Summability of perturbation expansions in disordered systems: Results for a toy model

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We study a simple model for a disordered ferromagnet in zero dimensions. By analytic means and by direct computation of 200 terms in the perturbation expansion for the free energy, we show that the series is not Borel summable for any finite disorder. We discuss the significance of this result for more complicated systems.

I. INTRODUCTION

It is well known that perturbative calculations of critical exponents using the ϵ expansion[1] are asymptotic[2] rather than convergent. In fact, this seems to be generally the case for perturbation expansions in physics. If one is expanding a physical quantity f in powers of a parameter g, i.e.,

$$f(g)=\sum_K A_K g^K , \tag{1}$$

then the coefficient A_K typically has the form

$$A_K = cK^b a^K K![1+O(1/K)] \tag{2}$$

for large K. The series has zero radius of convergence because of the factor of $K!$. One can, nonetheless, try to sum up the series using a Borel transform.[2] Here one looks at the function obtained from the series without the factors of $K!$, namely

$$F_B(g)=\sum_K \frac{A_K}{K!} g^K , \tag{3}$$

which has a radius of convergence $1/|a|$, and one puts back the $K!$ to recover $f(g)$ by a Laplace transform, i.e.,

$$f(g)=\int_0^\infty e^{-t}F_B(gt)dt . \tag{4}$$

One therefore needs to determine F_B along the entire positive real axis, not just inside its radius of convergence. This can be carried out either by a Padé analysis[3] or, as we shall see in Sec. IV below, by a conformal transformation.[4] A necessary condition for this to work is that $F_B(g)$ has no singularity on the positive real axis, otherwise the integral in Eq. (4) is ill defined. This means that the parameter a in Eq. (2) cannot be positive because the singularity closest to the origin would then be on the positive axis. Even if this is not the case there could be subdominant terms which spoil Borel summability. For example, if

$$A_K = K![c_1(-a)^K + c_2 b^K][1+O(1/K)] \tag{5}$$

with a,b real and $a>b>0$, then $F_B(g)$ has poles at $-1/a$ and $1/b$. Hence the factors of b^K in Eq. (5) spoil Borel summability even though they are exponentially small compared with the $(-a)^K$ terms.

Expansions for critical exponents of pure, i.e., nonrandom, systems turn out to be Borel summable and the Borel transformation has been used to obtain very accurate exponents.[5] However, the question of Borel summability of expansions for disordered systems has not been discussed very much, at least to our knowledge, apart from some work on the percolation problem.[6,7] We have no general solution to this problem but have found it very useful to investigate a simple model in detail and this paper reports the results of our investigations. The model has just a single degree of freedom, so there is no phase transition and we are simply looking at a perturbation expansion of the free energy. We studied it because the free energy can be computed exactly and we can obtain a large number of terms (200) in the perturbation expansion. We shall see that the high-order behavior is surprisingly rich and is not Borel summable for any finite disorder.

The plan of the paper is as follows. In Sec. II we introduce the model while Sec. III attempts to look at the high-order behavior using a saddle-point method combined with the replica trick. As we shall see this has been only partially successful. Consequently, we have also numerically computed the coefficients in the expansion (up to 200 terms for various values of the disorder) and present these results in Sec. IV. They clearly show a rather complicated behavior which is not Borel summable. In Sec. V we summarize our results and discuss their possible physical significance. We also comment on their relevance to more realistic models with many degrees of freedom. Some technical aspects of the replica saddle-point method are given in Appendix A, while Appendix B describes the technique for numerical evaluation of the coefficients.

II. THE MODEL

We wish to find a very simple model of a disordered system where the free energy can be obtained exactly and where a large number of coefficients in the expansion can be computed. We also have in mind the problem of phase transitions in disordered ferromagnets, for which a Ginzburg-Landau-Wilson (GLW) "ϕ^4" Hamiltonian is the starting point.[8] In fact, the high-order behavior of a GLW Hamiltonian is very closely related to the behavior of the corresponding model with just a single degree of freedom, see, e.g., Ref. 7. We therefore study the following model for a single "soft spin." The Hamiltonian is given by

$$H=\tfrac{1}{2}(1+\psi)\phi^2+\frac{u}{4}\phi^4 , \tag{6}$$

where ϕ represents the spin, ψ the quenched disorder, and $u>0$ for stability. From H one obtains the partition function, Z, by integrating over ϕ, i.e.,

$$Z(\psi)=\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}e^{-H} . \tag{7}$$

The random variable ψ has a Gaussian distribution of variance $2w$ so the average free energy, f, is given by

$$-f=\int_{-\infty}^{\infty}\frac{d\psi}{\sqrt{4\pi w}}e^{-\psi^2/4w}\ln Z(\psi) . \tag{8}$$

Note that $f=0$ if $u=w=0$.

It is clearly straightforward to compute numerically the double integral in Eqs. (7) and (8) to obtain f for any choice of u and w. One can also expand f in powers of these quantities. We have found it more convenient to fix the ratio between w and u, i.e.,

$$w=\lambda u , \tag{9}$$

so f can be expanded in terms of the single variable u, the coefficients being polynomials in λ, i.e.,

$$-f=\sum_{K=1}^{\infty}A_K(\lambda)u^K . \tag{10}$$

The purpose of this paper is to compute the $A_K(\lambda)$ for large K.

Equation (8), of course, describes the quenched free energy. If, instead, we average Z rather than $\ln Z$ over ψ we obtain the annealed partition function, $Z_{\rm ann}$. Performing the integral over ψ first, one has

$$Z_{\rm ann}=\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}\exp\left[-\frac{\phi^2}{2}-(u-w)\frac{\phi^4}{4}\right] , \tag{11}$$

which is just the partition of a pure system with u replaced by $u-w$. Hence the annealed problem is undefined unless $\lambda<1$. However, the quenched model is well defined for any positive λ.

III. SADDLE-POINT APPROACH

For pure systems one can determine the form of the high-order coefficients by a saddle-point technique.[2] To apply such an approach here we first use the replica trick[9] to average over the disorder, i.e.,

$$-f=\lim_{n\to 0}([Z^n]_{\rm av}-1)/n , \tag{12}$$

where $[\cdots]_{\rm av}$ is the average over ψ written out explicitly for $[\ln Z(\psi)]_{\rm av}$ in Eq. (8). Clearly, one has

$$[Z^n]_{\rm av}=\int_{-\infty}^{\infty}\prod_\alpha\left[\frac{d\phi_\alpha}{\sqrt{2\pi}}\right]\exp\left[-\tfrac{1}{2}\sum_\alpha\phi_\alpha^2-\frac{u}{4}\sum_\alpha\phi_\alpha^4+\frac{\lambda u}{4}\left[\sum_\alpha\phi_\alpha^2\right]^2\right] , \tag{13}$$

where $\alpha=1,\ldots,n$, is a replica label and, expanding $[Z^n]_{\rm av}$ as

$$[Z^n]_{\rm av}=\sum_K A_K(n,\lambda)u^K \tag{14}$$

[where $A_K(n,\lambda)=nA_K(\lambda)$ as $n\to 0$], the coefficient $A_K(n,\lambda)$ is given by

$$A_K(n,\lambda)=\frac{(-1)^k}{K!4^K}\int D\phi\exp\left[-\tfrac{1}{2}\sum_\alpha\phi_\alpha^2\right]\times\left[\sum_\alpha\phi_\alpha^4-\lambda\left[\sum_\alpha\phi_\alpha^2\right]^2\right]^K . \tag{15}$$

Here $\int D\phi$ is a short-hand notation for $\int_{-\infty}^{\infty}\prod_\alpha(d\phi_\alpha/\sqrt{2\pi})$. We do the ϕ_α integrals by steepest descents for large K. First, multiply the ϕ_α by $\sqrt{K}$ so

$$A_K(n,\lambda)=\frac{(-1)^K K^{n/2}}{K!4^K}e^{2K\ln K}\int D\phi\, e^{Kf\{\phi_\alpha\}} , \tag{16}$$

where

$$f\{\phi_\alpha\}=-\tfrac{1}{2}\sum_\alpha\phi_\alpha^2+\ln\left[\sum_\alpha\phi_\alpha^4-\lambda\left[\sum_\alpha\phi_\alpha^2\right]^2\right] . \tag{17}$$

The extrema of $f\{\phi_\alpha\}$ are where

$$\frac{\partial f}{\partial \phi_\alpha}=\phi_\alpha\left[-1+\frac{4\left[\phi_\alpha^2-\lambda\sum_\beta \phi_\beta^2\right]}{\sum_{\alpha'}(\phi_{\alpha'})^4-\lambda\left[\sum_{\alpha'}\phi_{\alpha'}^2\right]^2}\right]=0\ . \tag{18}$$

The solutions have the form

$$\phi=(\pm\phi_c,\pm\phi_c,\ldots,\pm\phi_c,\ \ 0,0,\ldots,0)\ , \tag{19}$$

where the number of $\pm\phi_c$ terms is s and the number of zero terms is $n-s$;

$$\phi_c^2=4/s \tag{20}$$

and at the saddle point f takes the value f_s where

$$f_s=-2+\ln 16+\ln\left[\frac{1}{s}-\lambda\right]\ . \tag{21}$$

There are $2^{s\,n}C_s$ solutions for a given s. Next, we discuss the Gaussian fluctuations about the saddle points. Taking the second derivative of f and inserting the saddle point values for ϕ from Eqs. (19) and (20) we find

$$\frac{\partial^2 f}{\partial\phi_\alpha\partial\phi_\beta}=\begin{cases}\dfrac{2\delta_{\alpha\beta}}{1-\lambda s}-\dfrac{2\lambda}{1-\lambda s}-\dfrac{4}{s}, & \alpha,\beta\le s\\ 0, \ \ \alpha\le s,\beta>s & \\ -\dfrac{\delta_{\alpha\beta}}{1-\lambda s}, \ \ \alpha,\beta>s\ . & \end{cases} \tag{22}$$

The corresponding eigenvalues are

$$-2,\ \ 2/(1-\lambda s),\ \ -1/(1-\lambda s)\ ,$$

with degeneracies $1, s-1$, and $n-s$, respectively. Naively, we are looking for the absolute maximum of f and require that all eigenvalues are negative. Actually the $n\to 0$ limit makes things more complicated and one should include all saddle points, but let us ignore this complication for the moment. For $\lambda<1$ the maxima are the $s=1$ solutions. Integrating out the Gaussian fluctuations, multiplying by the number of solutions, and letting $n\to 0$ we find a contribution $A_K^{(1)}(\lambda)$ to the coefficient $A_K(\lambda)$, where

$$A_K^{(1)}(\lambda)=\frac{(-1)^K K!}{\sqrt{2\pi K}}\frac{[4(1-\lambda)]^K}{\sqrt{1-\lambda}}\ . \tag{23}$$

The superscript "1" indicates that this is the contribution from the $s=1$ saddle points. Note that it is of the expected form given in Eq. (2) and the coefficients alternate in sign. However, the other saddle points do contribute as well. From the discussion in Appendix A and our numerical results we believe that the complete result for $A_K(\lambda)$ is

$$A_K(\lambda)=[A_K^{(1)}(\lambda)+A_K^{(\infty)}(\lambda)][1+O(1/\sqrt{K})]\ , \tag{24}$$

where

$$A_K^{(\infty)}(\lambda)=\frac{-K!(4\lambda)^K}{\sqrt{\pi}K^{3/2}}\exp(-\gamma\sqrt{K}+\sigma)\cos(\mu\sqrt{K}+\delta)\ . \tag{25}$$

The values of the coefficients γ, σ, μ, and δ will be discussed in Sec. IV and Appendix A. Here we just note three points about Eqs. (23)–(25). Firstly, both terms in Eq. (24) contribute for $\lambda<1$ but only $A_K^{(\infty)}(\lambda)$ is present for $\lambda>1$. Secondly, $A_K^{(\infty)}(\lambda)$ dominates for $\lambda>\frac{1}{2}$, whereas $A_K^{(1)}$ dominates if $\lambda\le\frac{1}{2}$. Finally, the oscillations of the cosine in Eq. (25) have a longer and longer period as K increases. This means that the Borel transform F_B has a singularity on the positive real axis. We believe this is an essential singularity. An argument for this is obtained by considering the related series $f(x)=\sum_K \sin(a\sqrt{K})x^K$ and replacing the sum by an integral [M. Nauenberg (private communication)].[10] This can be reduced to a Gaussian integral whose evaluation gives an essential singularity at $x=1$. For $\lambda<\frac{1}{2}$, $A_K^{(\infty)}(\lambda)$ is exponentially small compared with $A_K^{(1)}(\lambda)$ but the (essential) singularity of the Borel transformation on the positive axis means that the series is not summable for any nonzero λ. In the next section, we give numerical results which strongly support the results in Eqs. (23)–(25).

IV. NUMERICAL RESULTS

As discussed in Appendix B, it is reasonably straightforward to compute the coefficients $A_K(\lambda)$ iteratively from the lower-order terms. However, in practice the final result is a sum of terms each of which is much larger in magnitude than the final answer. Hence, there is a problem with round-off errors in going to very high order. To circumvent this we have done every stage of the calculation exactly by expressing the results as rational fractions. To cope with the enormous integers involved the program was written using MACSYMA. One is then limited by computer time and memory. We have been able to determine up to 200 terms for several choices of λ.

Figure 1 plots the ratio of the exact coefficient $A_K(\lambda)$ to the asymptotic result from the $s=1$ saddle point given by Eq. (23) for several values of λ. Note that for $\lambda=0.5$, 0.4, 0.2, and 0.1 the ratio approaches 1 as expected from the discussion in Sec. III. However, for $\lambda=0.6$ the ratio ap-

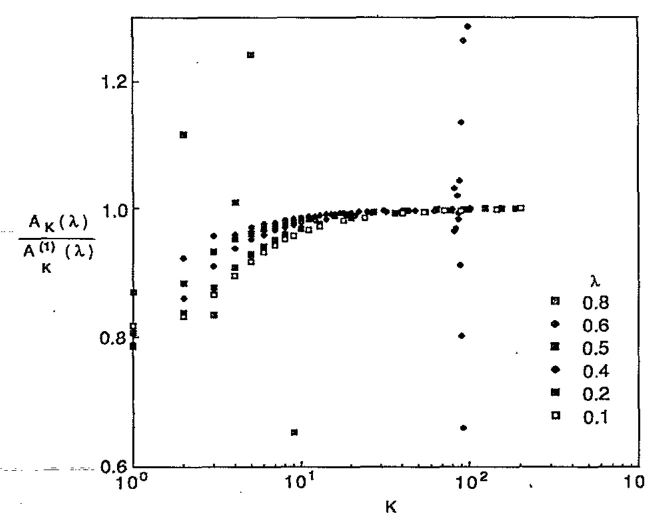

FIG. 1. Plot of the ratio of the Kth term, $A_K(\lambda)$, to the prediction from the $s=1$ saddle point, $A_K^{(1)}(\lambda)$, for several values of λ.

pears to be tending to unity but suddenly deviates after about 80 terms. For $\lambda=0.8$ the ratio never approaches very close to unity and becomes very different for $K \gtrsim 8$. This is precisely the behavior expected from Eqs. (23)–(25), where the ratio should tend to 1 for $\lambda \leq 0.5$. For λ only slightly greater than 0.5 the ratio will be close to unity for moderate values of K, because of the $\exp(-\gamma\sqrt{K})$ term in Eq. (25), but will differ strongly from this value at larger K, where the λ^K factor strongly outweighs the $(1-\lambda)^K$ factor in Eq. (23).

Next we check that our results are of the form in Eq. (25), in the region where the ratio is very different from unity. In Appendix A we give arguments according to which

$$\gamma=\left[\frac{2R}{\lambda}\right]^{1/2}\sin\left[\frac{\theta}{2}\right],\quad \sigma=\frac{\ln 2}{4\lambda}$$

$$\mu=\left[\frac{2R}{\lambda}\right]^{1/2}\cos\left[\frac{\theta}{2}\right],\quad \delta=\frac{3\pi}{4\lambda} \tag{26}$$

where

$$R=[(\ln 2)^2+(3\pi)^2]^{1/2}=9.450\,23\ldots$$

and (27)

$$\theta=\tan^{-1}(3\pi/\ln 2)=1.497\,38\ldots .$$

This is tested in Fig. 2, which shows results for $\lambda=1$. There is a strong similarity between the data and the theory, though the two do not agree precisely, since the data are clearly decreasing in magnitude as well as oscillating. In fact, this can be compensated for simply by a very small change in R to $R=9.607$, as shown in Fig. 3, where now the agreement is excellent. The calculation in Appendix A is rather analogous to that in Ref. 7 for the percolation problem. It was subsequently shown[11] that some of the coefficients in the result of Ref. 7 are slightly in error, although the basic structure is correct, and we suspect the same may be true here, though we have so far

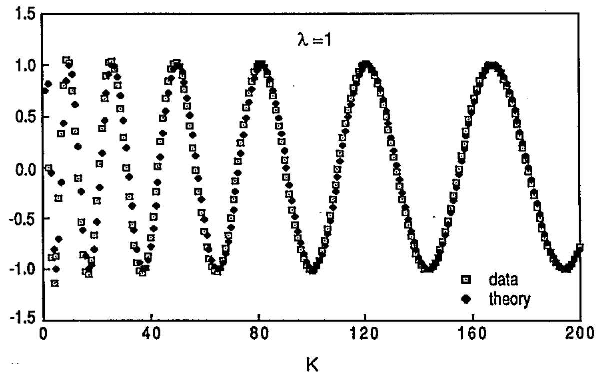

FIG. 3. Same as for Fig. 2 but with the value of R in Eq. (27) changed slightly to 9.607. The agreement is now excellent.

been unable to carry through a calculation like that of Ref. 11 for the present problem. Similar data are shown in Fig. 4 for $\lambda=0.6$ for the choice $R=9.615$, $\theta=1.495$, again very close to the values in Eq. (27).

We feel that the results in Figs. 1–4 give a rather convincing verification of the form of high-order behavior described by Eqs. (23)–(25). It now remains to discuss how accurate an answer for f one can get from the series, given that the series is not Borel summable, so there must be some error even if one could compute all the terms. A technique which is useful for series with simple oscillatory behavior like Eq. (23) is that of a conformal transformation.[4] For Eq. (23) one would introduce a new variable,

$$y[4(1-\lambda)u]=\frac{[1+4(1-\lambda)u]^{1/2}-1}{[1+4(1-\lambda)u]^{1/2}+1}, \tag{28}$$

and reexpress the series for f in terms of y, i.e.,

$$-f=\sum_K K!B_K(\lambda)y^K, \tag{29}$$

where we have explicitly taken out the factor of $K!$. The

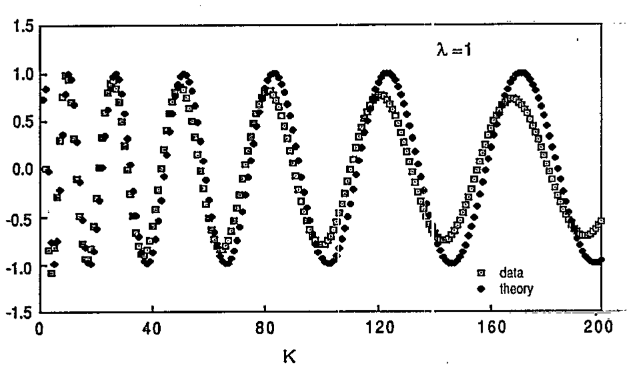

FIG. 2. The diamonds plot $\cos(\mu\sqrt{K}+\delta)$ against K where μ and δ are given in Eqs. (26) and (27). The squares plot $A_K(\lambda=1)/[c\exp(-\gamma\sqrt{K}+\sigma)]$ where $c=-K!(4\lambda)^K/(\pi^{1/2}K^{3/2})$ and γ and σ are given by Eqs. (26) and (27). The two sets of data would agree if the calculations in Appendix A were completely correct.

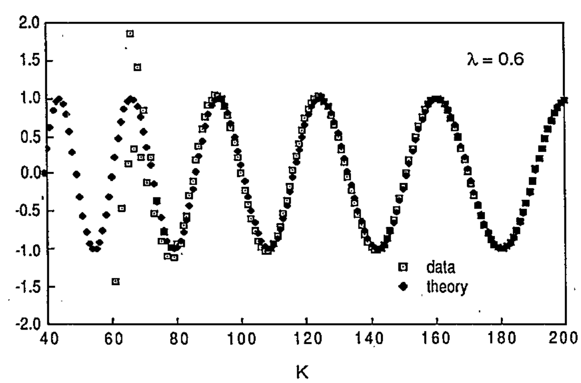

FIG. 4. Similar to Figs. 2 and 3 but the squares plot $[A_K(\lambda=0.6)-A_K^{(1)}(0.6)]/[c\exp(-\gamma\sqrt{K}+\sigma)]$ with $R=9.615$, $\theta=1.495$. The agreement is very good, which shows that where the $\lambda=0.6$ data in Fig. 1 start to deviate from a ratio of unity they then have the behavior in Eq. (25).

motivation for introducing y is that the leading singularity in the Borel transform is at $y=-1$ while the integral in Eq. (4) goes from $y=0$ to $y=1$. Hence it is not necessary to know F_B outside its radius of convergence, unless other singularities come into play. The Borel transform, Eqs. (3) and (4), now take the form

$$-f=\sum_K B_K(\lambda)I_K[4(1-\lambda)u] \ , \tag{30}$$

where

$$I_K(x)=\int_0^{\infty} e^{-t}[y(xt)]^K dt \ . \tag{31}$$

Notice that the $B_K(\lambda)$ are independent of u. The I_K may be evaluated by a saddle-point method with the results

$$I_K(x)=\left[\frac{4\pi}{3}\right]^{1/2}\left[\frac{K^2}{x}\right]^{1/6} \times\exp\left[-3\left[\frac{K^2}{x}\right]^{1/3}+\frac{1}{3x}\right][1+O(K^{-2/3})] \ . \tag{32}$$

Hence provided the B_K grow less fast than $\exp(K^{2/3})$ successive partial sums in Eq. (30) will converge to the correct answer. This is certainly the case for pure systems.[12] However, the Borel transform of the new term, Eq. (25), which appears for random systems, has a singularity on the positive axis at $u=1/(4\lambda)$, which causes the B_K to grow exponentially with K, since the point $u=-1/(4\lambda)$ becomes $y=(1-\sqrt{\lambda})/(1+\sqrt{\lambda})$ in the transformed plane and hence the B_K vary as $[(1+\sqrt{\lambda})/(1-\sqrt{\lambda})]^K$. Presumably, this will be modulated by exponential and cosine factors similar to those in Eq. (25). Figure 5 plots $\log_{10}|B_K|$ against K for $\lambda=0.6$ and $K\le 200$ and clearly shows the expected straight-line behavior. The best fit, also shown, has slope 0.81 close to $(1+\sqrt{\lambda})/(1-\sqrt{\lambda})$, which is equal to 0.79 for $\lambda=0.6$.

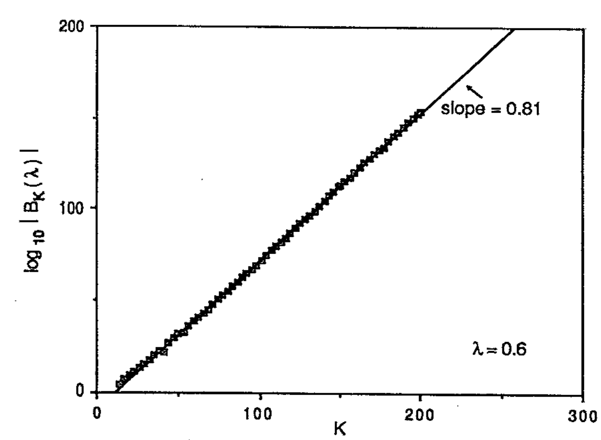

FIG. 5. A plot of the logarithm of the transformed coefficients $B_K(\lambda)$ defined in Eq. (29) against K for $\lambda=0.6$. The best fit to the data has slope 0.81 compared with the expected value of $(1+\sqrt{\lambda})/(1-\sqrt{\lambda})$ which is 0.79.

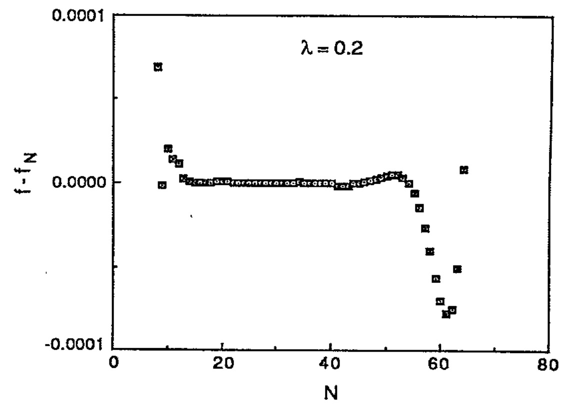

FIG. 6. A plot of the difference between the exact free energy f, and the estimate from N terms in the perturbation expansion, f_N, defined by Eq. (33). The data are for $u=1$ and $\lambda=0.2$.

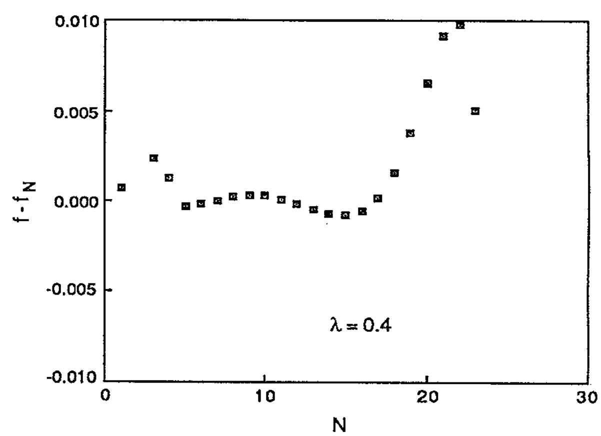

FIG. 7. Same as Fig. 6 but for $\lambda=0.4$.

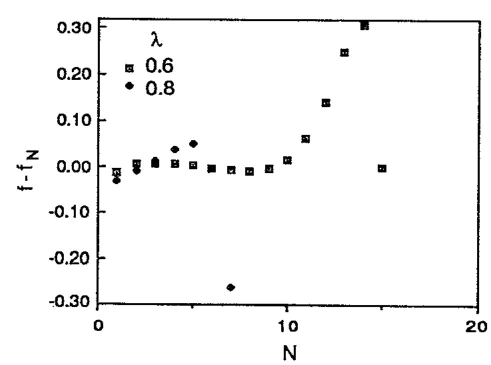

FIG. 8. Same as Fig. 6 but for $\lambda=0.6$ and 0.8.

Defining the sum of the first N terms in Eq. (30) by f_N, i.e.,

$$-f_N=\sum_{K=1}^{N} B_K(\lambda)I_K[4(1-\lambda)u]\ , \tag{33}$$

we show the difference f_N-f for $u=1$ and several values of λ in Figs. 6–8. For small λ, one sees there is a wide range N, where f_N is very close to the exact result but for larger λ no range of N gives a good result.

V. DISCUSSION

We have shown, fairly convincingly, we believe, that the series expansion for the free energy of our toy model of a disordered ferromagnet is not Borel summable. The series exists and the coefficients are finite but a complete knowledge of the series is not enough to obtain the free energy exactly, at least with the resummation techniques known to us. The Borel transform gives a reasonably accurate result for small disorder if one stops the expansion at an appropriate point. Adding more terms gives a worse value. In this respect the Borel transform itself is rather like the original asymptotic expansion.

If the perturbation expansion of the toy model has this rich behavior and is not Borel summable, we suspect that the same will be true of realistic models for disordered ferromagnets. If this is so, then the ϵ expansion would not converge to the exact exponents even if one could compute an infinite number of terms. However, the fixed point values of u and λ in three dimensions do seem to lie in a region where the perturbation theory gives some useful information,[12] and in practice, the chief error probably arises from the shortness of the series rather than lack of Borel summability.

It was pointed out by Griffiths[13] that there are singularities in random systems everywhere below the transition temperature of the corresponding pure system. For static quantities these are weak essential singularities,[14] though they do give rise to stronger effects in dynamics.[15] It seems tempting to relate lack of Borel summability to Griffiths singularities. However, the present model has no Griffiths singularities because it describes only a single degree of freedom, so the connection, if any, is not very clear to us. In our model, lack of summability presumably occurs because there is a finite probability that the coefficient of ϕ^2 in Eq. (6) is negative, in which case the subsequent expansion in powers of u is meaningless. There *is* an essential singularity as $u\to 0$ in the averaged free energy, due to this mechanism. Using the very questionable replica methods of Appendix A, we estimate that the nonperturbative part of the free energy has the form

$$f_{\text{nonpert}}\sim 4\sqrt{\pi\lambda u}\ \exp\left[-\frac{1}{4\lambda u}-\frac{1}{\lambda}\left[\frac{R}{2\lambda}\right]^{1/2}\sin(\theta/2)+\frac{\ln 2}{2\lambda}\right]\sin\left[\frac{1}{\lambda}\left[\frac{R}{2u}\right]^{1/2}\cos(\theta/2)+\frac{3\pi}{2\lambda}\right]\left[1+O\left[\frac{1}{\sqrt{u}}\right]\right]\ , \tag{34}$$

the only part of which we are certain is correct is the leading term $\exp(-1/4\lambda u)$. Essential singularities of this form will ensure the breakdown of Borel summability. For realistic models for disordered ferromagnets (i.e., for dimension >1), the essential singularities are the Griffith singularities. It remains to give a clear analytic derivation of our results, which would probably mean doing away with the replica trick. We leave this for future work.

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APPENDIX A

In this Appendix, the replica trick will be used in an attempt to derive Eq. (25) for $A_K^{(\infty)}(\lambda)$. Unfortunately, our calculation is incomplete and is only included in the hope that it may stimulate others to produce a satisfactory treatment.

Assembling results from Eqs. (16)–(22),

$$A_K(n,\lambda)\equiv n\,A_K(\lambda)=\frac{e^{2(K\ln K-K)}}{K!}K^{-n/2}4^K S\ , \tag{A1}$$

where

$$S=\sum_{s=1}^{n} {}^{n}C_s 2^{s/2}\frac{(-1)^{(s-1)/2}}{(1-\lambda s)^{(1-n)/2}}\left[\lambda-\frac{1}{s}\right]^{k}, \tag{A2}$$

correct to the level of Gaussian fluctuations about each saddle. Note that the contributions to S from even values of s are pure imaginary, so it is not obvious whether $A_K(\lambda)$ derived from Eqs. (A1) and (A2) is real—as, of course, it should be.

As emphasized by McKane,[7] whose approach we shall closely follow, the sum in expressions like Eq. (A2) must first be carried out before setting $n=0$:

$$S\equiv\sum_{s=1}^{n} f(s,n;K)=\oint_c \frac{f(z,n;K)dz}{e^{2\pi i z-1}}\ , \tag{A3}$$

where c is a contour which surrounds the positive integers $1,2,\ldots,n$. Equation (A3) is valid if $f(z,n;K)$ is analytic within the contour c. In our case the function $f(z,n;K)$ has a branch cut at $z=1/\lambda$. For $\lambda>1$ this causes no difficulties. However, for the physically interesting case when $\lambda < 1$, we must modify (A3) by separating out the terms in the sum with $s = 1, 2, \ldots, m$, where m is the smallest integer less than $1/\lambda$. As $K \to \infty$, these terms are dominated by the $s = 1$ term and so this is the only one we retain. The sum over the remaining terms from $s = m+1, \ldots, n$ can be carried out using the contour deformations given by McKane[7] (after one recognizes that the sum can be extended to include $s = n+1, n+2, \ldots, n+m$ as the extra terms give no contribution, since ${}^nC_{n+i} = 0, i > 0$). The result is

$$S \approx S^{(1)} + S^{(\infty)} , \tag{A4}$$

$$S^{(1)} = n 2^{1/2} (\lambda - 1)^K / (1-\lambda)^{1/2} , \tag{A5}$$

$$S^{(\infty)} = -\frac{\pi i n}{2} \int_{\delta-\infty}^{\delta+\infty} \frac{dz}{\sin^2 \pi z} f(z, 0; K) + O(n^2) , \tag{A6}$$

and $1/\lambda < \delta < m+1$. Note that for $\lambda > 1$, there would be no need to separate off the term with $s = 1$ and so $S = S^{(\infty)}$. Expressing ${}^nC_s$ in terms of gamma functions, one finds after some algebra that

$$S^{(\infty)} = \pm n \int_{\delta-\infty}^{\delta+\infty} \frac{dz}{z^{3/2}} \frac{1}{e^{i\pi z} - e^{-i\pi z}} \exp\left[\pm i \frac{\pi}{2} z + \frac{z}{2} \ln 2 + (K - \tfrac{1}{2})[\ln\lambda + \ln(1 - 1/\lambda z)] \right] . \tag{A7}$$

The ambiguity in the signs arise from the fact that $(-1)^{(s-1)/2}$ can be written as $(e^{\pm i\pi})^{(s-1)/2}$. The final result should not depend on the sign choice adopted.

For large K, the integral in Eq. (A7) can be evaluated by steepest descents. Setting $z \to \sqrt{K} z$ and retaining terms up to $O(1)$ gives

$$S^{(\infty)} = \pm n \frac{\lambda^{K-1/2}}{K^{1/4}} \int_{\delta/\sqrt{K}-i\infty}^{\delta/\sqrt{K}+i\infty} \frac{dz}{z^{3/2}} \frac{1}{e^{i\pi\sqrt{K}z} - e^{-i\pi\sqrt{K}z}} \exp\left[\sqrt{K} \left[\pm i \frac{\pi}{2} z + \tfrac{1}{2}\ln 2 - \frac{1}{\lambda z} \right] - \frac{1}{2\lambda^2 z^2} + O\left[\frac{1}{\sqrt{K}} \right] \right] . \tag{A8}$$

There are two saddles, one of which is in the upper half-plane, while the other is in the lower half-plane. Adopting the $+$ signs in (A8) gives

$$S^{(\infty)} \approx S_u + S_l , \tag{A9}$$

$$S_u = \frac{n\sqrt{\pi}\lambda^K}{K^{1/2}} \exp\left[-i \left[\frac{2RK}{\lambda} \right]^{1/2} e^{i\theta/2} + \frac{R}{4\lambda} e^{i\theta} \right] \left[1 + O\left[\frac{1}{\sqrt{K}} \right] \right] , \tag{A10}$$

$$S_l = \frac{n\sqrt{\pi}\lambda^K}{K^{1/2}} \exp\left[-i \left[\frac{2R_l K}{\lambda} \right]^{1/2} e^{-i\theta_l/2} + \frac{R_l}{4\lambda} e^{-i\theta_l} \right] \left[1 + O\left[\frac{1}{\sqrt{K}} \right] \right] , \tag{A11}$$

where

$$R = [(\ln 2)^2 + (3\pi)^2]^{1/2} , \quad R_l = [(\ln 2)^2 + (\pi)^2]^{1/2} , \tag{A12}$$

$$\theta = \tan^{-1}(3\pi/\ln 2) , \quad \theta_l = \tan^{-1}(\pi/\ln 2) . \tag{A13}$$

For large K, S_l is numerically much greater than S_u so the natural approach would be to set $S^{(\infty)} \approx S_l$. There now arises two (probably related) difficulties. $S^{(\infty)}$ according to (A9) is not real. If one makes the *ad hoc* assertion that $S^{(\infty)} = 2\,\mathrm{Re} S_l$, one only obtains poor agreement with the calculated numerical coefficients. However, setting $S^{(\infty)} = 2\,\mathrm{Re} S_u$ gives an approximation for $A_K^{(\infty)}(\lambda)$, of the form of Eq. (25), which is in quite good agreement with the numerical coefficients (see Figs. 2 and 3). We feel that this must be more than a numerical coincidence, but we are at a loss to explain why the nominally dominant contribution from the saddle in the lower half-plane should be discarded, and equally baffled by the fact that our expression for S is not real. We suspect that progress is only likely to be made through a calculation similar to that of Ref. 11.

APPENDIX B

In this appendix we describe how the coefficients in the expansion were generated numerically.

We let $\phi \to (1+\psi)^{-1/2}\phi$ in Eq. (7) so

$$Z(\psi) = \frac{1}{(1+\psi)^{1/2}} \int \frac{d\phi}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\phi^2 - \frac{v}{4}\phi^4 \right] , \tag{B1}$$

where

$$v = u/(1+\psi)^2 . \tag{B2}$$

Expanding the $\exp(-v\phi^4/4)$ factor in Eq. (B1) and doing the integrals one finds

$$Z(\psi)/Z_0(\psi) = 1 + \sum_{n=1}^{\infty} a_n v^n , \tag{B3}$$

where

$$Z_0(\psi) = (1+\psi)^{-1/2} , \tag{B4}$$

and

$$a_n = (-1)^n \frac{(4n-1)!!}{4^n n!} . \tag{B5}$$

In practice, we determined a_n recursively in terms of a_{n-1}, i.e., $a_n = -(4n-1)(4n-3)a_{n-1}/(4n)$. Next we have to take the logarithm of Z and find

$$\ln Z(\psi) - \ln Z(\psi_0) = \sum_{n=1}^{\infty} b_n v^n , \tag{B6}$$

where the b_n are generated recursively in terms of the lower-order b's and the a_n from the formula

$$b_n = a_n - \frac{1}{n}\sum_{m=1}^{n-1} m b_m a_{n-m} , \tag{B7}$$

which is derived by equating the logarithmic derivative of Eq. (B3) to the derivative of Eq. (B6) and multiplying by the denominator.

So far the value of ψ has been kept fixed. We next expand v^n in powers of ψ, i.e.,

$$\begin{aligned} v^n &= u^n/(1+\psi)^{2n} \\ &= u^n \sum_{K=0}^{\infty} (-1)^K \frac{(2n+K-1)!\psi^K}{(2n-1)!K!} \end{aligned} \tag{B8}$$

and average over the ψ with the result that

$$[v^n]_{\text{av}} = u^n \sum_{m=0}^{\infty} w^m \frac{(2n+2m-1)!}{(2n-1)!m!} \tag{B9}$$

We can now put all the results together to find

$$-f = \sum_{n,m} c_{n,m} u^n w^m , \tag{B10}$$

where

$$c_{n,m} = b_m \frac{(2n+2m-1)!}{(2n-1)!m!} \tag{B11}$$

for $n \geq 1$, and $c_{0,m}$ is obtained from the expansion of $\ln Z_0(\psi)$, i.e.,

$$[\ln Z_0(\psi)]_{\text{av}} = \sum_{m=1}^{\infty} c_{0,m} w^m , \tag{B12}$$

so $c_{00}=0$ and

$$c_{0,m} = \frac{(2m)!}{2^{m+2} m m!} \tag{B13}$$

for $m \geq 1$. Writing $w = \lambda u$ Eq. (B10) becomes

$$-f = \sum_{K=1}^{\infty} A_K(\lambda) u^K , \tag{B14}$$

where

$$A_K(\lambda) = \sum_{n=0}^{K} c_{K-n,n} \lambda^n . \tag{B15}$$

The terms in Eq. (B15) alternate in sign and are each generally much bigger than the final answer. Hence round-off errors limit the number of terms one can calculate if one used real arithmetic. To get round this difficulty we note that all the quantities involved in the calculation are rational fractions (as long as λ is rational). We therefore did the calculation by manipulating rational fractions, using MACSYMA to perform the reductions to lowest common denominator and to keep track of the enormous integers generated. Storage and computer time requirements then limited us to 200 terms. Similar manipulations were used to generate the transformed coefficients $B_K(\lambda)$ in Eq. (29) in terms of the $A_K(\lambda)$. This is quite simple since the inverse transformation to Eq. (28) is

$$(1-\lambda)u = \frac{y}{(1-y)^2} , \tag{B16}$$

so u^K can be trivially expanded in powers of y using the binomial expansion.

[1]D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (McGraw-Hill, New York, 1978).

[2]L. N. Lipatov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 116 (1977) [JETP Lett. **25**, 104 (1977)]; *ibid.* **72**, 411 (1977) [*ibid.* **45**, 216 (1977)]; E. Brezin, J. C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D **15**, 1544 (1977).

[3]G. A. Baker, *Essentials of Padé Approximants* (Academic, New York, 1975).

[4]See e.g., J. Zinn-Justin, Phys. Rep. **70**, 109 (1981); J. Zinn-Justin, in *Calculation of Critical Exponents from Field Theory, Proceedings of the 1980 Summer Institute on Phase Transitions, Cargèse, Corsica, France, 1980* (Plenum, New York, 1981).

[5]J. C. le Guillou and J. Zinn-Justin, Phys. Rev. B **21**, 3976 (1980); G. A. Baker, B. G. Nickel, and D. I. Meiron, *ibid.* **17**, 1365 (1978).

[6]A. Houghton, J. S. Reeve, and D. J. Wallace, Phys. Rev. B **17**, 2956 (1978).

[7]A. J. McKane, J. Phys. A **19**, 453 (1986).

[8]T. C. Lubensky, Phys. Rev. B **11**, 3573 (1975).

[9]G. Grinstein and A. H. Luther, Phys. Rev. B **13**, 1329 (1976); V. J. Emery, *ibid.* **11**, 239 (1975); S. F. Edwards and P. W. Anderson, J. Phys. F **6**, 1927 (1975).

[10]M. Nauenberg (private communication).

[11]E. Brézin and A. J. McKane (unpublished).

[12]I. O. Maier and A. I. Sokolov, Fiz. Tverd. Tela (Leningrad) **25**, 3454 (1984) [Sov. Phys.—Solid State **26**, 2076 (1984)]; G. Jug, Phys. Rev. B **27**, 609 (1983).

[13]R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[14]A. B. Harris, Phys. Rev. B **12**, 203 (1975).

[15]M. Randeria, J. P. Sethna, and R. G. Palmer, Phys. Rev. Lett. **54**, 1321 (1985).