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Dynamics of random Ising ferromagnets in the Griffiths phase

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The dynamics of a random Ising ferromagnet is considered, for a general distribution $P(J)$ of exchange interactions, in the Griffiths phase. The autocorrelation function $C(t)$ is found to have the previously derived asymptotic form, $C(t) \sim \exp[-A(\ln t)^{d/(d-1)}]$, where d is the spatial dimension, only if $-\ln[P(J)]$ grows faster than $J^{d/(d-1)}$ as $J \to \infty$. Otherwise, the asymptotic form of $C(t)$ depends explicitly on the form of $P(J)$ at large J.

A number of recent papers[1-6] have dealt with the dynamics of random magnetic systems in the "Griffiths phase," which refers to the temperature regime between the transition temperature for magnetic long-range order in the random system and the highest possible transition temperature allowed in principle by a rare statistical fluctuation of the disorder over the whole system. The latter temperature is called the "Griffiths temperature" T_G. Thus, for a ferromagnet with site or bond dilution, T_G is the critical temperature of the undiluted system; for a ferromagnet with a bounded distribution of exchange interactions, T_G is the critical temperature obtained when all bonds take the maximum value. For a ferromagnet with an unbounded distribution of exchange interactions, the Griffiths phase extends to infinite temperature.

For a random Ising ferromagnet, the result[1,4,5]

$$C(t) \sim \exp[-A(\ln t)^{d/(d-1)}] \tag{1}$$

has been obtained for the asymptotic behavior of the spin autocorrelation function in the Griffiths phase. In Eq. (1), d is the spatial dimension, and A depends on the system parameters (temperature, concentration of missing sites or bonds, etc.). The physics behind Eq. (1) concerns the dominance, as $t \to \infty$, of large regions in which, due to rare statistical fluctuations in the disorder, the exchange interactions have values characteristic of an ordered phase at the given temperature. Because these regions are finite they do relax, but only slowly due to their large size. While Eq. (1) is technically derived as an inequality,[5] the right-hand side being a *lower bound* on $C(t)$, we believe, and will assume hereafter, that it gives the correct asymptotic time dependence, provided the disorder distribution satisfies the condition discussed below.

In this paper, we pay special attention to the case where the distribution $P(J)$ of the exchange interactions is unbounded. We show that, provided that $P(J)$ decreases sufficiently rapidly for large J, i.e., faster than $\exp[-J^{d/(d-1)}]$, Eq. (1) is recovered. If $P(J)$ decreases more slowly, however, Eq. (1) no longer holds, and the asymptotic form of $C(t)$ depends explicitly on the large-J form of $P(J)$. In particular, for the class of distributions $P_n(J) \sim \exp[-(J/J_0)^n]$, we recover Eq. (1) for $n > d/(d-1)$, but obtain $C(t) \sim \exp\{-[(T/2J_0)\ln t]^n\}$ for $n < d/(d-1)$. For example, the simple exponential distribution, $n=1$, belongs (for any d) to the latter class and yields power-law relaxation, $C(t) \sim t^{-(T/2J_0)}$.

The calculations, which follow the method of Refs. 1–6, are most simply presented within a variational framework which we believe, however, gives correct asymptotic forms of $C(t)$. For concreteness we will, in the first instance, consider the distributions

$$P_n(J) = (n/J_0)(J/J_0)^{n-1}\exp[-(J/J_0)^n] , \tag{2}$$

where the algebraic prefactor is merely for computational convenience. More general distributions will be discussed subsequently. A lower bound for $C(t)$ is constructed as follows. Consider a compact region, of linear dimension L, in which all bonds exceed J in value. The probability of a given site belonging to such a region is of order (neglecting algebraic prefactors)

$$p(L) \sim \left[\int_J^\infty dJ' P_n(J')\right]^N = \exp[-L^d(J/J_0)^n] , \tag{3}$$

where N is the number of bonds in the region and we have ignored a constant factor of order unity between N and L^d. This region will relax at least as slowly as an isolated system of size L^d in which all bonds are equal to J. (Any coupling to spins outside the region should not change this result, since the spins outside will typically relax much faster than those inside.) The latter has relaxation time

$$\tau(J) \sim \exp[\sigma(J)L^{d-1}/T] . \tag{4}$$

In the Arrhenius formula (4), the activation barrier $\sigma(J)L^{d-1}$ is simply the free energy of the domain wall that has to be passed through the system in order to reverse its magnetization: $\sigma(J)$ is the surface tension of a pure Ising model with exchange constant J. Obviously Eq. (4) requires $J > J_c(T)$, where $J_c(T)$ is the critical exchange constant for a pure Ising model at temperature T. The best lower bound on $C(t)$ is obtained by maximizing $C(t)$ with respect to both J and L.

Combining (3) and (4) gives, for the mean autocorrelation function,

$$C(t) \geq \max_{J,L} \{\exp[-L^d(J/J_0)^n - te^{-\sigma(J)L^{d-1}/T}]\} . \tag{5}$$

(Note that replacing $\max_L$ by $\sum_L$ would give the same asymptotic behavior, as the sum would be dominated by the largest term for large t.) It is convenient to eliminate L in favor of a new variable $x = \sigma(J)L^{d-1}/T$. Then

$$C(t) \geq \max_{J,x} \{\exp[-A(J)x^{d/(d-1)} - te^{-x}]\} , \tag{6}$$

where

$$A(J)=(J/J_0)^n[T/\sigma(J)]^{d/(d-1)}. \tag{7}$$

Maximizing $C(t)$ with respect to J requires minimizing $A(J)$. To see whether there exists a nontrivial minimum, we consider the limiting behavior for $J\to J_c$ and $J\to\infty$. As $J\to J_c$, $A(J)$ diverges as

$$A(J)\sim(J-J_c)^{-d\nu},$$

where ν is the correlation length exponent of the pure system, and we have used the scaling law $\sigma\propto(J-J_c)^{(d-1)\nu}$. For large J (i.e., $J\gg J_c\sim T$), $\sigma(J)\to 2J$, and

$$A(J)\sim J^{n-d/(d-1)}.$$

Thus if $n>d/(d-1)$, then $A(J)$ is an increasing function of J for large J, as well as for $J\to J_c$, and the absolute minimum of $A(J)$ must occur at some finite value $J^*(T)$. In fact $\sigma=Jf(T/J)$ implies

$$J^*=aT,$$

$$A(J^*)=b(T/J_0)^n,$$

where a,b depend only on n and d.

Returning to (6), with $J=J^*$, and maximizing with respect to x, yields $x^*\simeq\ln t$ to leading order for large t, and

$$C(t)\geq\exp[-b(T/J_0)^n(\ln t)^{d/(d-1)}], \tag{8}$$

retaining only leading order terms in the exponent. Thus, the standard result (1) is recovered for $n>d/(d-1)$. To interpret the above results physically, we note that $x^*=\sigma(L^*)^{d-1}/T\simeq\ln t$ implies that $L^*\sim[(T/\sigma)\ln t]^{1/(d-1)}$, i.e., the *size* of the dominant regions increases with time. Note that the same time dependence is obtained on maximizing with respect to L at any fixed $J>J_c$. Maximizing with respect to J merely optimizes the amplitude A in Eq. (1).

For $n<d/(d-1)$, $A(J)$ tends to zero for large J, and so has no minimum for finite J. To interpret this, we return to (5) and set $\sigma=2J$, since we anticipate that large J will dominate for large t. Eliminating J in favor of $y=(2J/T)L^{d-1}$ yields

$$C(t)\geq\max_{L,y}\{\exp[-(Ty/2J_0)^nL^{d-n(d-1)}-te^{-y}]\}. \tag{9}$$

For $n<d/(d-1)$, it is clear that $L=1$ maximizes $C(t)$, i.e., *isolated strong bonds dominate at long times.* Setting $L=1$ in (9), and maximizing with respect to y, yields, to leading order for large t, $y^*\sim\ln t$ and

$$C(t)\geq\exp[-(T/2J_0)^n(\ln t)^n], \tag{10}$$

where we have once more retained only the leading order term in the exponent. Since $y^*=2J^*/T\simeq\ln t$, the dominant value J^* of J increases with t as $J^*\sim(T/2)\ln t$.

With the realization that isolated strong bonds dominate for $n<d/(d-1)$, we can generalize Eq. (10) for any distribution such that $-\ln P(J)$ grows more slowly than $J^{d/(d-1)}$ for $J\to\infty$. Since the relaxation time for a very strong bond $J\gg T$ is $\tau=\exp(2J/T)$ we obtain

$$C(t)\geq C_1(t)\equiv q\int dJ\,P(J)\exp(-te^{-2J/T}), \tag{11}$$

where q is the (mean) number of nearest neighbors of a site. (The factor q appears because there are $q/2$ bonds per site, and each frozen bond freezes two spins. We assume, of course, nearest-neighbor interactions only.) Although we have written Eq. (11) as an inequality, we expect it to be asymptotically exact for long times.

For large t, asymptotic analysis of the integral may be employed. Three regimes must be considered. (i) If $P(J)$ vanishes faster than a simple exponential, i.e., as $\exp[-(J/J_0)^n]$ with $1<n<d/(d-1)$, conventional steepest descent methods yield

$$C_1(t)\simeq qP(J^*)\left(\frac{\pi TP(J^*)}{|P'(J^*)|}\right)^{1/2}\times\exp[(T/2)P'(J^*)/P(J^*)], \tag{12}$$

$$J^*=\frac{T}{2}\ln\left(\frac{2P(J^*)t}{T|P'(J^*)|}\right)\simeq\frac{T}{2}\ln t, \tag{13}$$

where $P'(J)$ means dP/dJ. (ii) For $P(J)$ vanishing slower than an exponential, but faster than a power law, one obtains

$$C_1(t)\simeq q\left[P\left(\frac{T}{2}\ln t\right)\right]^2\left|P'\left(\frac{T}{2}\ln t\right)\right|^{-1}. \tag{14}$$

(iii) The power-law form $P(J)\sim B/J^{\alpha}$ (with $\alpha>1$) yields

$$C_1(t)\simeq\left(\frac{qB}{(\alpha-1)}\right)\left(\frac{T}{2}\ln t\right)^{-(\alpha-1)}. \tag{15}$$

For classes (i) and (ii), the leading time dependence is captured by

$$C(t)\sim P\left(\frac{T}{2}\ln t\right),$$

since the factors involving $P((T/2)\ln t)/|P'((T/2)\ln t)|$ give only powers of $\ln t$. For the special case of a pure exponential distribution, $P(J)=(1/J_0)\exp(-J/J_0)$, we find $C_1(t)\simeq q(T/2J_0)\Gamma(T/2J_0)t^{-T/2J_0}$, where $\Gamma(x)$ is the gamma function. This result reduces to (12) for $T\gg 2J_0$ and to (14) for $T\ll 2J_0$.

It is interesting that for distributions in classes (ii) and (iii), the decay of $C(t)$ is *slower than power law* in time [much slower for class (iii)]. It seems certain that conventional dynamic scaling,[7] which predicts power-law decay at T_c, $C(t)\sim t^{-2\beta/z\nu}$, where β and z are the order-parameter exponent and dynamical exponent respectively, must break down for these systems, since the decay is already slower than a power law in the Griffiths phase.

All of the above results are specific to Ising ferromagnets, and it might be wondered if similar dependence on $P(J)$ occurs in Heisenberg systems, for which the result $C(t)\sim\exp(-At^{1/2})$ has recently been derived[3,4] by assuming the dominance at long times of large, strongly correlated regions. In this case, however, isolated strong bonds are ineffective in hindering relaxation, since there are no activation barriers. Relaxation occurs instead by diffusion of the magnetization vector, and large correlated regions are essential for slow relaxation.[4]

We conclude by reconsidering the case of dilution disorder, to obtain a tighter lower bound on $C(t)$ than the result obtained earlier by one of us.[4] For dilution disorder, with sites (or bonds) occupied at random with probability p, slow relaxation is due to large, rare regions in which the concentration p' of occupied sites or bonds exceeds locally the critical value $p_c(T)$ necessary for the formation of an ordered phase in the bulk. The analog of Eq. (6) is

$$C(t) \geq \max_{p',x} \{\exp[-A(p')x^{d/(d-1)} - te^{-x}]\} , \quad (16)$$

where

$$A(p') = f(p')[T/\sigma(p')]^{d/(d-1)} ,$$

$$f(p') = p' \ln \left[\frac{p'}{p} \right] + (1-p') \ln \left[\frac{1-p'}{1-p} \right] .$$

Here $\sigma(p')$ is the surface tension of a bulk system with site or bond occupation probability p', and $\exp[-Nf(p')]$ is the probability that, in a region containing N sites and bonds, Np' will be occupied. Equation (16) requires that we minimize $A(p')$ with respect to p'. If the minimum occurs at $p' = p^*$, Eq. (16) yields

$$C(t) \geq \exp[-A(p^*)(\ln t)^{d/(d-1)}]$$

(retaining only the leading order term in the exponent). In Ref. 4 it was stated that $p^* = 1$, i.e., that the dominant contributions for long times come from regions of fully occupied sites or bonds. However, this is by no means clear, since $f(p')$ approaches the limiting value $f(1) = \ln(1/p)$ from below with infinite slope.[8] If $p^* < 1$, the previous results[4] should be modified by replacing $A(1)$ by $A(p^*)$, yielding a better lower bound on $C(t)$. The qualitative conclusions, however, are unchanged.

In summary, the result $C(t) \sim \exp[-A(\ln t)^{d/(d-1)}]$, for the asymptotic decay of the spin autocorrelation function, holds only if $P(J)$ falls off sufficiently rapidly at large J, such that large quasiordered regions dominate the dynamics. Otherwise, the long-time dynamics are dominated by isolated strong bonds, and the result depends explicitly on the large J form of $P(J)$. In the latter regime, $C(t) \sim P((T/2)\ln t)$ gives the leading time dependence for most distributions $P(J)$.

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