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Linear span

In linear algebra, the **linear span** (also called the **linear hull** or just **span**) of a set of vectors in a vector space is the intersection of all linear subspaces which each contain every vector in that set. The linear span of a set of vectors is therefore a vector space. Spans can be generalized to matroids and modules.

For expressing that a vector space V is a span of a set S, one commonly uses the following phrases: S spans V; V is spanned by S; S is a **spanning set** of V; S is a generating set of V.

Contents

Definition

Given a vector space *V* over a field *K*, the span of a set *S* of vectors (not necessarily infinite) is defined to be the intersection *W* of all subspaces of *V* that contain *S*. *W* is referred to as the subspace *spanned by* *S*, or by the vectors in *S*. Conversely, *S* is called a *spanning set* of *W*, and we say that *S spans W*.

Alternatively, the span of *S* may be defined as the set of all finite linear combinations of elements of *S*, which follows from the above definition.

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i \,\middle|\, k \in \mathbb{N}, v_i \in S, \lambda_i \in \mathbf{K} \right\}.$$

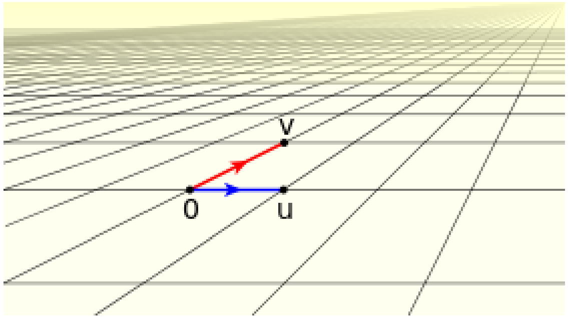
In particular, if *S* is a finite subset of *V*, then the span of *S* is the set of all linear combinations of the elements of *S*. In the case of infinite *S*, infinite linear combinations (i.e. where a combination may involve an infinite sum, assuming such sums are defined somehow, e.g. if *V* is a Banach space) are excluded by the definition; a generalization that allows these is not equivalent.

Examples

The real vector space $\mathbf{R}^3$ has {(-1,0,0), (0,1,0), (0,0,1)} as a spanning set. This particular spanning set is also a basis. If (-1,0,0) were replaced by (1,0,0), it would also form the canonical basis of $\mathbf{R}^3$.

Another spanning set for the same space is given by {(1,2,3), (0,1,2), (−1,1/2,3), (1,1,1)}, but this set is not a basis, because it is linearly dependent.

The set {(1,0,0), (0,1,0), (1,1,0)} is not a spanning set of $\mathbf{R}^3$; instead its span is the space of all vectors in $\mathbf{R}^3$ whose last component is zero. That space (the space of all vectors in $\mathbf{R}^3$ whose last component is zero) is also spanned by the set {(1,0,0), (0,1,0)}, as (1,1,0) is a linear combination of (1,0,0) and (0,1,0). It does, however, span $\mathbf{R}^2$.



The cross-hatched plane is the linear span of **u** and **v** in $\mathbf{R}^3$.

The empty set is a spanning set of {(0, 0, 0)} since the empty set is a subset of all possible vector spaces in $\mathbf{R}^3$, and {(0, 0, 0)} is the intersection of all of these vector spaces.

The set of functions x^n where n is a non-negative integer spans the space of polynomials.

Theorems

Theorem 1: The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

This theorem is so well known that at times it is referred to as the definition of span of a set.

Theorem 2: Every spanning set S of a vector space V must contain at least as many elements as any linearly independent set of vectors from V.

Theorem 3: Let V be a finite-dimensional vector space. Any set of vectors that spans V can be reduced to a basis for V by discarding vectors if necessary (i.e. if there are linearly dependent vectors in the set). If the axiom of choice holds, this is true without the assumption that V has finite dimension.

This also indicates that a basis is a minimal spanning set when V is finite-dimensional.

Generalizations

Generalizing the definition of the span of points in space, a subset X of the ground set of a matroid is called a *spanning set* if the rank of X equals the rank of the entire ground set.

The vector space definition can also be generalized to modules.[1] Given an R-module A and any collection of elements $a_1,\ldots,a_n$ of A, then the sum of cyclic modules,

$$Ra_1 + \cdots + Ra_n = \left\{ \sum_{k=1}^{n} r_k a_k \middle| r_k \in R \right\}$$

consisting of all R-linear combinations of the given elements a_i, is called the submodule of A spanned by $a_1,\ldots,a_n$. As with the case of vector spaces, the submodule of A spanned by any subset of A is the intersection of all the submodules containing that subset.

Closed linear span (functional analysis)

In functional analysis, a closed linear span of a set of vectors is the minimal closed set which contains the linear span of that set.

Suppose that X is a normed vector space and let E be any non-empty subset of X. The **closed linear span** of E, denoted by $\overline{\operatorname{Sp}}(E)$ or $\overline{\operatorname{Span}}(E)$, is the intersection of all the closed linear subspaces of X which contain E.

One mathematical formulation of this is

$$\overline{\operatorname{Sp}}(E) = \{u \in X | \forall \epsilon > 0\, \exists x \in \operatorname{Sp}(E) : \|x - u\| < \epsilon\}.$$

The closed linear span of the set of functions x^n on the interval [0, 1], where n is a non-negative integer, depends on the norm used. If the L^2 norm is used, then the closed linear span is the Hilbert space of square-integrable functions on the interval. But if the maximum norm is used, the closed linear span will be the space of continuous functions on the interval. In either case, the closed linear span contains functions that are not polynomials, and so are not in the linear span itself. However, the cardinality of the set of functions in the closed linear span is the cardinality of the continuum, which is the same cardinality as for the set of polynomials.

Notes

The linear span of a set is dense in the closed linear span. Moreover, as stated in the lemma below, the closed linear span is indeed the closure of the linear span.

Closed linear spans are important when dealing with closed linear subspaces (which are themselves highly important, consider Riesz's lemma).

A useful lemma

Let X be a normed space and let E be any non-empty subset of X. Then

(a) $\overline{\operatorname{Sp}}(E)$ is a closed linear subspace of X which contains E,

(b) $\overline{\operatorname{Sp}}(E) = \overline{\operatorname{Sp}(E)}$, viz. $\overline{\operatorname{Sp}}(E)$ is the closure of $\operatorname{Sp}(E)$,

(c) $E^\perp = (\operatorname{Sp}(E))^\perp = (\overline{\operatorname{Sp}(E)})^\perp$.

(So the usual way to find the closed linear span is to find the linear span first, and then the closure of that linear span.)

See also

- Affine hull
- Convex hull

Notes

1. Lane, Saunders Mac; Birkhoff, Garrett (1999-02-28). *Algebra: Third Edition* (https://www.amazon.co.uk/Algebra-Third-AMS-Chelsea-Publishing/dp/0821816462). EDS Publications Ltd. p. 168. ISBN 9780821816462.

References

- M.I. Voitsekhovskii (2001) [1994], "Linear hull", in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Lankham, Isaiah; Nachtergaele, Bruno; Schilling, Anne (13 February 2010). "Linear Algebra - As an Introduction to Abstract Mathematics" (PDF). University of California, Davis. Retrieved 27 September 2011.
- Brian P. Rynne & Martin A. Youngson (2008). *Linear Functional Analysis*, page 4, Springer ISBN 978-1848000049.

External links

- Linear Combinations and Span: Understanding linear combinations and spans of vectors, khanacademy.org.
- "Linear combinations, span, and basis vectors". *Essence of linear algebra*. August 6, 2016 – via YouTube.