ELSEVIER Physica A 263 (1999) 222–233

PHYSICA A

Phase Transitions and Singularities in Random Quantum Systems

Daniel S. Fisher

Physics Department, Harvard University, Cambridge MA 02138

Abstract

Random quantum systems that exhibit unusual behavior associated with "infinite randomness" fixed points are discussed, focusing on the random quantum Ising model. This system undergoes a transition at zero temperature from a phase with infinite susceptibility and continuously variable exponents to a ferromagnetic phase via a quantum critical point characterized by "tunneling scaling" with energy Ω and length scales, L, related by $\ln\Omega \sim L^{\psi}$. Exact results in one dimension and a scaling picture in higher dimensions are derived from a simple renormalization group. Other random quantum critical points and quantum disordered phases that can exhibit similar features are discussed briefly.

Quenched randomness can affect the collective behavior of systems with many degrees of freedom in a wide variety of ways. These can crudely be characterized by how random a system appears as it is looked at on larger and larger length scales or, roughly equivalently, probed at lower and lower frequencies. The simplest—and best understood—are situations in which the system behaves less and less random on larger and larger scales. This occurs in many ordered phases and at some critical points, such as the normal to superfluid transition of a helium film on a disordered substrate. In renormalization group (RG) language, such systems, although random on small scales, are controlled by *pure* fixed points; the randomness is effectively averaged out on large scales. More interesting behavior occurs when the randomness is competitive at all scales; this occurs when a system is controlled by a random fixed point. Such systems, which include spin glasses, critical points of certain random magnets and electron localization, have received a great deal of attention in the past few decades. In striking contrast to pure systems, many of the properties of these systems are dominated by rare spatially localized "active" regions.[1] An extreme limit of this would occur in a system which appeared more and more random on larger and larger length scales; the fixed point in such a case would have, in a sense, *infinitely* strong randomness. The large scale low frequency properties would be characterized by extremely broad distributions of physical properties, such as relaxation times or local susceptibilities. Such broad distributions are known to occur for the *dynamical* properties of many classical random systems. Although these systems are controlled by conventional finite randomness fixed points, the randomness causes a broad distribution of free energy barriers with the times to overcome these depending exponentially on the

0378-4371/99/$ – see front matter
PII: S0378-4371(98)00498-1

barrier height.[1][2][3] When the typical size of the barriers grows with length scale, this gives rise to extremely broad distributions of time scales [4] and to phenomena often used to characterize systems as "glassy": on *any* macroscopic time scale, almost all processes are either much faster—and hence in local equilibrium—or much slower and hence far from equilibrium, with only a few processes—consisting of a low density of active regions—evolving on the time scale being probed.

In quantum mechanics, energetics and dynamics are inextricably linked. Thus it should not be surprising that an extreme separation of *energy* scales that is characteristic of infinite randomness fixed points can occur due to the competition between randomness and quantum fluctuations. This paper will briefly review some of the recent progress in understanding simple random quantum mechanical systems whose properties are controlled by such "infinite randomness" fixed points.

We will focus initially on the simplest of all random quantum systems which can exhibit a phase transition: the random quantum ferromagnetic Ising model with Hamiltonian

$$\mathbb{H} = -\sum_{(ij)} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h_i\sigma_i^x - H\sum_i \mu_i\sigma_i^z \tag{1}$$

where the $\{J_{ij}\}$ are random positive interactions, the random transverse fields $\{h_i\}$ cause the quantum fluctuations, and H is an ordering field that couples to the (positive) magnetic moments $\{\mu_i\}$; H will be set to zero unless otherwise specified. At temperature T, such a quantum system in d-dimensions is equivalent to a $d+1$ dimensional classical Ising model with the couplings independent of the "time" direction in which it has extent $\frac{1}{T}$. The pure system at zero temperature thus has, in any dimension, a quantum phase transition as h is decreased which is in the universality class of the $d+1$ dimensional classical Ising model. We are interested in the low and zero temperature behavior of the random system—both the nature of the zero-temperature phases and the ordering transition. Although zero-temperature quantum critical points can, of course, not be reached, they typically control behavior over a substantial range of temperature and parameter space.

Some insight into possible low T behavior can be gleaned by considering a toy problem: a one dimensional system in which all the $\{h_i\}$ are equal and the nearest-neighbor J_{ij} are equal to J with probability p or equal to 0 with probability $1-p$. This breaks the system into segments of length L so it clearly has no phase transition. But the low T behavior is nevertheless interesting: if $h < J$, the pure system would be ordered so that long segments are "trying" to order and can thus have a large susceptibility. This can be simply estimated: the ground state of a segment is the symmetric combination of the up and down ferromagnetic states—essentially corresponding to the collective spin of the segment pointing in the $+x$ direction. The gap to the lowest excited state—the antisymmetric combination of up and down or, equivalently, the segment spin in the $-x$ direction—is $\tilde{h}(L) \sim J\left(\frac{h}{J}\right)^L$ which acts like an effective transverse field on the segment. The segment thus acts like a free spin with magnetic moment L and Curie susceptibility for temperatures in the range $\tilde{h}(L) << T << J$, but is non-magnetic for $T << \tilde{h}(L)$. Since the density of segments of length L is p^L, the density of "active" segments at temperature

T is $p^{\ln(\frac{J}{T})/\ln(\frac{J}{h})}$ yielding a susceptibility

$$\chi \sim \frac{1}{T^{\gamma}} \tag{2}$$

(ignoring $\ln T$ factors) with

$$\gamma = 1 - \ln(1/p)\,/\ln(J/h) \tag{3}$$

i.e. a continuously variable power law! The specific heat is similarly singular at low temperatures.

But, one would think, a system of decoupled segments is obviously a pathological special case of Eq. (1). Or is it? In fact, similar behavior of the susceptibility - indeed even its origin is terms of (almost) decoupled clusters - is the generic low temperature nature of the random quantum Ising model on the disordered side of the quantum critical point. [5][6] Somehow, the system renormalizes at low energies towards a decoupled system characterized by extremely broad distributions of effective couplings and effective transverse fields.

This behavior, as well as that *at* the quantum critical point and in the ordered phase can be understood via a simple renormalization group (RG) that is a generalization [5][6] of one introduced by Ma, Dasgupta and Hu. [7] Our discussion here follows that of Mau et al.[6] As we are interested in physics characteristic of broad distributions of couplings, we first focus on the low energy behavior in this limit and later consider the regime of validity. The basic strategy is to find the strongest coupling of the $\{J_{ij}\}$ and $\{h_i\}$ and minimize the corresponding term in the Hamiltonian. The degrees of freedom associated with this maximum energy scale, which we denote Ω_0, are then frozen at lower energy scales. If the strongest coupling is a field, say h_k, then the spin σ_k is put in its local ground state, i.e. in the x-direction, causing it to become non-magnetic. Effective interactions are then generated between neighboring spins of k; but, as all other nearby couplings are likely to be much smaller than h_k, these can be treated by second order perturbation theory. This yields new interactions

$$\tilde{J}_{ij} \approx J_{ij} + J_{ik}J_{kj}/h_k \approx \max(J_{ij}, J_{ik}J_{kj}/h_k) \tag{4}$$

where we can use the maximum rather than the sum since one of the two terms is likely to be much bigger than the other. If on the other hand, the strongest coupling is an interaction, say J_{kl}, then the two spins are combined together into a cluster which has two ground states (both up or both down) and can thus be represented again by a spin but now with a magnetic moment $\mu_{(kl)} = \mu_k + \mu_l$, the sum of the magnetic moments of the clusters—initially single spins—of which it is made. The effective field on the cluster (kl) is

$$\tilde{h}_{(kl)} \approx h_k h_l / J_{kl} \tag{5}$$

and the interactions of other clusters with the new one are

$$\tilde{J}_{i(kl)} \approx \max(J_{ik}, J_{il}). \tag{6}$$

Since all the new couplings are smaller than the initial Ω_0, we have decreased the energy scale to a smaller maximum energy, Ω. The process is now iterated using, at each stage, the effective couplings at scale Ω. Note that the decimations change the structure of the "lattice", so that we should consider the spins to be the vertices of a somewhat random graph with the RG modifying the spatial structure.

In the quantum disordered phase, under renormalization the fields eventually tend to dominate the bonds and at small Ω almost all the decimations are cluster annihilations with the effective interactions connecting them becoming weaker and weaker; the system thus renormalizes to a collection of asymptotically uncoupled clusters with a broad distribution of effective fields—directly analogous to the gaps $\tilde{h}(L)$ of segments in our toy example. In the ordered phase, in contrast, the interactions tend to dominate the fields at low energies, and most decimations are thus of bonds; eventually this causes an *infinite cluster* to form. The zero temperature quantum transition between these phases is thus a novel kind of percolation with the annihilation and aggregation of clusters competing at all energies at the critical point. But this competition is very delicate, indeed, locally it is barely evident: at low energies, each cluster either has a field much bigger than its interactions with other clusters, or has one strongly dominant interaction. The other couplings are just enough to prevent the system from decoupling at any scale.

This cluster RG procedure is clearly approximate, but it has the potential of becoming better and better if the distributions of the effective couplings broaden without bound as the energy scale, Ω, is lowered. As we shall see, this is exactly what happens at the quantum critical point so that the cluster RG yields asymptotically exact results in the critical regime, i.e. at low energies at and near the critical point. In two or three dimensions, the RG can be carried out numerically [6] starting with only short-range interactions $\{J_{ij}\}$, but in one dimension a special property enables direct analytical progress [5]: the effective couplings remain *independent* under renormalization. Insight gleaned from the 1-d results, combined with the general structure of the RG, yield a picture that is consistent with the recent numerical RG results of Mau et al [6] as well as recent Monte Carlo simulations. [18][19]

We focus first on the critical point. At each stage of the RG, an effective field, say $\tilde{h}_i$, is a product of some number, f_i, of fields on a set (not necessarily distinct) of original sites, divided by $f_i - 1$ original interactions which connect these sites. The typical f_i clearly grows under renormalization at criticality, thus, although these sets of original couplings are *not* independent (even in 1-d), one would guess that the width of the distribution of $\ln \tilde{h}_i$ will diverge as $\Omega \to 0$: indeed, at the critical point, the distribution of $\ln(\frac{\Omega_0}{\tilde{h}_i})/\ln(\frac{\Omega_0}{\Omega})$ and likewise $\ln(\frac{\Omega_0}{\tilde{J}_{ij}}/\ln(\frac{\Omega_0}{\Omega})$ tend to limiting fixed point distributions. [Note that Ω_0 simply sets the basic energy scale.] The asymptotic exactness then follows immediately: all other couplings near to an about-to-be-decimated coupling will almost surely be much weaker than Ω as $\Omega \to 0$.

The typical diameter, L, of a cluster at scale Ω, scales as a power of f_i and hence of $\ln(\frac{\Omega_0}{\Omega})$. Thus, in contrast to conventional classical and quantum critical points at which typical frequencies (or energies) scale as $\Omega \sim L^{-z}$, here the scaling is very different:

$$\ln\left(\frac{\Omega_0}{\Omega}\right) \sim L^{\psi} \tag{7}$$

with some exponent ψ. This type of scaling, which arises from the necessity of virtual excitations to high energies to flip clusters, we call "tunneling scaling".

The magnetic properties of a cluster are dominated by the μ_i active spins in the cluster—i.e. those which have not yet been decimated; typically

$$\mu \sim \left[\ln \frac{\Omega_0}{\Omega}\right]^{\phi}. \tag{8}$$

with some exponent ϕ. Spins which are active at *some* scale in the same cluster will coherently flip together as the cluster spin flips and hence be almost fully correlated. The spin correlation function $C_{ij} \equiv < \sigma_i^z \sigma_j^z >$ is thus of order one if i and j are in the same cluster at some Ω. These rare pairs dominate the average correlation function yielding, at criticality

$$\overline{C_{ij}} \sim \frac{1}{|\mathbf{r_i} - \mathbf{r_j}|^{\eta}} \tag{9}$$

with

$$\eta = 2(d - \phi\psi). \tag{10}$$

Although the average correlations thus appear similar to those at conventional random critical points—such as a classical three dimensional random exchange Ising ferromagnet—the physics is quite different: in conventional random systems, the pairs of spins which dominate average critical correlations are *themselves* weakly correlated, and the correlations between a typical pair of spins also falls off as a power of distance—albeit a larger power than the average correlations. [8] By contrast, at the random quantum critical point, the typical critical correlations between widely separated spins are very weak and broadly distributed: typically

$$-\ln C_{ij} \sim \kappa_{ij} |\mathbf{r_i} - \mathbf{r_j}|^{\psi} \tag{11}$$

with a random order one coefficient, κ_{ij}. These correlations arise from perturbative effects - tunneling processes - ignored in the simple RG.

Deviations from the quantum critical point are of three types: *Thermal fluctuations* can be understood by simply stopping the RG when $\Omega \approx T$. Almost all decimated processes then have energies much larger than T and are hence frozen, while almost all remaining processes involve couplings much less than T and thus have negligible effect on the equilibrium properties. The clusters at $\Omega = T$ are thus virtually independent spins with moments $\{\mu_i\}$. The susceptibility is then

$$\chi \sim \frac{\left[\ln\left(\frac{\Omega_0}{T}\right)\right]^{2\phi - \frac{d}{\psi}}}{T}. \tag{12}$$

The effects of an *ordering magnetic field*, H, can be handled similarly by renormalizing until H times a typical moment, μ, is of order Ω. The decimated clusters are non-magnetic, while the remaining clusters will almost all be fully polarized by the field. Thus the magnetization at the quantum critical point will be

$$M \sim \frac{1}{\left[\ln\left(\frac{\Omega_0}{H}\right)\right]^{\frac{d}{\psi}-\phi}} \tag{13}$$

in contrast to the power law scaling between M and H at conventional critical points.

Finally, and most interesting, are deviations from criticality at $H = T = 0$. If the system is pushed slightly off critical—by, say, increasing all the fields $\{h_i\}$ by δ, the RG flows will move significantly away from the critical fixed point when $\delta\left[\ln\left(\frac{\Omega_0}{\Omega}\right)\right]^{\lambda}$ is of order one. This yields a correlation length

$$\xi \sim |\delta|^{-\nu} \tag{14}$$

with $\nu = \frac{1}{\lambda\psi}$ determined by the RG eigenvalue, λ; this is conjectured to be related to the scaling of the typical number of original fields, f_i, involved in a cluster effective field. Thus, as at conventional critical points, we have three basic exponents relating magnetization (usually η/ν, here $\phi\psi$), deviations from critical (ν) and frequency (usually z, here ψ) to length scale. Alternatively, in terms of a percolation picture, the active spins in a cluster can be considered as constituting a fractal set with fractal dimension $d_f = \phi\psi$.

The main difference between the random quantum Ising critical behavior and conventional critical points arises from the logarithmic connection between frequency or energy and length scale Eq. (7), parametrized by ψ. The consequences of this for the near critical behavior are rather striking.

In the zero-temperature *quantum disordered phase* with small positive δ, clusters much larger than the correlation length are exponentially rare. Nevertheless, pairs of spins active at some scale in such anomalously rare clusters will dominate the *average correlations* yielding

$$\overline{C_{ij}} \sim e^{-|\mathbf{r_i}-\mathbf{r_j}|/\xi} \tag{15}$$

at long distances (with subdominant prefactors—see 1-d result below). The *typical correlations* - indeed those between almost all pairs of spins - fall off as a more rapid exponential with

$$\frac{-\ln C_{ij}}{|\mathbf{r_i}-\mathbf{r_j}|} \to \frac{1}{\xi_{typ}} \tag{16}$$

with probability one as $|\mathbf{r_i}-\mathbf{r_j}| \to \infty$. In conventional random systems ξ_{typ} is a fixed fraction of ξ for small δ. But here, the two correlation lengths are very different with

$$\xi_{typ} \sim |\delta|^{-(1-\psi)\nu} \sim \xi^{1-\psi} << \xi \tag{17}$$

so that ψ is a measure of the deviation from conventional scaling. [1] Because of the broad distribution of effective fields inherited from the higher energy critical fluctuations, there is

[1] Note that it is ξ not ξ_{typ}, whose exponent ν can be proven to satisfy $\nu \geq \frac{2}{d}$; indeed $\nu_{typ} = \nu(1-\psi)$ is less than $\frac{2}{d}$ in one dimension; see [9].

still a broad—although finite width—distribution of energy scales in the weakly disordered phase. This yields a scaling between energy, Ω, and length scale, L—the distance between surviving clusters at scale Ω—of

$$\Omega \sim L^{-z} \tag{18}$$

with z a continuous variable exponent which diverges as $\delta \to 0$ as

$$z \sim \delta^{-\psi\nu}. \tag{19}$$

Concommitantly, exactly as occurs in the toy finite-segment model, the $T = 0$ magnetization scales as

$$M \sim H^{\frac{1}{z}} \tag{20}$$

and the low T susceptibility as $\chi(T) \sim T^{\frac{1}{z}-1}$ (times powers of δ and $\ln H$ or $\ln T$). The weakly disordered phase thus behaves like a line of zero temperature critical points! This is an extreme manifestation, due to the quantum mechanics, of the weak thermodynamic Griffiths singularities [10][11] and the stronger dynamical effects [3] in classical random paramagnets that are caused by rare strongly coupled regions. As the system is taken further away from the critical point, at some δ the susceptibility will become finite [with the analytic χH part then dominating $M(H)$]; but this is not a particularly special point: it occurs even in the toy random-segment model.

In the zero temperature *ordered phase* for small negative δ, the renormalization can be continued past the crossover scale, $\xi \sim |\delta|^{-\nu}$, down to zero energy. What remains will be a single infinite cluster of active spins—fractal on scales smaller than ξ but uniform on larger scales. These active spins have polarization of order one and constitute the spontaneous magnetization, M_0, which scales as

$$M_0 \sim (-\delta)^{\beta} \tag{21}$$

with

$$\beta = \nu(d - \phi\psi). \tag{22}$$

In one dimension, the infinite cluster only forms at zero energy; at any positive T it will fall apart into finite clusters since there can be no long range order at finite temperature. But for $d > 1$, the infinite cluster will form at a non-zero energy scale Ω_∞ (although it will continue to grow by agglomeration of finite clusters for $\Omega < \Omega_\infty$)[6]. Only at temperatures $T > \Omega_\infty$ will the infinite cluster fall apart and the system be paramagnetic. For temperatures below Ω_∞, the infinite cluster is held together by interactions that are substantially stronger than T and the spontaneous magnetization will survive. Thus for small δ, the ferromagnetic transition occurs at

$$T_c \sim \Omega_\infty \underset{\cdot}{\sim} e^{-K(-\delta)^{-\nu\psi}} \tag{23}$$

with some (non-universal) constant K.

So far, we have discussed a heuristic picture of the behavior of strongly random quantum Ising ferromagnets based on a simple, but initially approximate, RG. Several questions immediately arise: can the consequences of the RG be demonstrated more convincingly? Can the claim of asymptotic exactness in the quantum critical regime be firmly established? How general are the results? for weakly random Ising ferromagnets? for other random quantum critical points? and, qualitatively, for other random quantum phases? We will briefly address each of these in turn.

For the one dimensional random quantum Ising model the approximate RG, as mentioned earlier, has a great simplifying feature: if the couplings are initially independent, the renormalized effective couplings will also be independent—although the moments and lengths of clusters will be correlated with their effective fields, an important feature of the physics and of the analysis.[5] The RG flows then become non-linear integro-differential equations for two distributions: that of the cluster variables and that of the interaction variables.

By the vagaries of fortune, these turn out to have a family of exact solutions which contain both the critical behavior and the full crossovers from critical to the weakly ordered and weakly disordered phases for small δ. A great number of exact results can thereby be obtained in the scaling limit of low energies and long distances near the quantum critical point, *even* in the presence of a small ordering field, H. The structure discussed above is found to obtain with the exponents

$$\psi = \frac{1}{2}, \quad \phi = \frac{\sqrt{5}+1}{2}, \quad \text{and } \nu = 2. \tag{24}$$

Remarkably, the zero temperature scaling function for the magnetization as a function of H can be found in closed form: for $|\delta|$ and $\frac{1}{\ln(\Omega_0/H)}$ small[2]. The scaling variable is $\gamma = \delta \ln(\Omega_0/H)$ in terms of which

$$M \sim \left[\ln\left(\frac{\Omega_0}{H}\right)\right]^{\phi-2} \left[\frac{\gamma^2\alpha}{\sinh^2\gamma} + \frac{e^{-\gamma}}{\sinh\gamma}\left(\phi\gamma\alpha + \gamma^2\frac{d\alpha}{d\gamma}\right)\right] \tag{25}$$

with $\alpha(\gamma) \equiv |\gamma|^{\phi} Q_{\phi-1}(\coth\gamma)$ a smooth even function of γ and Q_μ the Legendre function of order μ. The small H limits in the ordered and disordered phases can be found and are of the form discussed above [e.g. Eq. (20)]—but with the appropriate $\ln H$ factors. It is interesting to note that the analogous scaling function for M as a function of $\frac{T-T_c}{H^{1+\gamma/\beta}}$ is not known for the *pure* $2-d$ (or equivalently quantum 1-d) Ising model! The fact that *more* exact results can be obtained in the random than in the non-random system is surprising; it is a consequence of the extreme separation of energy scales near the critical point.

Computation of the average correlation function can be reduced to analysis of a linear ordinary differential equation from which the asymptotic form in the scaling limit $\xi \to \infty$, $r >> \xi$, can be derived. In the disordered phase,

$$\overline{C_{or}} \sim \frac{\delta^{7/3-2\phi}}{r^{5/6}} \exp\left[-\frac{r}{\xi} - \frac{3}{2}\left(\frac{2\pi^2 r}{\xi}\right)^{\frac{1}{3}}\right] \tag{26}$$

[2] The basic energy scale Ω_0 is just a correction to scaling

quite different from the usual Ornstein-Zernicke result.

These results, and others, are asymptotically exact consequences of the cluster RG, but are they exact results for the actual 1-d random quantum Ising model? While there is no proof at this point, the evidence overwhelmingly favors the affirmative answer. Many properties of the 1-d random quantum model were computed exactly almost thirty years ago by McCoy and Wu [11][12] [3]. Indeed, the existence of a phase with M scaling as a continuously varying power of H is a consequence of this work. All quantities which can be computed by both the cluster RG and by these or other exact [13][14][29] methods agree and numerical computations [15][29][16] provide further support. While there is no rigorous computation of the interesting exponent ϕ, the RG approximations involved in obtaining it are very similar to those involved in obtaining the results that do agree with rigorous methods. We thus believe that, like the Kosterlitz-Thouless predictions of exact exponents from an approximate RG, the predicted $\phi = \frac{1+\sqrt{5}}{2}$ should be exact.

For dimensions greater than one, there are neither exact solutions nor exact quantitative predictions—other than scaling laws—from the cluster RG. But the RG can be implemented numerically to yield arbitrarily accurate, at least in principle, exponents and other quantities. This has recently been carried out in two and three dimensions by Mau, Motrurich and Huse [6]; in both cases at the critical point the system renormalizes at low energies to a fixed point with infinitely broad distributions of couplings scaling in the manner described above. [4]

For $d < 4$, weak randomness is a *relevant* perturbation at the pure quantum critical point (by the Harris criterion [17]) so we expect that—as in 1-d—the critical behavior will be controlled by the infinite randomness fixed point for *any* amount of randomness. [5] In two dimensions preliminary estimates [6] yield $\psi \sim 0.2$ - less than in 1-d but clearly non-zero - and $\phi \sim 4$. Recent Monte Carlo simulations with moderate randomness in two dimensions [18][19] are consistent with the general picture discussed above although they suggest a somewhat larger, but probably not inconsistent, value of ψ [18]. Pich and Young [18] find that at the critical point typical correlations decay as Eq. (11) and energy scales as Eq. (7). The *average* correlations at criticality, in contrast, decay as a power law with exponent $\eta \approx 2.0$. The data are substantially less consistent with conventional scaling although great care must be taken. [31] Rieger and Kawashima [19] [6] studied the behavior in the disordered phase obtaining a dynamic exponent z varying continuously from 2 to 10, consistent with diverging at the critical point in the expected manner. Analytic progress in two dimensions would seem unlikely except for one observation:

[3] Actually McCoy and Wu studied the closely related classical anisotropically random 2-d Ising model—named for them.

[4] It is important to note that the approximate cluster RG contains the seeds of its own potential failure: if instead of broadening indefinitely, the widths of distributions of the $\ln \tilde{h}$ or $\ln \tilde{J}$ had *saturated* (or narrowed) at the critical point, this would have indicated a failure of the approximation and, presumably, conventional $\Omega \sim L^{-z}$ scaling.

[5] The behavior for $d \geq 4$ is less clear; weak randomness is naively irrelevant but rare regions may, nevertheless, always drive the system to strong effective randomness.

[6] The para-ferro transition temperature found in [19] appears to decrease less rapidly than expected from Eq. (23), but this maybe a result of errors in the location of the quantum critical point.

since the extreme-randomness-dominated critical behavior is a type of *two dimensional* percolation, conformal field theoretic methods might conceivably be useful.

At this stage, it appears that the scaling scenario discussed in this paper is indeed correct for random quantum Ising ferro-magnets; certainly in one dimension, very likely in two dimensions and probably also in three dimensions. But what other random quantum systems will exhibit similar critical behavior?

It has been shown that in one dimension all random quantum Potts models are in the same universality class as the Ising ($q = 2$) case, with the q–state degrees of freedom riding on top of the underlying percolation structure.[20] This should also apply in higher dimensions at least for strong randomness. [7] [30]

The critical points of quantum Ising *spin glasses* will also be in the same universality class as ferromagnets: at low energies, the strongest couplings dominate at each scale and the frustration is irrelevant.[6] A uniform z-field, H, will couple differently, however, and of course the ordered phase will, except in 1-d, be very different from the ferromagnetic case. Numerical simulations of two and three dimensional quantum spin glasses found what appears to be more conventional scaling: with a range of $\Omega \sim L^{-z}$ scaling in the disordered phase but z *saturating* at the critical point. [27][26] These may suffer, however, from the same problems of analysis as ref. [31]

In one dimension there have been several other quantum transitions studied with qualitatively similar critical scaling, including the ordering transition in a random Ising antiferromagnet with the total z-magnetization conserved—i.e. a random XXZ model,[21] and the transition in a random spin -1 antiferromagnetic chain from the topologically ordered Haldane valence bond state to a disordered phase.[22] In both of these random antiferromagnetic systems, the disordered phase is a "random singlet phase" first analyzed by Ma, Dasgupte and Hu [7] using an RG which motivated that used here. In this random-singlet phase, most of the spins are paired in singlets at low temperatures but a small density remain which will be paired into even more weakly bound singlets—via virtual excitations of the intervening pairs—as the temperature is lowered. The scaling of energy and length is again logarithmic tunneling scaling, Eq. (7), with $\psi = \frac{1}{2}$.[21] Although qualitatively similar behavior occurs in higher dimensional random antiferromagnets over a range of temperatures,[25] it probably does not persist to low energies; the infinite randomness fixed point appears to be unstable [23] and the system will form a different type of state.[24]

One of the challenging open questions is whether *phases* with tunneling scaling can occur in random quantum systems in dimensions greater than one. But whether or not this occurs, it is clear that disordered random quantum phases can exhibit, because of proximity to random quantum critical points, strange behavior such as divergent susceptibilities and continuously variable energy versus length exponents.[26][27] This is strikingly different from disordered phases in pure or random classical systems.

In what experimental systems the kind of phenomena we have discussed here might be found is largely an open question. But in addition to the kind of random magnets we have discussed, there are potentially many other possibilities, including disordered heavy

[7] Weak randomness in 3-d should result in a first order transition for Potts models as is the case for the pure systems

fermion alloys [32], hydrogen-bonding solids, helium in porous media and on irregular substrates, metal insulator transitions[33], etc. which might exhibit some or all of the unusual features caused by randomness and quantum mechanics.

As a final note, it is worth mentioning that the one-dimensional random quantum Ising ferromagnet discussed in this paper is essentially the only random system on a realistic lattice that can undergo a phase transition—classical or quantum—for which many exact analytic results can be obtained. As such, it is a good testing ground for many of the more general ideas about random systems—especially the crucial role played by localized rare regions—developed by David Huse, this author and others.[28]

I would like to thank my collaborators on parts of this work—Siun-Chuon Mau, Olexei Motrunich and especially David Huse and Peter Young for their substantial contributions to my understanding of this subject. This work was supported in part by the National Science Foundation via grant DMR 9630064, Harvard University's MRSEC, and DMS 9304586.

References

[1] Such phenomena in spin glass ordered phases are discussed in D.S. Fisher and D.A. Huse, Phys. Rev B**38**, 386 (1988).

[2] Broad distributions of barriers are found at critical points in classical magnets with a random ordering field, see D.S. Fisher, Phys. Rev. Lett. **56**, 416 (1986) and J. Villain, J. Physique, **46**, 1843 (1985).

[3] In disordered phases, see e.g. M. Randeria, J.P. Sethna and R.G. Palmer, Phys. Rev. Lett. **54**, 1321 (1985).

[4] D.S. Fisher, J. Appl. Phys.**61**, 3672 (1987).

[5] D.S. Fisher, Phys. Rev. B**51**, 6411 (1995).

[6] S-C Mau, O. Motrunich, D.A. Huse and D.S. Fisher, in preparation

[7] S.K. Ma, C. Dasgupta and C.-K. Hu, Phys. Rev. Lett. **43**, 1434 (1979); C. Dasgupta and S.K. Ma, Phys. Rev. B**22**, 1305 (1980).

[8] A good discussion of this behavior at classical random critical points is in A.W.W. Ludwig, Nuc. Phys. B**330**, 639 (1990).

[9] J.T. Chayes, L. Chayes, D.S. Fisher, T. Spencer, Comm. Math. Phys. **120**, 501 (1989).

[10] R.B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).

[11] B.M. McCoy, Phys. Rev. B**188**, 1014 (1969).

[12] B. M. McCoy and T. T. Wu, Phys. Rev. **176**, 631 (1968); **188**, 982 (1969).

[13] R.H. McKenzie, Phys. Rev. Lett. **77**, 4804 (1996).

[14] R. Shankar and G. Murthy, Phys. Rev. B**36**, 536 (1987).

[15] A.P. Young and H. Rieger, Phys. Rev. B**53**, 8486 (1996).

[16] D.S. Fisher and A.P. Young, Phys. Rev. B, **58**, 9131 (1998).

[17] A. B. Harris, J. Phys. C **7**, 1671 (1974).

[18] C. Pich and A. P. Young, cond-mat/9802108.

[19] H. Rieger and N. Kawashima, cond-mat/9802104.

[20] T. Senthil and S.N. Majumdar, Phys. Rev. Lett. **76**, 3001 (1996)

[21] D.S. Fisher, Rev. B**50**, 3799 (1994) and references therein.

[22] C. Monthus, O. Golinelli and Th. Jolicoeur, Phys. Rev. Lett. **79**, 3254 (1997); R. A. Hyman and K. Yang, Phys. Rev. Lett. **78**, 1783 (1997).

[23] D.A. Huse, private communication.

[24] A different type of disordered random quantum phase was found in one dimension by: E. Westerberg, A. Furusaki, M. Sigrist and P.A. Lee, Phys. Rev. Lett. **75**, 4302 (1982).

[25] R.N. Bhatt and P.A. Lee, Phys. Rev. Lett. **48**, 344 (1982).

[26] H. Rieger and A. P. Young, Phys. Rev. Lett. **72**, 4141 (1994).

[27] M. Guo, R. N. Bhatt and D. A. Huse, Phys. Rev. Lett. **72**, 4137 (1994).

[28] For an introduction, see, e.g. D.S. Fisher, in Proceedings of NATO Advanced Study Institute *Phase Transitions and Relaxation in Systems with Competing Energy Scales*, T. Riste and D. Sherrington, eds., Kluwer (Amsterdam 1993) and references therein.

[29] F. Iglói and H. Rieger, Phys. Rev. B**57**, 11404 (1998) and Phys. Rev. Lett. **78**, 2473 (1997).

[30] T. Senthil and S. Sachdev, Phys. Rev. Lett. **77**, 5292 (1996) showed that infinite randomness critical points also exist in several rather special circumstances in dimensions $d > 1$.

[31] A. Crisanti and H. Rieger, [J. Stat. Phys. **77**, 1087 (1994)] found apparently conventional $\Omega \sim L^{-z}$ scaling from simulations of the 1-d random quantum Ising model but larger systems show tunneling scaling. [15, 17, 29]

[32] See e.g., A.H. Castro-Neto, G. Castillo and B. Jones, cond-mat/9710123.

[33] D. Belitz and T. Kirkpatrick, Phys. Rev. B**52**, 13922 (1995).