k_m}} Z(\mathbf{J})$. Then:

$$\frac{\delta}{\delta J_{k_1}} G(\mathbf{J}) = \frac{\delta}{\delta J_{k_1}} [Z(\mathbf{J}) \cdot f(\mathbf{J})]$$

where $f(\mathbf{J})$ is the complicated function coming from the other derivatives. By the product rule:

$$\begin{aligned} \frac{\delta}{\delta J_{k_1}} G(\mathbf{J}) &= \left(\frac{\delta}{\delta J_{k_1}} Z(\mathbf{J}) \right) f(\mathbf{J}) + Z(\mathbf{J}) \left(\frac{\delta}{\delta J_{k_1}} f(\mathbf{J}) \right) \\ &= Z(\mathbf{J}) \left(\sum_i \Delta_{k_1 i} J_i \right) f(\mathbf{J}) + Z(\mathbf{J}) \left(\frac{\delta}{\delta J_{k_1}} f(\mathbf{J}) \right) \end{aligned}$$

Now set $\mathbf{J} = 0$. The first term vanishes because it's linear in J . We are left with:

$$F = \left. \frac{\delta}{\delta J_{k_1}} f(\mathbf{J}) \right|_{\mathbf{J}=0}$$

But $f(\mathbf{J})$ was defined as the result of applying the derivatives $\frac{\delta}{\delta J_{k_2}} \cdots \frac{\delta}{\delta J_{k_m}}$ to $Z(\mathbf{J})$. So $\frac{\delta}{\delta J_{k_1}} f(\mathbf{J})$ is the result of applying *all* m derivatives.

By our inductive assumption, we know what $G(\mathbf{J}) = Z(\mathbf{J})f(\mathbf{J})$ looks like when we set $\mathbf{J} = 0$: it's the sum over all pairings of $\{k_2, \dots, k_m\}$. The action of $\frac{\delta}{\delta J_{k_1}}$ on this sum is to "contract" k_1 with one of the other indices.

More precisely, the derivative $\frac{\delta}{\delta J_{k_1}}$ can hit:

1. A factor of J inside $f(\mathbf{J})$, which corresponds to pairing k_1 with the index associated with that J .
2. It cannot hit the explicit J -independent terms (the Δ s) because they are constants.

When the derivative hits a specific J_{k_p} (for $p \in \{2, \dots, m\}$), it brings down a factor of 1 and removes that J_{k_p} . This process effectively pairs k_1 with k_p , giving a factor $\Delta_{k_1 k_p}$, and leaves behind the sum over all pairings of the remaining $m - 2$ indices.

Since this can happen for any $p \in \{2, \dots, m\}$, we get:

$$\langle x_{k_1} x_{k_2} \dots x_{k_m} \rangle_0 = \sum_{p=2}^m \Delta_{k_1 k_p} \langle x_{k_2} \dots \hat{x}_{k_p} \dots x_{k_m} \rangle_0$$

where the hat means that index k_p is omitted. The expectation value on the right is over $m - 2$ variables, for which the theorem holds by the inductive hypothesis. This recursive formula is Wick's theorem: it says the m -point function is the sum over all ways to pair the first index with another index, multiplied by the $(m-2)$ -point function for the remaining indices.

This completes the inductive proof.

Part 2: Proof for Operator-Valued Fields (QFT)

In Quantum Field Theory, the statement is for time-ordered products of free fields.

2.1 Setup and Assumptions

Let $\phi(x)$ be a free quantum field. It can be decomposed into positive and negative frequency parts:

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$$

where $\phi^{(+)}$ contains annihilation operators and $\phi^{(-)}$ contains creation operators. The vacuum $|0\rangle$ is annihilated by $\phi^{(+)}$.

The time-ordered product is:

$$T\{\phi(x_1)\phi(x_2)\} = \theta(t_1 - t_2)\phi(x_1)\phi(x_2) + \theta(t_2 - t_1)\phi(x_2)\phi(x_1)$$

The propagator is:

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle = G_F(x - y)$$

2.2 The Core Idea: Normal Ordering and Contractions

The proof relies on two concepts:

1. **Normal Ordering ($: : \rangle$):** All creation operators are moved to the left of all annihilation operators. The vacuum expectation value of a normal-ordered product is zero.

$$\langle 0 | : \phi(x_1) \dots \phi(x_n) : | 0 \rangle = 0$$

2. **Contraction:** The contraction of two fields $\phi(x)$ and $\phi(y)$ is defined as:

$$\text{contraction}\{\phi(x)\phi(y)\} \equiv \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle - \langle 0 | : \phi(x) \phi(y) : | 0 \rangle = G_F(x-y)$$

Wick's Theorem (Operatorial Form):

The time-ordered product of n fields is equal to the normal-ordered product plus the sum of all possible normal-ordered products with one contraction, plus the sum of all possible normal-ordered products with two contractions, etc., until all fields are contracted.

$$T \{ \phi(x_1) \dots \phi(x_n) \} = : \phi(x_1) \dots \phi(x_n) : + \sum_{\text{single pairs}} :$$

where the last term is the sum over all complete pairings (fully contracted term), which is just a c-number.

2.3 Proof Sketch by Induction

Base Case ($n=2$):

We need to show:

$$T \{ \phi(x) \phi(y) \} = : \phi(x) \phi(y) : + \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

This can be verified by checking the four possible time-orderings for $x^0 > y^0$ and $y^0 > x^0$. For example, if $x^0 > y^0$:

$$T\{\phi(x)\phi(y)\} = \phi(x)\phi(y) = (\phi^{(+)} + \phi^{(-)})(\phi^{(+)} + \phi^{(-)})$$

$$: \phi(x)\phi(y) := \phi^{(-)}\phi^{(-)} + \phi^{(-)}\phi^{(+)} + \phi^{(+)}\phi^{(-)} + \phi^{(+)}\phi^{(+)} \quad (\text{Wait, this is just } T\{$$

Let's be more careful. The normal ordering moves all $\phi^{(-)}$ to the left:

$$: \phi(x)\phi(y) := \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(-)}(y)\phi^{(+)}(x) + \phi^{(+)}(x)\phi^{(+)}(y)$$

Now compare with the original product $\phi(x)\phi(y)$:

$$\phi(x)\phi(y) = \phi^{(-)}(x)\phi^{(-)}(y) + \phi^{(-)}(x)\phi^{(+)}(y) + \phi^{(+)}(x)\phi^{(-)}(y) + \phi^{(+)}(x)\phi^{(+)}(y)$$

The difference is in the third term:

$$\phi(x)\phi(y) - : \phi(x)\phi(y) := \phi^{(+)}(x)\phi^{(-)}(y) - \phi^{(-)}(y)\phi^{(+)}(x) = [\phi^{(+)}(x), \phi^{(-)}(y)]$$

For a free field, this commutator is a c-number (proportional to the identity operator) and is equal to the propagator $\langle 0 | [\phi^{(+)}(x), \phi^{(-)}(y)] | 0 \rangle = \langle 0 | \phi(x)\phi(y) | 0 \rangle$ for $x^0 > y^0$. A similar check works for $y^0 > x^0$. Thus the base case holds.

Inductive Step:

Assume the theorem holds for n fields. Consider $T\{\phi(x_1) \dots \phi(x_n)\phi(y)\}$.

We use a clever trick: without loss of generality, assume y^0 is later than all x_i^0 (we can always permute the fields under the T-product). Then:

$$T\{\phi(x_1) \dots \phi(x_n)\phi(y)\} = T\{\phi(x_1) \dots \phi(x_n)\}\phi(y)$$

Now, write $\phi(y) = \phi^{(+)}(y) + \phi^{(-)}(y)$. The annihilation part $\phi^{(+)}(y)$ commutes with all earlier-time fields, so it can be pushed through to act on the vacuum. The creation part $\phi^{(-)}(y)$ does not commute.

Applying the inductive hypothesis to $T\{\phi(x_1) \dots \phi(x_n)\}$, we get a sum of normal-ordered terms with 0, 1, 2, ... contractions. When we multiply by $\phi(y) = \phi^{(+)} + \phi^{(-)}$

, we use the identity:

$$: ABC \dots : \phi^{(-)} =: ABC \dots \phi^{(-)} :$$

$$: ABC \dots : \phi^{(+)} =: ABC \dots \phi^{(+)} : + \sum [\phi^{(+)}, (\text{one field in } ABC \dots)] : (\text{rema} :$$

where the commutator $[\phi^{(+)}, \phi(x_i)]$ is again a c-number and is exactly the contraction for $y^0 > x_i^0$.

This process generates:

1. Terms where $\phi(y)$ is not contracted (the $: \dots \phi(y) :$ parts).
2. Terms where $\phi(y)$ is contracted with one of the $\phi(x_i)$ (coming from the commutator terms).

This is precisely the statement of Wick's theorem for $n+1$ fields. The sum over all such possibilities gives all pairings involving the new field y .

Summary

Both proofs, though different in context, rely on the same fundamental idea: