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et RAYMOND STORA

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MATHEMATICAL ASPECTS OF THE PHYSICS OF DISORDERED SYSTEMS*

Jürg FRÖHLICH

Theoretical Physics, ETH-Hönggerberg
CH-8093 Zürich, Switzerland

Lecture 5 in collaboration with

A. BOVIER and U. GLAUS

Theoretical Physics, ETH-Hönggerberg
CH-8093 Zürich, Switzerland

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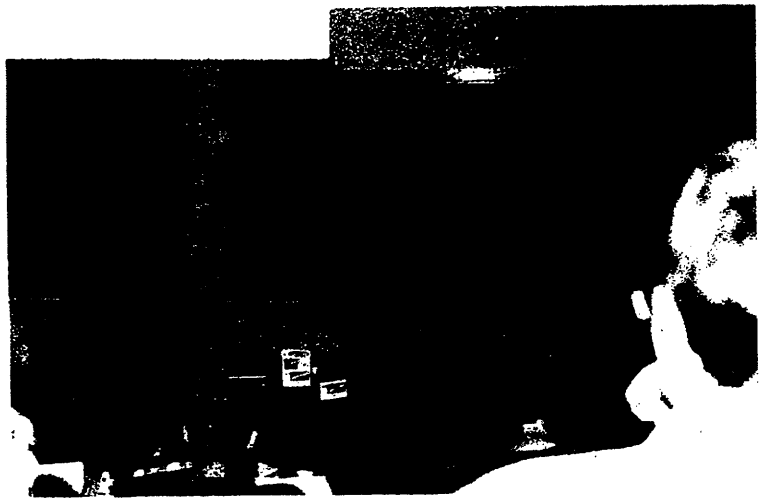

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1. General introduction to the problems

1.1. Some general comments

Disorder, frustration, turbulence and chaos are characteristic features of our time period. They appear to penetrate many different layers of our existence. It is therefore no accident that, in recent years, there has been a lot of research activity into the mathematics and physics of
- disordered systems (in condensed matter physics);
- (chaotic behaviour of) dynamical systems;
- turbulence (in fluid dynamics).

In the past fifteen years, say, these subjects of theoretical and experimental research have become immensely popular and have reached a fairly mature stage, although many of the important issues are really still wide open—at least when looked upon from a very slightly mathematical point of view.

One may have different opinions as to how important the *mathematical physics* of disordered systems, dynamical systems and fluid dynamics is. For a mathematical physicist, though, these subjects offer an opportunity to confront himself, intellectually, with some basic themes of contemporary science.

As an encouraging feature of our times, I should like to draw attention to the fact that mathematics and theoretical physics are on *converging trajectories* once again. We hope that this trend will result in a genuine and fruitful new encounter, and not in a collision. People in both fields, *and in between*, have some responsibility to achieve that goal.

The lectures of Tom Spencer and myself are intended to survey a small fraction of the more *mathematical aspects of disordered systems theory*. These aspects are certainly not particularly central for the future of that subject, but we feel it is always worthwhile to find out what, in a scientific endeavour, can be motivated or understood in a way that does not leave further doubts. The choice of material reflects to a large extent our area of competence and our personal taste. We have no ambition to give a complete survey. (We do what we can.)

It is hoped that our efforts are not completely useless and that they are acceptable as a tiny contribution to scientific culture, not unlike a good painting of Mont Blanc may represent a tiny, yet worthwhile contribution to general culture, although a mountaineer who wants to climb Mont Blanc may actually prefer a map of the area over a good painting of Mont Blanc.

1.2. Topics in disordered systems theory

It may be useful to distinguish the following four main areas:
(A) Mechanisms for the creation of disorder.
(B) Static aspects and equilibrium properties of disordered systems.
(C) Dynamical aspects of disordered systems; relaxation to equilibrium, metastability.
(D) Transport in disordered systems.

The least developed topic might well be topic (C), although there are now some beginnings in that direction.

In the lectures of Tom Spencer and myself, the main emphasis is on topics (B) and (D). Some more specialized themes concerning (A) and (B) appear in accompanying lecture and seminar notes by various people.

We proceed to sketch some theoretical problems met in the study of topics (A) through (D).

(A) *Mechanisms for the creation of disorder*
One typical instance of creation of disorder is the *melting of a crystal lattice*. We mention this as just one example of an order–disorder transition; it is a fairly challenging and difficult example.

It is fair to say, we think, that there is no fundamental (microscopic) theory of crystallization and of melting, yet, but there are amusing phenomenological approaches towards understanding melting. One popular such approach is to describe disorder relative to a perfect crystal lattice in terms of *defects*, namely *dislocations* and *disclinations*. Disorder is created when these defects are generated, and these defects are generated by thermal motion and mechanical deformations. There is an elegant differential-geometric description of dislocations and disclinations. An imperfect crystal can be viewed as a cell complex equipped with an affine connection. A *dislocation* then is a locus of *torsion*, while *disclinations* correspond to *curvature* [1]. (Part of) This discrete geometrical structure can be described with the help of Regge calculus [2].

Dislocations and disclinations are defects of dimension $\nu - 2$, where ν is the dimension of the underlying lattice (or cell complex). Thus they are point defects for $\nu = 2$, and line defects for $\nu = 3$.

One approach towards a theoretical description of *melting* is to view this process as a *condensation of defects*. In this view of melting, the question whether the melting transition is continuous or first order and whether there might exist phases intermediate between the completely ordered and the completely disordered phase, depends on fairly detailed properties of the effective interactions between defects. There are several indications that three-dimensional melting is weakly first order.

The mere existence of the melting transition, described as a condensation of defects, can be understood, heuristically, with the help of a simple energy–entropy argument and has been proven rigorously in the framework of simple models.

In two dimensions, the mean energy of an isolated point defect in a square area of diameter l is proportional to $\log l$. The total number of possible positions is proportional to l^2, i.e. the entropy grows logarithmically in l. Hence the free energy behaves like

$$F = E - TS \sim \text{const.} \log l - kT \,\text{const.}' \log l \,. \tag{1}$$

Thus, for T large enough, a dilute system of bound point defects becomes unstable in the thermodynamic limit, i.e. defects unbind and form a plasma.

It should be emphasized that in two-dimensional systems with regular short-range interactions, a crystal lattice is unstable and *translational invariance remains unbroken at all temperatures* (Mermin's theorem) [3], although *directional ordering* is possible. Eq. (1) is an appropriate ansatz for the description of the *unbinding of disclination pairs*.

In three dimensions, dislocations are line defects with a self-energy roughly proportional to their length, l. In a cubic area of diameter $\sim$ const. l, the number of possible configurations of a single dislocation loop of length l is clearly proportional to exp[const. l], so the entropy grows linearly in l. The free energy thus behaves like

$$F \sim \text{const.}\, l - kT \,\text{const.}'\, l \,. \tag{2}$$

Hence, for T large enough, a dilute system of dislocation loops becomes unstable in large volumes, i.e. dislocation loops condense and the crystal lattice melts [4].

Another mechanism for generating disorder is *dilution*. Consider a crystalline system with the property that atoms at the sites of some regular sublattice can, in principle, be replaced by another type of atoms or molecules. Let p denote the probability that the atom at a site of that sublattice is substituted by another atom (we then say that the site is "occupied"), and suppose that the events that *different sites* are occupied or remain empty are all *independent* of each other. The process so obtained is called *Bernoulli site percolation* [5]. One is interested, for example, in understanding the structure of the random connected sets of occupied sites. In particular, one may ask whether there are ∞ connected sets of occupied sites, what the probability is that a given site belongs to an infinite cluster, and how these quantities depend on p, etc.

The percolation problem just described has some natural generalizations. If we view a perfect ν-dimensional (crystalline) lattice as a cell complex, it is natural to introduce the notion of percolation of k-cells with $k \leq \nu$ [6]. These percolation processes are interesting in their own right, but are, in several instances, important in the study of other problems. For example, two-cell percolation is a toy problem in studying lattice gauge theory, but it is also important as a tool in the study of bond percolation in three dimensions. The reason is that k-cell percolation and $(\nu - k)$-cell percolation are *dual* to each other, in the sense of Kramers–Wannier duality. Three-dimensional bond percolation, in turn, is important in the study of *dilute magnets*.

Percolation is a special case of the so-called *q-states Potts models* which is obtained when one sets $q = 1$ in the Fortuin–Kasteleyn representation of the Potts models. Potts models associated with k-cells can be defined for arbitrary k, $1 \leq k \leq \nu - 1$. They exhibit transitions as the temperature is varied, and the interesting fact is that the nature of the transition changes from continuous to first order, as q is increased from $q = 1$ towards $q = \infty$ [7]. Among the fascinating aspects of q-states Potts models associated with k-cells $(2 \leq k \leq \nu - 2)$ is their rich random-geometrical and random-topological structure [8].

The theory of site and bond percolation is, as mentioned, an essential tool in the study of *dilute (ferro) magnets* [9]. These are systems doped with some density, p, of (ferro) magnetic ions which have short-range (ferro) magnetic (exchange) interactions. The magnetic properties of such systems depend not only on the usual thermodynamic parameters such as temperature, but also on p. In the simplest models, the magnetic moments of two ions are correlated only if they belong to the same site- or bond-connected cluster. This shows why site or bond percolation is

relevant in the analysis of such magnets. This will be discussed in detail in lecture 3.

Related to dilute ferromagnets are *dilute antiferromagnets*. If the exchange coefficient is nearest neighbour and negative and if the lattice contains two interpenetrating sublattices (e.g. even and odd), then the ground state and (for $\nu > \nu_\ell$) the low-temperature equilibrium states are antiferromagnetic if the external magnetic field vanishes. In zero magnetic field the classical antiferromagnet is, in fact, equivalent to the ferromagnet. However, the behaviour of a dilute antiferromagnet changes drastically when an external magnetic field is turned on. For suitably chosen exchange couplings, the dilute antiferromagnet in a uniform magnetic field is nearly equivalent to a *ferromagnet* in a *random staggered magnetic field*. This transformation (originally proposed by Aharony and Fishman [10]) will be described in lecture 4. The models thus obtained are the so-called *random field ferromagnets* (RFFM) which have attracted a lot of experimental and theoretical interest in the past few years. For large anisotropy, the magnetic moments can be described by Ising spins and one obtains the random field Ising model (RFIM) studied in lecture 4. We shall sketch an argument explaining why the lower critical dimension, ν_ℓ, of the RFIM is most probably two. The ordinary Ising model has lower critical dimension one, thus *a random magnetic field enhances disorder*. One reason why the RFIM has attracted much theoretical activity is that there have been two conflicting arguments, one predicting $\nu_\ell = 2$ (Imry and Ma [11]) and the other one predicting $\nu_\ell = 3$ (based on results of Young and of Parisi and Sourlas [12]). The controversy has, to a considerable extent, been settled in recent work of Imbrie [13], following work of Chalker and of Fisher, Fröhlich and Spencer [14]. The apparently wrong prediction that $\nu_\ell = 3$ was based on the celebrated dimensional reduction technique which relates an RFFM in D dimensions to an ordinary ferromagnet (in zero field) in dimension $\nu = D - 2$. There is now some mathematical understanding of the circumstances that make dimensional reduction fail in the RFIM. The exact solution of the RF spherical model suggests that dimensional reduction may be correct, qualitatively for RF models described, approximately, by Gaussian spin wave theory. More noteworthy is the observation that dimensional reduction appears to work for a $g\phi^4 - w\phi^3$ lattice theory in an *imaginary random magnetic field*. The critical behaviour of this theory in dimension D is related to the Lee–Yang edge singularity in the $(\nu = D - 2)$-dimensional Ising model (Parisi and Sourlas [15]). It would be interesting to nail this down more precisely. Remarkable is the fact that a theory that has as academic an appearance as the $g\phi^4 - w\phi^3$ theory in an imaginary random magnetic field describes something as concrete as branched polymers (Parisi and Sourlas [15]). All this leads to the prediction of precise values of the critical exponents of branched polymers in three dimensions (predictions which are fairly well confirmed by numerical experiments).

The upper critical dimension for branched polymers is eight which is related to the fact that the Hausdorff dimension of branched polymers reaches four in high dimensions, so that the intersection probability of two branched polymers tends to 0 in dimension ≥ 8. Above dimension 8, the critical exponents have the values $\nu = 1/4$, $\eta = 0$. In the disguise of *lattice animals*, branched polymers arise in the study of the cluster shapes of bond percolation processes. Thus percolation theory helps in understanding random magnetic systems (dilute magnets) and, in turn, random magnetic systems help to understand some aspects of the percolation problem.

It is remarkable that the branched polymer exponents $\nu = 1/4$, $\eta = 0$ are also the mean-field values of some critical exponents of a large class of (discrete) *random-surface theories*. In recent times, random-surface theories have come to play a fairly prominent role in statistical physics and disordered systems theory. Numerous problems in statistical physics actually lead fairly directly to the study of statistical fluctuations of random surfaces [16]. Among them we wish to mention the following *examples*:

(1) Crystal growth; statistics of crystal surfaces.

(2) Domain walls and interfaces in magnets, spin glasses and alloys (see also lecture 3).

(3) Role of gases of domain walls and their fluctuations in order–disorder transitions; of domain walls and interfaces and their fluctuations in (uniaxial) commensurate–incommensurate transitions, etc.

(4) Critical behaviour in surface models and uses of "critical" surfaces of Hausdorff dimension >2 in the catalysis of chemical reactions.

(5) Wetting and unpinning.

(6) Surface structures in soap foam, in emulsions, in systems of membrane-like polymers, etc. (nice examples of disordered systems).

There are now some encouraging beginnings in the direction of a statistical mechanics of random surfaces [6, 8, 16–19]. Some basic phenomena, like surface roughening [18], surface crumpling [16, 17],

collapse of surfaces into (branched) polymers [17], or breathing (the converse of collapse) [16, 17, 19], etc. have been isolated, and there are simple models in which these phenomena can be studied mathematically. There are fairly interesting models for which problems (1) through (6) above have found complete or partial solutions.

So far we have isolated the following mechanisms for the creation of disorder:

(i) Order–disorder transition via condensation of topological defects.
(ii) Creation of disorder via dilution.
(iii) Creation of disorder by random magnetic fields.
(iv) Creation of disorder by domain wall wandering and other surface fluctuations.

We now wish to discuss a fifth such mechanism:

(v) Creation of disorder through *frustration*.

This mechanism has been a focal point of recent theoretical interest: It is fundamental in the study of spin glasses and real glasses. As an example of a fully frustrated system, consider a two-dimensional antiferromagnet on a triangular lattice. On the vertices of each triangle, there are Ising spins which feel antiferromagnetic exchange interactions (fig. 1.1). No matter how the spins are oriented, there are two vertices in each triangle with the property that the spins at those vertices receive conflicting instructions as to whether to be up or down. As a result there are very many groundstates or states very nearly degenerate with a groundstate in this system, causing a violation of the third law of thermodynamics [20], i.e. the residual entropy at 0 temperature is strictly positive. This might be a rather typical feature of frustrated systems. It suggests that low-temperature phase diagrams in such systems may be very complicated.

A typical system with random frustration is a spin glass [21]: At each site of the lattice $\mathbb{Z}^\nu$ there is attached an Ising spin. Two spins σ_i and σ_j

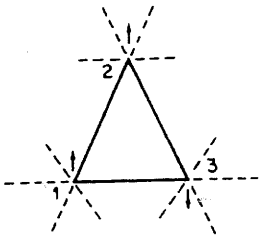

Fig. 1.1.

interact by an exchange force of strength J_{ij}, and J_{ij} is a random variable with distribution $d\rho(J_{ij})$ such that the support of $d\rho$ is not entirely contained in the positive half axis. If $d\rho(J_{ij}) = \delta(J_{ij})\, dJ_{ij}$, for $|i-j|>1$, then the interactions are nearest neighbour and one can form the quantity

$$J_p = \prod_{\langle i,j\rangle \in \partial p} J_{ij} , \tag{3}$$

where p is a plaquette (unit square) in $\mathbb{Z}^\nu$. We set

$$\tau_p = \operatorname{sign} J_p . \tag{4}$$

A plaquette for which $\tau_p = -1$ is called *frustrated*. Let c be a unit cube of $\mathbb{Z}^\nu$. Clearly

$$\prod_{p\in\partial c} \tau_p = 1 , \tag{5}$$

for all c. (This is the analogue of the homogeneous Maxwell equations.) Therefore, the plaquettes p for which $\tau_p = -1$ are dual to closed surfaces, Σ, of dimension $\nu - 2$. (In two dimensions they are isolated spots, in three dimensions they are dual to loops.) Now on every loop $\mathfrak{L}$ interlacing a closed surface Σ dual to frustrated plaquettes, there is some site j such that the spin at j receives conflicting instructions from its neighbours as to whether to be up or down. This is easily verified.

If the *external magnetic field*, h, *vanishes*, configurations of exchange constants, $\{J_{ij}\}$, fall into equivalence classes: two configurations J and J' are equivalent if and only if there exists α,

$$\alpha\colon j \in \mathbb{Z}^\nu \mapsto \alpha_j \in \{+1, -1\} , \tag{6}$$

such that

$$J'_{ij} = \alpha_i J_{ij} \alpha_j . \tag{7}$$

By changing the variables

$$\sigma'_j = \alpha_j \sigma_j , \tag{8}$$

one sees that two spin glass systems with equivalent exchange constants

are completely equivalent, for $h = 0$. They describe identical physics. The transformations (7) and (8) are called *gauge transformations*. Note that the quantities J_p and τ_p are *invariant* under gauge transformations. Thus systems related to each other by gauge transformations are equivalent, as long as h vanishes. It is convenient, therefore, to introduce gauge-invariant correlations. Let Γ_{ij} be a path in $\mathbb{Z}^\nu$ starting at i and ending at j. We define

$$C(\Gamma_{ij}) = \sigma_i \Big(\prod_{\langle k,l\rangle \in \Gamma_{ij}} J_{kl} \Big) \sigma_j \,, \tag{9}$$

and, as a special case,

$$C(\langle i, j\rangle) = \sigma_i J_{ij} \sigma_j \,. \tag{10}$$

Let Σ be a $(\nu-2)$-dimensional surface dual to frustrated plaquettes, and let $\mathfrak{L}$ be an arbitrary loop interlacing Σ. Then

$$\prod_{\langle i,j\rangle \in \mathfrak{L}} C(\langle i, j\rangle) = \prod_{\langle i, j\rangle \in \mathfrak{L}} J_{ij} < 0 \,. \tag{11}$$

Thus Σ is the boundary of a sheet or domain wall, γ, of dimension $\nu - 1$ with the property that, for each bond $\langle i, j\rangle$ dual to γ,

$$C(\langle i, j\rangle) < 0 \,.$$

In the case of nearest-neighbour interactions, the total energy of a configuration of domain walls $\{\gamma_1, \gamma_2, \ldots\}$ is given by

$$H(\{\gamma_1, \gamma_2, \ldots\}) = \sum_n E(\gamma_n) \,, \tag{12}$$

with

$$E(\gamma) = 2 \sum_{\langle i,j\rangle \text{ dual to } \gamma} |J_{ij}| \,. \tag{13}$$

Every equivalence class of exchange couplings, J, determines a unique configuration of $(\nu-2)$-dimensional surfaces, $\Sigma_1, \Sigma_2, \ldots$, dual to frustrated plaquettes which are *boundaries of domain walls*. The problem of calculating the groundstate energy of a spin glass can be understood as the problem of choosing domain walls $\gamma_1, \gamma_2, \ldots$ in such a way that

$$H(\{\gamma_1, \gamma_2, \ldots\}) \text{ is } \textit{minimal}, \text{ given } \Sigma_1, \Sigma_2, \ldots \,. \tag{14}$$

It is easy to see from that that there are, in general, lots of states whose energies are almost identical to the groundstate energy, because the choice of the domain walls is non-unique, given a tiny energy uncertainty. One may thus expect that the entropy at zero temperature is positive, or, at least, that it rises very sharply near $T = 0$, and that, at low temperatures, there are enormously many (*meta*) *stable states* of *enormous life-time* related to configurations whose *energy* is *very close* to the *groundstate energy* but separated from the groundstate by *very high energy barriers*.

While the spin glass problem in zero magnetic field is already very difficult, it appears to be really hard to analyze spin glass phase diagrams in the presence of a variable external magnetic field. In this situation there is *no gauge invariance*, and the gauge-invariant formalism described above is quite useless. It is quite safe to expect that, in large enough dimension, there are lots of transitions as the magnetic field is varied (devil's staircases?), but all we know for sure is that if the strength of the external magnetic field is above some critical value, frustration becomes irrelevant, ordering sets in, and the equilibrium state is unique.

It is interesting to note that the signs of the exchange couplings are determined by *Bernoulli bond percolation*:

$$\text{sign } J_{ij} = \begin{cases} 1 \,, & \text{with probability } p \,, \\ -1 \,, & \text{with probability } 1 - p \,, \end{cases}$$

where $p \equiv \int_0^\infty d\rho(J)$. The surfaces Σ dual to frustrated plaquettes are the "*boundaries*" *of clusters of* $(\nu-1)$*-cells dual to bonds* $\langle i, j\rangle$ *for which* $J_{ij} < 0$. The statistics of the surfaces Σ is therefore a *problem in bond percolation* (apparently a rather challenging one; see sect. 3.5). For fully frustrated spin glasses, $p = 1/2$.

Another aspect of the spin glass problem is the *destructive interference of interactions*. We recall that the Ruderman–Kittel interaction in three-dimensional spin glasses has exchange couplings

$$J_{ij} \sim \frac{\cos[p(i-j)]}{|i-j|^3} \,. \tag{15}$$

These couplings are thus of very long range ($\Sigma_j |J_{ij}|$ diverges). However, roughly speaking, the events that $J_{ij} = +J$ and that $J_{ij} = -J$ are equally

likely, because the magnetic ions are distributed randomly. This causes huge cancellations in the total exchange energy and enhances disorder so drastically that spin glass models are in disordered phases, with clustering of correlations, for long range couplings J_{ij} with the property that corresponding systems with couplings $|J_{ij}|$ are permanently ordered, or do not even behave thermodynamically. (A spin glass with couplings $\{J_{ij}\}$ with the property that $\overline{J_{ij}} = 0$, $\sup_i \Sigma_j \overline{J_{ij}^p} \leq \text{const.}$, uniformly in $p = 2, 3, \ldots$ where $\overline{F(J)} \equiv \int d\rho(J) F(J)$, is thermodynamically stable [22] and appears to have a unique Gibbs state at high temperatures [23]. For more detailed results see lectures 2 and 3.)

We have just learned that disorder and decay of correlations in spin glasses are enhanced by destructive interference of competing ferro- and antiferromagnetic interactions and by frustration. It is natural to ask whether similar mechanisms are at work in other systems, as well. It is expected [24] that frustration is an important aspect of *real glasses*. The principle of this can be understood quite easily: consider a substance made up of several species of atoms, $A_1, A_2, \ldots, A_m$ in ratios $n_1 : n_2 : \cdots : n_m$. Suppose that kn_1 atoms of type A_1, kn_2 atoms of type $A_2, \ldots, kn_m$ atoms of type A_m form a rather stable compound (an approximately rigid body), $\mathfrak{C}$. A low-temperature condensate of such compounds can be thought of as corresponding approximately to a closest-packing configuration of $\mathfrak{C}$'s. But, depending on the geometrical (steric) properties of $\mathfrak{C}$, there may or may not exist a configuration of density close to closest-packing density that forms a regular lattice. Indeed, configurations of (approximately) closest-packing density may correspond to aperiodic tilings of physical space. There are many regular compounds $\mathfrak{C}$ which cannot be arranged in a periodic array of high density (e.g. pentagons in the plane, ...). Since we have assumed that, at low temperature, a compound $\mathfrak{C}$ is, approximately, a rigid body, we may describe its position in space by centre of mass coordinates, x, and a frame (dreibein) $R \in \mathrm{SO}(3)$. (In systems with different isomers, R may be, more generally, an element of O(3).) Now consider two compounds $\mathfrak{C}$ and $\mathfrak{C}'$, with positions (x, R), (x', R'), with $|x - x'|$ small. Purely geometrical and energetic circumstances may favour R and R' to have a certain relative orientation to each other which is given by an element J of some subgroup $G \subseteq \mathrm{SO}(3)$ which depends on (x, R) and x'. It may happen that if one tries to line up n compounds $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$ at roughly equidistant positions $x_1, \ldots, x_n$ located along a loop $\mathfrak{L}$ in space, one is unable to choose orientations $R_1, \ldots, R_n$ such that

$$R_{x_{j+1}} = J(x_{j+1}, x_j) R_{x_j}, \tag{16}$$

for all j, with $x_{n+1} \equiv x_1$. The reason is very simple. Let

$$J_{\mathfrak{L}} = \prod_{j=1}^{n} J(x_{j+1}, x_j) . \tag{17}$$

By iterating (16) we see that, in order to obtain a perfect arrangement, we must require

$$J_{\mathfrak{L}} R_{x_1} = R_{x_1} , \tag{18}$$

hence

$$J_{\mathfrak{L}} = \mathbb{1} . \tag{19}$$

But a minimum-energy, approximately closest-packing configuration may have to violate (19) for many loops, $\mathfrak{L}$ (unless the compounds $\mathfrak{C}$ can be arranged in a dense, periodic array). To understand this one could try to describe the low-temperature properties of the system by an effective Hamilton function, H, given by

$$H(\{x_i, R_i\}) = \sum_c \phi_c(\{x_i, R_i\}, i \in c) , \tag{20}$$

where c is an arbitrary finite cluster of compounds, and

$$\phi_c(\{x_i, R_i\}, i \in c) \simeq 0$$

if

$$\max_{i,j \text{ in } c} |x_i - x_j| > \rho ,$$

for some finite ρ. Since the compounds $\mathfrak{C}$ have a positive diameter, D, the potentials ϕ_c vanish when the cardinality of c is larger than some finite integer $n_0 \sim (\rho/D)^3$. Unfortunately, while it appears possible to explicitly write down reasonable expressions for the potentials, ϕ_c, it looks horrendously complicated to even calculate approximate ground-states or estimate the residual entropy at $T = 0$. One would really have to rely on large-scale computer calculations.

If one is dealing with a system that has a phase where *translational disorder* is rather *weak*, i.e. the positions, x_i, of the compounds are essentially frozen at the sites of an irregular (possibly random) lattice,

Λ, but *orientational disorder* is *tolerated*, energetically, and rather large, then one might hope to describe that phase of the system by an effective theory on Λ. The possible orientations, R_x, at each site $x \in \Lambda$, would be chosen to belong to some (e.g. discrete) subset $\mathfrak{R}$ of SO(3), and to lowest order in an expansion in powers of $\{R_x\}$, $x \in \Lambda$, the Hamilton function would have the form

$$H \simeq \sum_{\langle x,y\rangle \subset \Lambda} \mathrm{tr}(R_x^{\mathrm{T}} \cdot J_{xy}^{\Lambda} R_y) , \tag{21}$$

where J_{xy}^{Λ} belongs to some (typically discrete) subgroup $G \subseteq \mathrm{SO}(3)$, for every bond $\langle x, y\rangle$ of Λ. *Frustration* then occurs on loops, λ, of the dual lattice with the property that, for loops $\mathfrak{L}$ of Λ interlacing λ,

$$J_{\mathfrak{L}}^{\Lambda} = \prod_{\langle x,y\rangle \in \mathfrak{L}} \circlearrowleft J_{xy}^{\Lambda} \neq \mathbb{1} . \tag{22}$$

The stable defects of configurations $\{R_x\}$ can be classified by homotopy ($\pi_0(G/H)$, $\pi_1(G/H)$, $\pi_2(G/H)$, where H is the "symmetry group" of the orientational configurations). Many concepts and ideas used in the analysis of the spin glass problem can be carried over to this situation, but in the present case the randomness of the couplings $\{J_{xy}^{\Lambda}\}$—if they are random at all—is coupled to the randomness of the lattice Λ which probably renders the analysis considerably more difficult.

Clearly, orientational ordering is possible without there being translational long-range order. But then the concept of frustration becomes rather vague and is useful, at best, to explain properties of the short-range ordering of large but finite "compounds of compounds". Anyhow, two-dimensional gases or liquids of compounds must generally be expected to exhibit long-range orientational ordering (dipolar-, quadrupolar-, . . . ordering), and this does not contradict the Mermin–Wagner theorem, as is well known.

This concludes our survey of the "mechanisms for the creation of disorder".

(B) *Static aspects and equilibrium properties of disordered systems*
Since my lectures are organized around this topic, it is best to present now a short outline of subsequent lectures. In lecture 2, I shall discuss the high-temperature and/or large magnetic field properties of disordered magnets and spin glasses. To be specific, let us consider the example of a dilute ferromagnet. The Hamilton function is given, for

example, by

$$H = -\sum_{\langle i,j\rangle \subset \mathbb{Z}^{\nu}} \sigma_i J_{ij} \sigma_j + h \sum_{i \in \mathbb{Z}^{\nu}} \sigma_i , \tag{23}$$

where $\sigma_j = \pm 1$, h (the magnetic field) is real and

$$J_{ij} = \begin{cases} 1 , & \text{with probability } p , \\ 0 , & \text{with probability } 1-p . \end{cases} \tag{24}$$

It is well known that there are no phase transitions in this system for $h \neq 0$. Let $T_c(p)$ denote the critical temperature of the model in zero magnetic field, $h = 0$, $T_c(1)$ being the critical temperature of the pure Ising model. We shall show that

$$T_c(p) < p T_c(1) ,$$

and

$$T_c(p) = 0 , \quad \text{for } p < p_c ,$$

where p_c is the percolation threshold of ν-dimensional bond percolation. More difficult is the proof that $T_c(p)$ is positive, for $p > p_c$. This is shown in lecture 3 (modulo the "usual suspects" of bond percolation in $\nu \geq 3$, known to hold when $\nu = 2$). Thus the phase diagram can be summarized as in fig. 1.2. In the disordered phase ($T > T_c(p)$), but for $T < T_c(1)$, one encounters the famous *Griffiths singularities*. We show that, as a function of h, the free energy $f(T, h)$ of the system has an

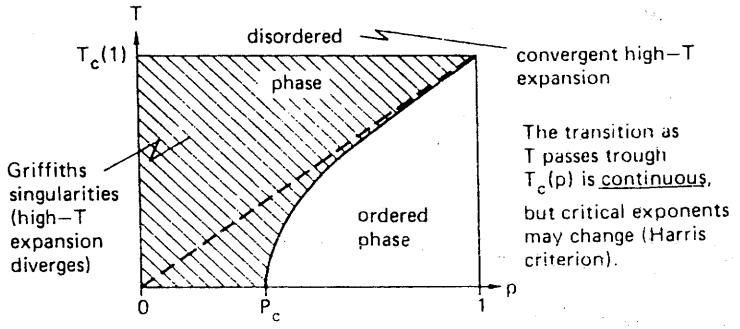

Fig. 1.2.

essential singularity at $h = 0$, for $T_c(p) \leq T < T_c(1)$, but is differentiable at $h = 0$, i.e. $m(h) = (\partial f(T, h)/\partial h)$ tends to 0, as $h \to 0$. One of our new results is that, for $T > T_0(p)$, with $T_c(p) < T_0(p) < T_c(1)$, $f(T, h)$ is actually C^∞ in h (Fröhlich and Imbrie [25]). It would be interesting to know whether $f(T, h)$ is quasi-analytic at $h = 0$, but we rather expect it is not.

The features of the high-temperature phase described above for the dilute ferromagnet appear to be generally valid for disordered magnets and spin glasses, and part of this picture will be established for a general class of models. We also show that ordering in dilute ferromagnets at low temperature is stable against introducing a sufficiently small density of antiferromagnetic exchange couplings, in a variety of models (lecture 3). However, when the density of antiferromagnetic bonds increases, frustration tends to increase as well, and curious phenomena may happen. It is known that the magnetization decreases and it may, in fact, decrease so much that its sign is opposite to the sign of the boundary condition. Curious things happen when neither ferro- nor antiferromagnetic exchange couplings percolate, but together they do. When the resulting system is highly frustrated, it is really a (dilute) spin glass. But when frustration is suppressed, i.e. frustrated loops have a very low density, the system behaves like an antiferromagnet. When such a system is put into a uniform magnetic field, its behaviour is described by the random field Ising model (RFIM). This is discussed in lecture 4, and the phase diagram of the RFIM is developed in some detail. In particular, we show that if the disorder in the magnetic field is large enough, then the equilibrium state of the RFIM is unique, and connected correlations decay at arbitrary temperatures. Uniqueness and clustering hold, of course, at high temperatures for arbitrarily weak disorder. However, for $\nu > 2$, small disorder and at low temperatures, long-range order is expected on the basis of the improved Imry–Ma argument (see refs. [26] and [14] for results) and Imbrie [13] has established this fact rigorously for $T = 0$ and $\nu \geq 3$. The conjectured phase diagram of the RFIM in $\nu > 2$ is then as in fig. 1.3. Here H^2 is the variance of the random magnetic field, h_j, and the mean of h_j is assumed to vanish, for all $j \in \mathbb{Z}^\nu$. It is expected that when the mean, h, of h_j is nonzero, then the equilibrium state is unique, and connected correlations decay, but this has only been shown for h large enough.

For spin glasses much less is known. The only safely established facts concern the disordered regime and a result concerning the positivity of the entropy at zero temperature in a somewhat artificial model. If the

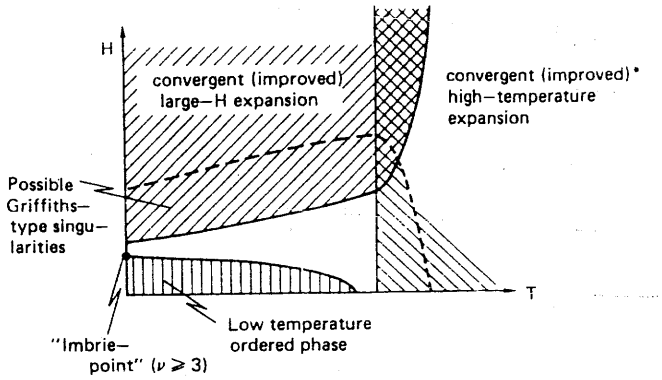

Fig. 1.3. The conjectured phase diagram of the RFIM in $\nu > 2$. Improved expansions are due to Fröhlich and Imbrie [25] and Berretti [27].

exchange couplings $\{J_{ij}\}$ are independent, bounded random variables with

$$\overline{J_{ij}} = 0 \quad \text{and} \quad \overline{|J_{ij}|^p} \leq \text{const.}\ |i - j|^{-2\alpha} \tag{25}$$

for $p = 2, 3, \ldots$, then there is *no ordering* (the Edwards–Anderson order parameter vanishes) *for arbitrary temperatures and magnetic fields, provided*

$$\alpha > 1, \quad \text{in } \nu = 1, \tag{26a}$$

$$\alpha \geq 2, \quad \text{continuous internal symmetry in } \nu = 2, \tag{26b}$$

and, for arbitrary ν, one strongly expects that there is a disordered high-temperature phase, provided $\alpha > (\nu/2)$. (See refs. [22] and [23] for further results.) It is expected that, in one dimension, the Ising spin glass has a transition if $\alpha < 1$ ($\alpha > 1/2$ is required to obtain thermodynamic behaviour; Kotliar, Anderson and Stein [28]). It is well known (and rigorous proofs exist) that in deterministic ferromagnets, the corresponding critical values of α are twice as big [29].

Some interesting conceptual problems arise when one tries to give a complete description of all possible Gibbs states of a spin glass, for a given sample of J_{ij}'s. It is conceivable that for example in one dimension, for $1/2 < \alpha < 1$, exotic Gibbs states exist which could be con-

structed by choosing $\{J_{ij}\}$-dependent boundary conditions in a class of b.c. of "measure 0" (to be somewhat vague).

In my last lecture, I shall derive the connection between Ising models in imaginary random magnetic fields and branched polymers discovered by Parisi and Sourlas and the application of this connection and the dimensional reduction technique to predict the values of the critical exponents for the three-dimensional branched polymer system. (The notes have been compiled by Bovier, Glaus and the author.) The mean field theory for branched polymers and other rigorous results would provide a natural gateway to the exciting topic of *random surfaces* and their appearances in statistical physics. But there is no room for a systematic account of these matters, except for a discussion of the intricacies and subtleties that one meets when one analyzes the fluctuations of the *interface in the RFIM*. In particular, we discuss why and how the Grinstein–Ma arguments fail at low temperatures, e.g. in dimension 4, and we discuss some conjectures concerning the interface of the three-dimensional RFIM (possible transition from logarithmic to power law fluctuations).

(C) *Dynamical aspects of disordered systems*

Consider, for example, a dilute magnet with ferromagnetic exchange couplings just barely above percolation threshold and a very low density of antiferromagnetic exchange couplings sprinkled in, in such a way that there is very little frustration. Then in dimension $\nu \geq 2$, we would expect that there is ordering at very low temperature. However, the spontaneous magnetization in the Gibbs state with + *boundary conditions* at ∞ may actually be *negative*, since a family of *finite* clusters of ferromagnetic couplings of fairly high density may be antiferromagnetically connected to the infinite cluster. If such a system is subject to a fairly strong homogeneous external magnetic field in the + direction, the total magnetization is positive and it remains positive when the external field is slowly turned off. It is plausible that, in this system, relaxation back to equilibrium, where the magnetization is negative, would take an astronomically long time. Similar reasoning applies to the RFIM and to spin glasses. One may thus expect as typical dynamical features of disordered magnets at low temperature:

- complicated hysteresis phenomena and path-dependence;
- freezing in metastable states of enormous life-time;
- very slow relaxation to equilibrium;
- slow decay of autocorrelation functions (in this connection, it has been proposed that $(1/f)$- or Flicker noise may describe how the system hops from one metastable state to another, but other spectra seem to appear, as well).

Of course, at high temperatures in the disordered phase, relaxation towards equilibrium occurs relatively rapidly, in particular relaxation times are *finite*.

The very properties of the dynamics of disordered systems at low temperatures, in particular the existence of many metastable states of very long life-time and the slow decay of autocorrelations in time, make Monte-Carlo simulations of the *equilibrium properties* of such systems very difficult and generally quite unreliable. But see, e.g., ref. [30]. (The model dynamics and the Monte-Carlo dynamics are in general chosen to be dissipative dynamics given by temperature-dependent stochastic processes whose invariant measures are given by the Gibbs states.) But what obstructs the Monte-Carlo method bothers the experimentalist even more. His experimental data tend to be strongly path-dependent, and he has a very hard time to actually "see" equilibrium. A certain lack of awareness of these circumstances has recently led to controversies about the correct interpretation of experiments done on the three-dimensional RFIM. Villain and Bruinsma et al. have clarified the situation by appealing to the dynamical facts described above and have developed a heuristic description of the time dependence of various quantities [31].

The challenge to the theoretician who wants to explain, theoretically, the outcome of experiments done on disordered systems at low temperatures is to understand dissipative dynamics in systems with ∞ many degrees of freedom and to develop a good theory of metastability.

The last part of this introduction is devoted to the fascinating subject of (D).

(D) *Transport in disordered systems*

This is a huge subject in itself. Among the fairly fashionable topics of transport theory, one finds:

(i) Classical and quantum mechanics of particles moving through random arrays of scatterers, or

(ii) moving in stochastically time-dependent potentials.

(iii) Transport of heat, sound, light, electric charge, etc. through disordered media (Anderson localization, Anderson transition, etc.).

(iv) Random walks in random environments.

A typical example of problem (i) is the Lorentz gas. A classical

model, closely related to independent work of J.T. Chalker, and on numerous discussions with J. Imbrie, but especially with Tom Spencer. Many of the mathematically rigorous results in lecture 5 are novel or new and are based on collaboration with A. Bovier, G. Felder and U. Glaus, but the punch line is a brief discussion of the Parisi–Sourlas results on branched polymers. Some of the rigorous work in lecture 5 is related to earlier work of M. Aizenman, D. Brydges, T. Spencer and myself, M. Aizenman and C.E. Newman and others. See also the notes of David Brydges for a review of work concerning related questions. The numerical results described in lecture 5 have been compiled by S. Carracciolo and U. Glaus.

(3) *Acknowledgements*. I thank M. Aizenman, J.T. Chayes, L. Chayes, G. Felder, D. Fisher, J. Imbrie, L. Russo and T. Spencer for enjoyable and fruitful collaborations which led to some of the results reviewed in the following and shaped my perspective. I owe a great deal to D. Brydges, E.H. Lieb, E. Seiler and T. Spencer for having gotten me interested in and taught statistical physics. Without my collaboration with Tom Spencer and our friendship I would never have gotten involved in work that I believe has been worth our efforts.

I also thank K. Osterwalder and R. Stora for having invited me to lecture at Les Houches and for having organized such a good school.

2. The "high-temperature" behaviour of disordered magnets

2.1. *Definition of models and main results*

We consider disordered magnets described, in an idealized manner, as classical lattice spin systems with coupling constants that are random variables. A typical Hamilton function of such a system in a box Λ contained in the lattice $\mathbb{Z}^\nu$ is given by

$$H_\Lambda = -\sum_{i,j \text{ in } \Lambda} J_{ij}\sigma_i\sigma_j + \sum_{j\in\Lambda} h_j\sigma_j\,, \tag{2.1}$$

and we usually suppose that

$$\sigma_j = \pm 1 \text{ (Ising spins)}, \quad \forall j \in \mathbb{Z}^\nu\,.$$

In (2.1) J_{ij} denotes the exchange coupling between σ_i and σ_j and is a

real-valued random function on $\mathbb{Z}^\nu \times \mathbb{Z}^\nu$ with distribution

$$\mathrm{d}R(J) \equiv \prod_{i,j} \mathrm{d}\rho_{|i-j|}(J_{ij})\,, \tag{2.2}$$

where the measures $\mathrm{d}\rho_{|j|}$ are probability measures. Furthermore, h_j is some inhomogeneous magnetic field, a random function on $\mathbb{Z}^\nu$ with distribution

$$\mathrm{d}L(h) \equiv \prod_j \mathrm{d}\lambda(h_j)\,, \tag{2.3}$$

where $\mathrm{d}\lambda$ is some probability measure on $\mathbb{R}$. We shall shortly consider different special cases.

Let

$$W = -\sum_{\substack{i\in\Lambda\\ j\in\Lambda^c}} J_{ij}\sigma_i\sigma_j \tag{2.4}$$

(the interaction energy between the spins in Λ and in Λ^c), and let $\mathrm{d}b_\Lambda$ be some probability measure on $\{\sigma_j\}$, $j\in\Lambda^c$, whose role is to impose boundary conditions (b.c.) on the system inside Λ which we regard as an idealized description of part of the experimental setup available to measure statistical properties of the system.

The equilibrium state of the system at inverse temperature β with b.c. $\mathrm{d}b_\Lambda$ is defined by

$$\mathrm{d}\mu_{\beta,b_\Lambda}(\sigma) = Z^{-1}_{\beta,b_\Lambda}\,\mathrm{e}^{-\beta H_\Lambda(\sigma)}\left(\int \mathrm{e}^{-\beta W(\sigma)}\,\mathrm{d}b_\Lambda(\sigma)\right)\prod_{j\in\Lambda}\mathrm{d}\sigma_j\,, \tag{2.5}$$

where $\mathrm{d}\sigma_j$ is the counting measure on $\{-1, 1\}$, and Z_{β,b_Λ} is the usual partition function chosen so that $\int \mathrm{d}\mu_{\beta,b_\Lambda}(\sigma) = 1$.

If F is a function on $\{\sigma_j\}$, $j\in\Lambda$, we denote by

$$\langle F\rangle_\beta \equiv \langle F\rangle_{\beta,b_\Lambda} \equiv \langle F\rangle_{\beta,b_\Lambda}(J,h) \tag{2.6}$$

the integral

$$\int \mathrm{d}\mu_{\beta,b_\Lambda}(\sigma)\,F(\sigma)\,,$$

i.e. $\langle F\rangle_\beta$ is the expectation value of F in $d\mu_{\beta,b_\Lambda}$ which is a random variable, since it still depends on $J \equiv \{J_{ij}\}$ and $h \equiv \{h_j\}$.

The *quenched expectation* of F is then given by

$$\overline{\langle F\rangle}_\beta \equiv \int \langle F\rangle_{\beta,b_\Lambda}(J,h)\, dR(J)\, dL(h)\,, \tag{2.7}$$

i.e. $\langle F\rangle_\beta$ is averaged over all possible samples. Next, we consider some specific models and summarize some results.

(1) *Random field Ising model, large disorder*
$J_{ij} = 0$ for $|i-j| \neq 1$; $J_{ij} = 1$ for $|i-j| = 1$; $\{h_j\}$ independent, identically distributed (i.i.d.) random variables with distribution $d\lambda(h_j)$ given by

$$d\lambda(h_j) \overset{\text{e.g.}}{=} (\sqrt{2\pi}H)^{-1} \exp\{-h_j^2/2H^2\}\, dh_j\,, \tag{2.8}$$

with

$$\beta\, e^{-\beta(H-\text{const.})} \ll 1\,. \tag{2.9}$$

Under these conditions, the thermodynamic limit

$$\langle F\rangle(h) \equiv \lim_{\Lambda \nearrow \mathbb{Z}^\nu} \langle F\rangle_{\beta,b_\Lambda}(h) \tag{2.10}$$

exists and is independent ot b.c., for dL – almost all h. Moreover, there exists a constant $m(\beta) > 0$ *independent* of h such that connected correlations have almost surely *tree decay* with decay rate $m(\beta)$.

(2) *High-temperature spin glass*
$J_{ij} = 0$ for $|i-j| \neq 1$;

$$d\rho(J_{ij}) \equiv d\rho_1(J_{ij})$$

$$\overset{\text{e.g.}}{=} (\sqrt{2\pi}\Delta)^{-1} \exp\{-(J_{ij}-\bar{J})^2/2\Delta^2\}\, dJ_{ij}\,, \tag{2.11}$$

when $|i-j| = 1$; $d\lambda(h_j)$ arbitrary; β small. (If supp $d\rho \subseteq [0,\infty)$, we call such a model a *dilute ferromagnet*.) The main results for these models are the existence of the thermodynamic limit, independence of b.c. and tree decay of connected correlations with decay rate $m(\beta) = m(\beta; d\rho, d\lambda)$ independent of J and h (for almost all J and h).

(3) *Low-temperature, predominantly ferromagnetic spin glass* (see also lecture 3)

$$J_{ij} \text{ as in (2)}, \quad \bar{J} = 1, \quad \Delta \ll 1, \quad \beta \gg 1, \quad h_j \equiv 0\,. \tag{2.12}$$

The main result for these models is that there are two extremal equilibrium states with opposite spontaneous magnetization and with the property that connected correlations have tree decay.

The results summarized in (1)–(3) have the common feature that they can be proven by means of improved "high-temperature" expansions which were recently developed by Imbrie and the author [1], and, in a weaker form, by Berretti [2]. We outline Berretti's expansion below. The more powerful tools in ref. [1] are technically rather involved and cannot be explained here. However, they share several features with the techniques, discussed in Tom Spencer's lectures, that were developed in ref. [3] to establish Anderson localization.

(4) *Random field Ising model, small disorder* (see also lecture 4)

$$J_{ij} \text{ as in (1)}, \quad \overline{h_j} = 0, \quad H^2 = \overline{h_j^2} \ll 1, \quad \beta \gg 1\,. \tag{2.13}$$

Quasi-theorem. For $\nu > 2$, there are two extremal equilibrium states with opposite spontaneous magnetization, and connected correlations have tree decay.

In the generality in which it is stated here, this result has not been proven, yet, but Imbrie has proven it for $\nu \geq 3$ and $\beta = \infty$; see ref. [4]. It is expected that a combination of the methods of refs. [1] and [4] would yield a proof of the quasi-theorem, above, for $\nu \geq 3$.

One may hope to show by means of energy–entropy considerations that in two dimensions there are no states with spontaneous magnetization, as suggested by the arguments in refs. [5, 6, 2]. In any event, the odds are in favour of the conjecture that *the lower critical dimension of the random field Ising model is* $d_\ell = 2$.

(5) *High-temperature spin glass, long-range interactions*
We assume that

$$\sup_i \sum_j \left| \int d\rho_{|i-j|}(J_{ij}) J_{ij} \right| < \infty\,, \qquad \sup_i \sum_j \|J_{ij}\|_\infty^p \leq k\,, \tag{2.14}$$

for some p-independent constant k and all $p = 2, 3, \ldots$. The distribution $d\lambda$ may be arbitrary, and β is required to be very small. Under these conditions we make the following

Conjecture. An improved high-temperature expansion converges almost surely and uniformly in Λ, and correlations have cluster decomposition properties.

This conjecture is still open, but a proof now appears to be within reach.

In one and two dimensions there are, however, some rigorous results. Let the Hamilton function be given by

$$H = -\sum_{i,j} J(i,j)|i-j|^{-\alpha} \mathbf{S}_i \cdot \mathbf{S}_j \,, \tag{2.15}$$

where the $J(i, j)$ are i.i.d. random variables with *mean* 0 and *variance* 1, for all i, j in $\mathbb{Z}^\nu$, and the spins S_i are unit vectors in $\mathbb{R}^N$, $N = 1, 2, 3, \ldots$. The result is:

Theorem.
(1) *For* $\nu = 1$, $N = 1, 2, 3, \ldots$ *and* $\alpha > 1$, *there is no spontaneous magnetization, and the equilibrium state is unique (in a sense explained later), for all* $\beta < \infty$. *See ref.* [7].
(2) *For* $\nu = 2$, $N = 2, 3, \ldots$ *and* $\alpha \geq 2$, *there is no spontaneous magnetization and no breaking of the* $O(N)$ *invariance, for all* $\beta < \infty$. *See ref.* [8].

It is expected that, in one dimension and for $N = 1$ (Ising) and $1/2 < \alpha < 1$, there are transitions. More about these matters appear in lecture 3.

Next, we wish to describe the main difficulties one meets when one tries to use high- or low-temperature expansion techniques to analyze disordered magnets in regions of thermodynamic parameters where they are expected to have only finitely many extremal equilibrium states with good cluster decomposition properties.

(A) *The Griffiths singularities* [10]
Let $J_{ij} = 0$ for $|i-j| > 1$, and let J_{ij} be a bounded random variable for $|i-j| = 1$ but $\Delta > 0$. Let

$$\overline{J_{ij}} = 1, \quad \max J_{ij} \equiv J_{\max} \gg 1 \,. \tag{2.16}$$

By ergodicity, there exist ∞ many connected regions, $\Omega(J, \delta, L)$, containing a cube with sides of length $L = 1, 2, 3, \ldots$ in the lattice such that

$$|J_{ij} - J| < \delta \ll 1 \,, \tag{2.17}$$

for any given $\delta > 0$ and $J \in [1 + 2\delta, J_{\max}]$. Let β_0 be the transition point (critical point) in the Ising model with $J_{ij} = 1$, for all nearest-neighbour pairs i, j. If we restrict the system to $\Omega(J, \delta, L)$, choose L very large and let Re β range over the interval $(\beta_0 J^{-1}, \beta_0)$, then the system on $\Omega(J, \delta, L)$ behaves like a low-temperature system, with a susceptibility of order L^ν. In particular, we expect singularities in the complex β-plane and in the complex h-plane close to the real axes and pinching the real axes, as $L \to \infty$. Nevertheless, the entire system in the thermodynamic limit remains in the single-phase region, as long as

$$\beta < \beta_0 \,,$$

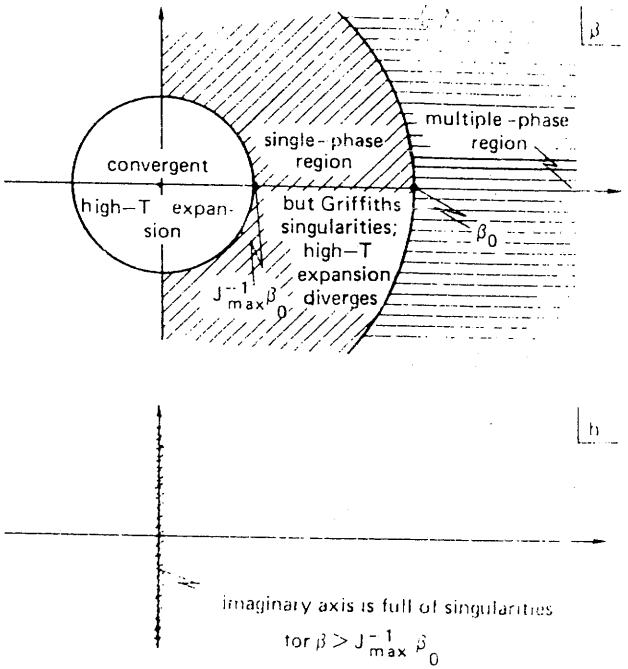

Fig. 2.1

since $\overline{J_{ij}} = 1$ (see sect. 2.2 for a more precise statement). The expected situation is described in fig. 2.1. Expansion in h around $h = 0$ converges for $\beta < J_{\max}^{-1}\beta_0$ and $|h|$ small. For $\beta > J_{\max}^{-1}\beta_0$ the whole imaginary axis in the complex h-plane is covered by singularities, but, for $\beta < \beta_0$, there is no spontaneous magnetization at $h = 0$, and the magnetization is C^∞ at $h = 0$. (Most of this can actually be proven [1, 10].) We also expect Griffiths-type singularities in the random field Ising model in the complex H-plane ($H^2 = h_j^2$) in regions where the expansion of ref. [1] actually converges, i.e. where the equilibrium state is unique and has cluster decomposition properties. The main problem is to develop expansion techniques which converge even in the presence of Griffiths singularities. Solutions of this problem have been found in refs. [1, 2].

(B) *Problems with long-range exchange couplings*
Consider the model described in (5). It is easy to check that under conditions (2.14), individual terms in the high-temperature expansion are all almost surely finite, but to establish almost sure convergence is non-trivial, because of combinatorial problems. The difficulty described here is not purely academic, since, e.g., the Ruderman–Kittel interaction is not absolutely summable. Mathematically, it is somewhat related to, though presumably easier than, the difficulties one meets when one tries to prove convergence of the Mayer expansion for dipole gases which were finally overcome with the help of renormalization group methods [11]. In $\nu = 1$ and 2 dimensions, presumably optimal results are now known.

2.2. *Griffiths' theorem on Griffiths singularities*

We now want to state and prove a rigorous result on the existence of the singularities described in (A) above in a simple example. We let Λ be a periodic box in $\mathbb{Z}^\nu$ (wrapped on a torus). The Hamilton function for a magnet in Λ is chosen to be

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j + h\sum_i \sigma_i\,, \tag{2.18}$$

with

$$J_{ij} = \left\{\begin{array}{l} 0\,, \quad |i-j| > 1\,, \\ 1\,, \quad \text{with probability } p \\ 0\,, \quad \text{with probability } 1-p \end{array}\right\} \quad |i-j| = 1\,, \tag{2.19}$$

or

$$J_{ij} = \begin{cases} J\tau_i\tau_j\,, & \text{for } |i-j| = 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{2.20}$$

with

$$\tau_i = 1\,, \quad \text{with probability } p$$

and

$$\tau_i = 0\,, \quad \text{with probability } 1-p\,.$$

For this system we show that when $\Lambda \nearrow \mathbb{Z}^\nu$ and for $\beta > \beta_0$, but β not too large, the imaginary axis in the complex h-plane contains a singularity at $h = 0$ (and is presumably full of singularities), but the spontaneous magnetization vanishes when $h \to 0$ along the real axis [10]. Moreover, the results of ref. [1] imply that the magnetization can be C^∞ in h at $h = 0$, in this situation.

Calculation of magnetization
To be specific, we choose J_{ij} as in (2.20), but (2.19) can be treated too. We call a site j occupied if and only if $\tau_j = 1$, otherwise it is called empty. Let C denote an arbitrary configuration of occupied sites in Λ (i.e. a family of disjoint, connected clusters of occupied sites). Furthermore,

$|C| \equiv$ number of sites belonging to C,

$P_{C,\Lambda} \equiv$ probability of occurrence of C,

calculated according to (2.20);

$M_\Lambda \equiv$ average magnetization per site in Λ,

$M_C \equiv$ average magnetization per site in C.

Then

$$M_\Lambda = |\Lambda|^{-1}\sum_{C\subseteq\Lambda} |C| P_{C,\Lambda} M_C\,. \tag{2.21}$$

Let

$$z \equiv e^{-2\beta h} \tag{2.22}$$

and let f_C denote the free energy per site of the system restricted to C, i.e.

$$\beta f_C = -|C|^{-1} \log Z_C \,, \tag{2.23}$$

where Z_C is the partition function. Then

$$M_C = \frac{\partial f_C}{\partial h} = \frac{\partial z}{\partial h} \cdot \frac{\partial f_C}{\partial z} = -2\beta z \frac{\partial f_C}{\partial z} \,. \tag{2.24}$$

Writing

$$e^{-\beta h \sigma_i} = e^{\beta h} e^{-\beta h(\sigma_i + 1)}$$

and summing over all values of σ_i, $i \in C$, we see that $e^{-\beta h|C|} Z_C$ *is a polynomial in* z *of degree* $|C|$. Hence

$$Z_C = \text{const.}\ z^{-(|C|/2)} \prod_{\alpha=1}^{|C|} (z - \zeta_\alpha(C)) \,, \tag{2.25}$$

where $\zeta_\alpha(C)$ is the αth zero of Z_C. The Lee–Yang theorem [12] tells us that

$$|\zeta_\alpha(C)| = 1, \quad \text{for all } \alpha \,. \tag{2.26}$$

By (2.23)–(2.25)

$$M_C = 2\beta z \frac{1}{\beta} \left(-\frac{1}{2z} + \frac{1}{|C|} \sum_{\alpha=1}^{|C|} \frac{1}{z - \zeta_\alpha(C)} \right) . \tag{2.27}$$

Inserting (2.27) into (2.21) we find

$$\begin{aligned} M_\Lambda &= -|\Lambda|^{-1} \sum_{C \subseteq \Lambda} |C| P_{C,\Lambda} \left(1 - 2z|C|^{-1} \sum_{\alpha=1}^{|C|} (z - \zeta_\alpha(C))^{-1} \right) \\ &= -p + 2z \sum_{\alpha=1}^{N_\Lambda} \eta_\alpha(\Lambda)(z - \zeta_\alpha)^{-1} \,, \end{aligned} \tag{2.28}$$

where p is as in (2.20), N_Λ is a finite integer, and

$$\eta_\alpha(\Lambda) = |\Lambda|^{-1} \left(\sum_{\substack{C:\ \zeta_\alpha(C) = \zeta_\alpha \\ \text{for some } \alpha}} m_\alpha(C)\, P_{C,\Lambda} \right) > 0 \,, \tag{2.29}$$

where $m_\alpha(C)$ is the number of times ζ_α occurs in $\{\zeta_\alpha(C)\}_{\alpha=1}^{|C|}$. It follows that

$$\begin{aligned} \sum_{\alpha=1}^{N_\Lambda} \eta_\alpha(\Lambda) &= |\Lambda|^{-1} \sum_{C \subseteq \Lambda} \left(\sum_{\alpha=1}^{|C|} P_{C,\Lambda} \right) \\ &= \sum_{C \subseteq \Lambda} |\Lambda|^{-1} |C| P_{C,\Lambda} \\ &= p \,. \end{aligned} \tag{2.30}$$

Proof of the main theorem. From (2.28) and (2.30) we conclude that, for $|z| \neq 1$, $M_\Lambda(z)$ is *bounded uniformly in* Λ. Moreover, for positive $z \neq 1$, $M_\Lambda(z)$ converges to a limit, as $\Lambda \nearrow \mathbb{Z}^\nu$, because the thermodynamic limit of the quenched free energy, $f_\Lambda(z)$, with periodic b.c. exists, for positive z, and

$$M_\Lambda(z) = -2\beta z \frac{\partial f_\Lambda}{\partial z} \,, \quad \text{for positive } z \neq 1 \,.$$

Thus, by Vitali's theorem,

$$M_\Lambda(z) \to M(z), \quad \text{as } \Lambda \nearrow \mathbb{Z}^\nu, \text{ for all } z, \text{ with } |z| \neq 1 \,,$$

where M is the magnetization in the thermodynamic limit, and $f_\Lambda(z)$ converges, as $\Lambda \nearrow \mathbb{Z}^\nu$, for all z, with $|z| \neq 0, 1$. Let $\beta_0 \equiv \beta_0(p = 1)$ be the transition point of the ν-dimensional (pure) Ising model.

Theorem 2.1 [10]. *For* $\beta > \beta_0$, $M(z)$ *cannot be continued analytically from* $\{z: |z| > 1\}$ *to* $\{z: |z| < 1\}$, *or conversely, (along the real axis).*

Proof. Assume the contrary. Then there is some real $x > 1$ and a $\rho > 0$, with $x - \rho < 1$, such that the Taylor series of $M(z)$ around $z = x$ has radius of convergence ρ, and the disc of convergence of that series contains an arc A,

$$A = \{z = e^{i\phi} : |\phi| < \delta\} ,$$

for some $\delta > 0$ (see fig. 2.2). Given $\beta > \beta_0$ and $\delta > 0$, we may choose Λ so large that A contains some singularities, ζ_a, of $M_\Lambda(z)$ (because for every $\varepsilon > 0$ there is a connected cluster, $\mathfrak{C}$, not winding around Λ, with the property that $M_{\mathfrak{C}}(z)$ has a singularity at some point ζ, with $|\zeta - 1| < \varepsilon$). The residue, η_a, of the pole at ζ_a, is given by (2.29). Let $\mathfrak{C}_a$ be some connected cluster of sites in Λ not winding around Λ and such that $M_{\mathfrak{C}_a}(z)$ has a singularity at ζ_a, i.e. ζ_a is a zero of $Z_{\mathfrak{C}_a}$. Let Ω be the smallest cube properly containing $\mathfrak{C}_a$, and suppose that Λ is a union of n disjoint translates, $\Omega(x)$ $(x \in \Lambda)$, of Ω. Then

$$\eta_a(\Lambda) \geq |\Lambda|^{-1} \sum_{\substack{x_1, \ldots, x_k \\ k = 1, \ldots, n}} \sum_{\substack{C:\ C \cap \Omega(x_i) \supseteq \mathfrak{C}_a(x_i) \\ i = 1, \ldots, k}}' k P_{C,\Lambda} ,$$

where $\mathfrak{C}_a(x)$ is obtained by translating $\mathfrak{C}_a$ by x. Now

$$\sum{}' P_{C,\Lambda} = P_{\mathfrak{C}_a}^k (1 - P_{\mathfrak{C}_a})^{n-k} ,$$

where Σ' ranges over all configurations C, such that

$$C \cap \Omega(x_i) \supseteq \mathfrak{C}_a(x_i), \quad \text{for } i = 1, \ldots, k ,$$

$$C \cap \Omega(x) \not\supseteq \mathfrak{C}_a(x), \quad \text{for } x \notin \{x_1, \ldots, x_k\} ,$$

and $x_1, \ldots, x_k$ are given, and

$$P_{\mathfrak{C}_a} = p^{|\mathfrak{C}_a|} (1 - p)^{|\partial \mathfrak{C}_a|} ,$$

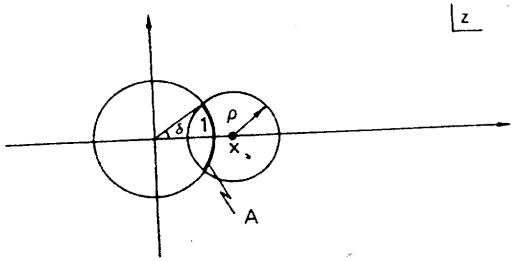

Fig. 2.2.

where $\partial \mathfrak{C}_a$ is the set of sites in Ω within distance 1 from $\mathfrak{C}_a$ and not contained in $\mathfrak{C}_a$. The theory of the Ising model at $\beta = 0$ thus shows that

$$\eta_a(\Lambda) \geq \frac{n}{|\Lambda|} P_{\mathfrak{C}_a} = |\Omega|^{-1} P_{\mathfrak{C}_a} \equiv P_a > 0 , \tag{2.31}$$

uniformly in Λ.

Lemma 2.2.

$$|M_\Lambda(r\zeta_a)| \geq 2 \frac{\eta_a(\Lambda)}{r - 1} \geq 2 \frac{P_a}{r - 1} ,$$

for $r > 1$, *uniformly in* Λ.

Proof. Set $\zeta_a = e^{i\phi_a}$ and meditate the situation of fig. 2.3. We set $r_{ab} e^{i\phi_{ab}} \equiv r\zeta_a - \zeta_b$. By convexity of the unit disk,

$$|\phi_{ab} - \phi_a| < \frac{\pi}{2} , \quad \text{for } r > 1 . \tag{2.32}$$

Hence, since $|u| \geq \mathrm{Re}(u e^{i\psi})$, for arbitrary $u \in \mathbb{C}$ and $\psi \in \mathbb{R}$,

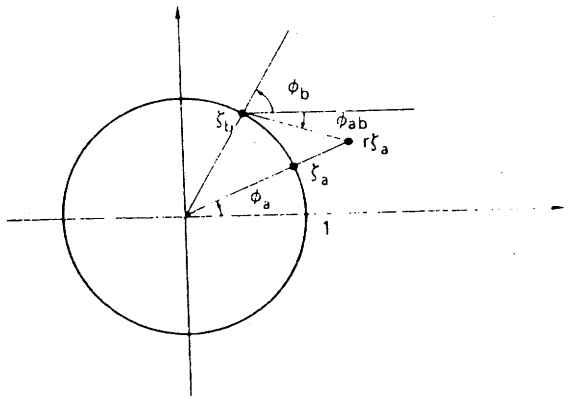

Fig. 2.3.

$$|M_\Lambda(r\zeta_a)| \geq 2r\,\mathrm{Re}((2r\zeta_a)^{-1}M_\Lambda(r\zeta_a)\,e^{i\phi_a})$$

$$= 2r\left(\frac{\eta_a(\Lambda)}{r-1} + \sum_{b\neq a}\frac{\eta_b(\Lambda)}{r_{ab}}\cos(\phi_a - \phi_{ab})\right)$$

$$\geq 2r\,\frac{\eta_a(\Lambda)}{r-1}, \quad \text{by (2.29) and (2.32)},$$

$$\geq 2\,\frac{P_a}{r-1}, \quad \text{by (2.31)}. \qquad \text{Q.E.D.}$$

Hence $M(r\zeta_a)$ diverges to $+\infty$, as $r \searrow 1$. This contradicts our assumption that $M(z)$ is holomorphic in some neighbourhood of the arc A introduced at the start of the proof of the theorem. By the $(h \to -h \Leftrightarrow z \to z^{-1})$-symmetry of the model, $M(z^{-1}) = -M(z)$, and hence $M(z)$ has no analytic continuation from $\{z: |z|<1\}$ to $\{z: |z|>1\}$ along the real axis, either. The proof of the theorem is thus complete.

Remarks

(1) For another rough sketch of Griffiths' result [10] and proof, see also ref. [13]. We emphasize that the above theorem proves that $M(z)$ has a *singularity at* $z = 1$. However, for small enough p and $\beta > \beta_0$, the results of ref. [1] prove that $M(z)$ *is* C^∞ *in* z, *for positive* z, *even* at $z = 1$.

(2) Isakov [14] has recently proved that in the pure Ising model, for $\beta > \beta_0$,

$$(n!)^{-1}\left(\frac{\partial^n}{\partial h^n} f\right)(h) = (n!)^{1/\nu - 1}\,\text{const.}^n .$$

(3) The above proof clearly extends to a wide variety of boundary conditions and other, more complicated distributions, $d\rho(J_{ij})$, with support on $[0, \infty)$. As an example, we choose

$$d\rho(J) = \left(\rho_0\delta(J) + \sum_{\alpha=1}^{\infty}\rho_\alpha\delta(J - J^\alpha)\right)dJ ,$$

with $0 < J_\alpha \nearrow +\infty$, as $\alpha \nearrow \infty$, and $\rho_\alpha > 0$, for all α, $\Sigma_{\alpha=0}^{\infty}\rho_\alpha = 1$. In this model $M(z)$ has a singularity at $z = 1$, for *all* values of β. But the results in ref. [1] show that $M(z)$ is C^∞ for positive z, even at $z = 1$, if β is small enough.

(4) It would be interesting to know whether $M(z)$ is quasi-analytic at $\{z: |z| = 1\}$.

Next we propose to estimate the transition points, $\beta_c(p)$, of dilute ferromagnets, such as those defined in (2.19), (2.20). [We characterize $\beta_c(p)$ by the property that $\lim_{z\downarrow 1} M(z) > 0$, for $\beta > \beta_c(p)$. Clearly $\beta_c(p = 1) = \beta_0$.] The following lemma appears in ref. [15]:

Lemma 2.3. *The equilibrium expectation,* $\langle\sigma_0\rangle_\beta \equiv \langle\sigma_0\rangle_\beta(J, h)$, *of the spin at the origin is separately concave in each* J_{ij}.

Proof.

$$\frac{\partial\langle\sigma_0\rangle_\beta}{\partial J_{ij}} = \langle\sigma_0\sigma_i\sigma_j\rangle_\beta - \langle\sigma_0\rangle_\beta\langle\sigma_i\sigma_j\rangle_\beta ,$$

hence

$$\frac{\partial^2\langle\sigma_0\rangle_\beta}{\partial J_{ij}^2} = \langle\sigma_0(\sigma_i\sigma_j)^2\rangle_\beta - \langle\sigma_0\sigma_i\sigma_j\rangle_\beta\langle\sigma_i\sigma_j\rangle_\beta$$

$$- \{\langle\sigma_0\sigma_i\sigma_j\rangle_\beta - \langle\sigma_0\rangle_\beta\langle\sigma_i\sigma_j\rangle_\beta\}\langle\sigma_i\sigma_j\rangle_\beta$$

$$- \langle\sigma_0\rangle_\beta\{\langle(\sigma_i\sigma_j)^2\rangle_\beta - \langle\sigma_i\sigma_j\rangle_\beta^2\}$$

$$= -2\langle\sigma_i\sigma_j\rangle_\beta\{\langle\sigma_0\sigma_i\sigma_j\rangle_\beta - \langle\sigma_0\rangle_\beta\langle\sigma_i\sigma_j\rangle_\beta\}$$

$$\leq 0 ,$$

by the first and second Griffiths inequality. (We have used the fact that $(\sigma_i\sigma_j)^2 = 1$!) Q.E.D.

This lemma permits us to apply Jensen's inequality when integrating $\langle\sigma_0\rangle_\beta(J, h)$ over J_{ij}, for arbitrary $\langle i, j\rangle$. Thus

$$\int\prod_{\langle i,j\rangle} d\rho(J_{ij})\,\langle\sigma_0\rangle_\beta(J, h) \leq \langle\sigma_0\rangle_\beta\left(\int\prod(d\rho(J_{ij})\,J_{ij}), h\right)$$

$$= \langle\sigma_0\rangle_\beta(\bar{J}, h) , \qquad (2.33)$$

where $\bar{J}$ is the mean of J_{ij}, and $\bar{J} = p$, in example (2.19). The right-hand side of (2.33) is the magnetization in a pure Ising model at temperature $\beta\bar{J}$. A simple application of the ergodic theorem shows that the