Unexpected behavior in out-of-equilibrium dynamics of mean field spin glasses

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in collaboration with
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Out of equilibrium dynamics in 2 mean-field spin glass models

- **Spherical mixed p-spin model**:
  - starting thermalized at a finite T′
  - T=0 relaxation dynamics
  - analytical solution with N=∞
  - Monte Carlo for small sizes

- **Viana-Bray model (SG on a RRG)**:
  - starting from a random configuration
  - Glauber dynamics at $T<T_c$
  - Monte Carlo with huge sizes and times

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Giampaolo Folena's PhD thesis (co-supervised by S. Franz and me)

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Billoire (SK), Bernaschi (GPU), Maiorano, Parisi

Main (unexpected) results

- **Spherical mixed p-spin model**:
  - $T=0$ dynamics goes below the threshold energy!
  - positive correlation with the initial configuration
  - complexity gives only a qualitative explanation of the asymptotic dynamics

- **Viana-Bray model (SG on a RRG)**:
  the dynamics with $T<T_c$ does not decorrelate from the initial <u>random</u> configuration

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~~Weak ergodicity breaking~~

Strong ergodicity breaking

Definition of models

- Fully connected spherical mixed p-spin model

$$H(\underline{\sigma}) = \sum_p c_p \sum_{i \leq i_1 < \ldots < i_p \leq N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p}$$

$$\underline{\sigma} \in \mathbb{R}^N : \sum_i \sigma_i^2 = N$$

$$\overline{J_p} = 0 \quad \overline{J_p^2} = \frac{p!}{2N^{p-1}}$$

$$\frac{1}{N} \overline{H(\underline{\sigma}) H(\underline{\tau})} = \frac{1}{2} \sum_p c_p^2 \, q_{\sigma\tau}^p \equiv f(q_{\sigma\tau}) \quad q_{\sigma\tau} = \frac{1}{N} \sum_i \sigma_i \tau_i$$

$$f(q) = \frac{q^3}{2} \quad \text{(pure)} \qquad f(q) = \frac{q^3 + q^4}{2} \quad \text{(mixed)}$$

admit an analytical solution

Definition of models

- Ising spin glass on a sparse random graph (SK-like)

$$H(\underline{\sigma}) = \sum_{(ij)\in E} J_{ij}\sigma_i\sigma_j$$

$$\underline{\sigma} \in \{-1, 1\}^N$$

$$P(J) = \frac{1}{2}\delta(J-1) + \frac{1}{2}\delta(J+1)$$

E is the edgeset of a random 4-regular graph

better for numerical simulations

Different universality classes

- Fully connected spherical SG models with $p>2$ undergo a **discontinuous** Random First Order Transition (RFOT)
- Ising SG models with pairwise interactions undergo a **continuous** Spin Glass transition
- ...but similar unexpected results!

Dynamical mean-field theory
(exact for fully-connected weakly interacting models)

- Langevin dynamics at T starting thermalized at T′

$$\dot{\sigma}_i(t) = -\frac{\partial H}{\partial \sigma_i}(\underline{\sigma}(t)) - \mu(t)\sigma_i(t) + \xi_i(t) + h_i(t)$$

$$\langle \xi_i(t)\xi_j(t')\rangle = 2T\delta_{ij}\delta(t-t') \qquad \underline{\sigma}(0) \sim \exp[-H(\underline{\sigma}(0))/T']$$

- Large N limit first

$$C(t,t') \equiv \frac{1}{N}\overline{\langle \underline{\sigma}(t)\cdot\underline{\sigma}(t')\rangle}$$

$$R(t,t') \equiv \frac{1}{N}\sum_{i=1}^{N} \left.\frac{\delta\overline{\langle\sigma_i(t)\rangle}}{\delta h_i(t')}\right|_{h=0}$$

Dynamical mean-field theory

(exact for fully-connected weakly interacting models)

- Closed set of equations in C(t,t') and R(t,t')

$$
\begin{aligned}
\partial_t C(t,t') =& -\mu(t)C(t,t') + 2TR(t',t) + \int_0^t ds f''(C(t,s))R(t,s)C(s,t') \\
& + \int_0^{t'} ds f'(C(t,s))R(t',s) + f'(C(t,0))C(t',0)/T' \\
\partial_t R(t,t') =& -\mu(t)R(t,t') + \delta(t-t') + \int_{t'}^t ds f''(C(t,s))R(t,s)R(s,t') \\
\mu(t) \equiv & \, T + \int_0^t ds f''(C(t,s))R(t,s)C(t,s) \\
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Smooth T dependence
Can be set to T=0

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Smooth T dependence Can be set to T=0

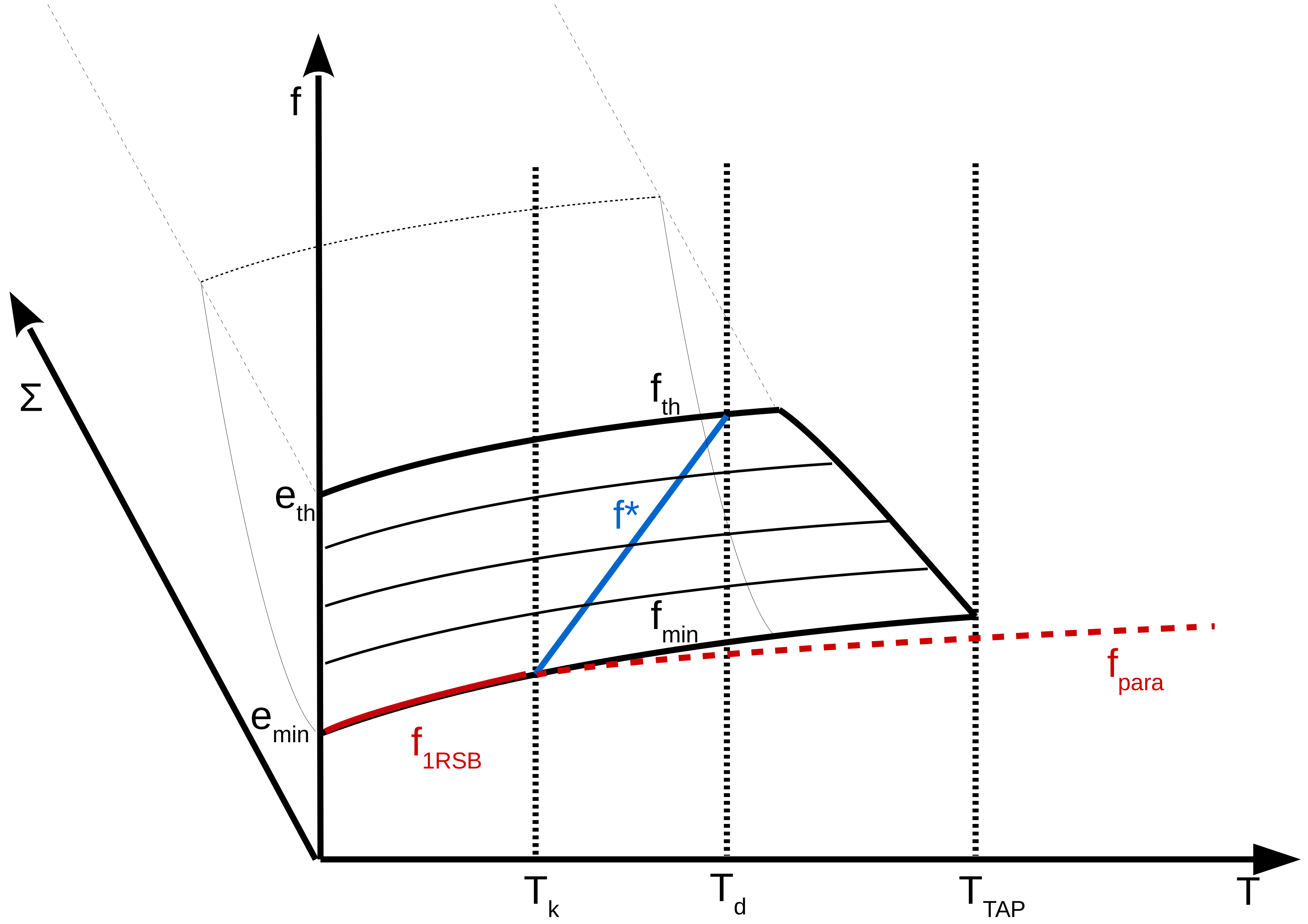
Dependence on the initial condition at T'

Cugliandolo-Kurchan solution

- Aging solution for $T'=\infty$ and $T<T_d$
  - weak long term memory
  - weak ergodicity breaking: $\lim_{t\to\infty} C(t,t') = 0 \quad \forall t'$
  - $C(t,t')$ plateau value is marginal: $f''(q_m)(1-q_m)^2 = T^2$
  - energy relaxes to threshold energy
  - modified fluctuation-dissipation relation

$$TR(t,t') = X[C(t,t')]\frac{\partial C(t,t')}{\partial t'}$$

Static-dynamic connection



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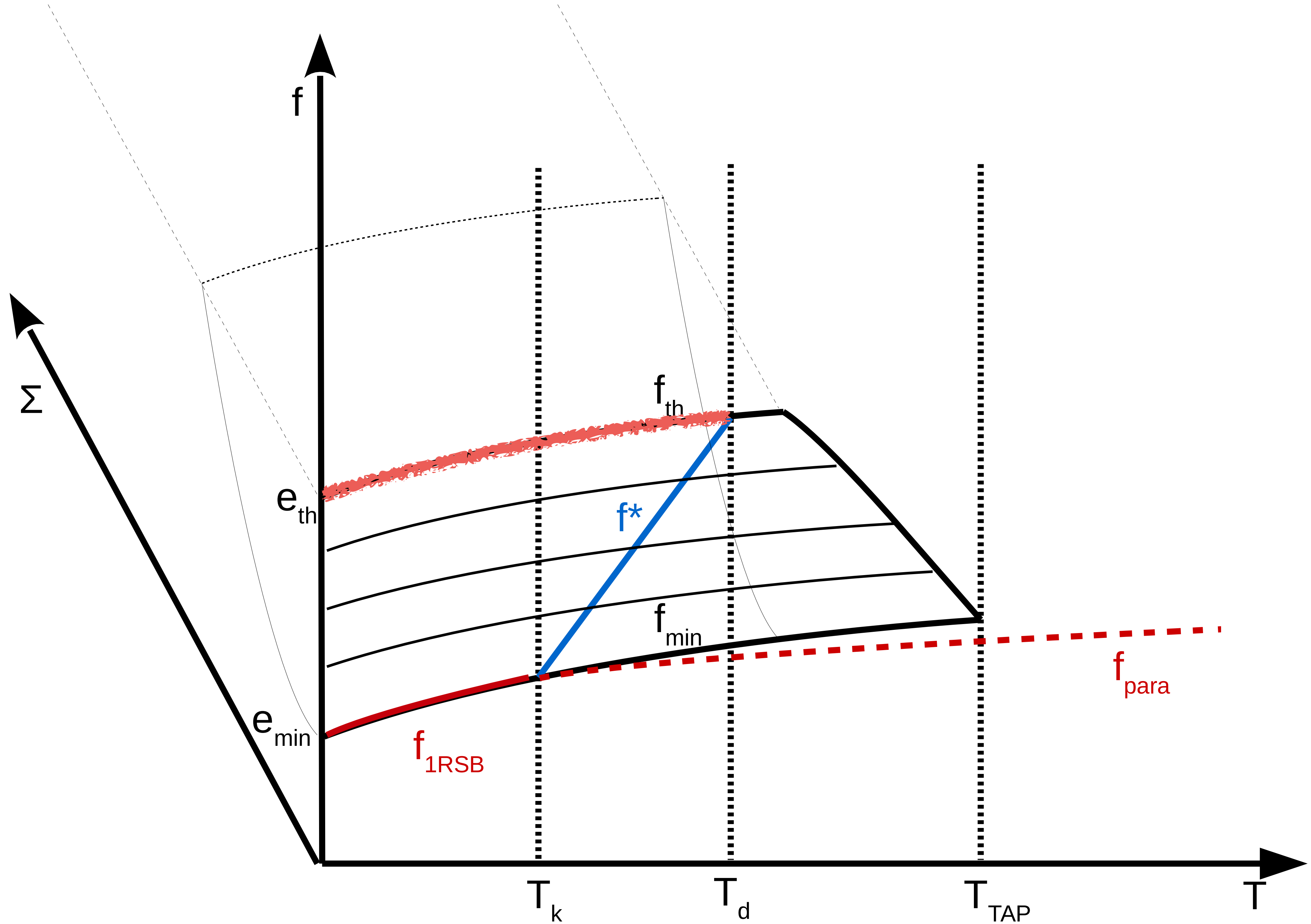

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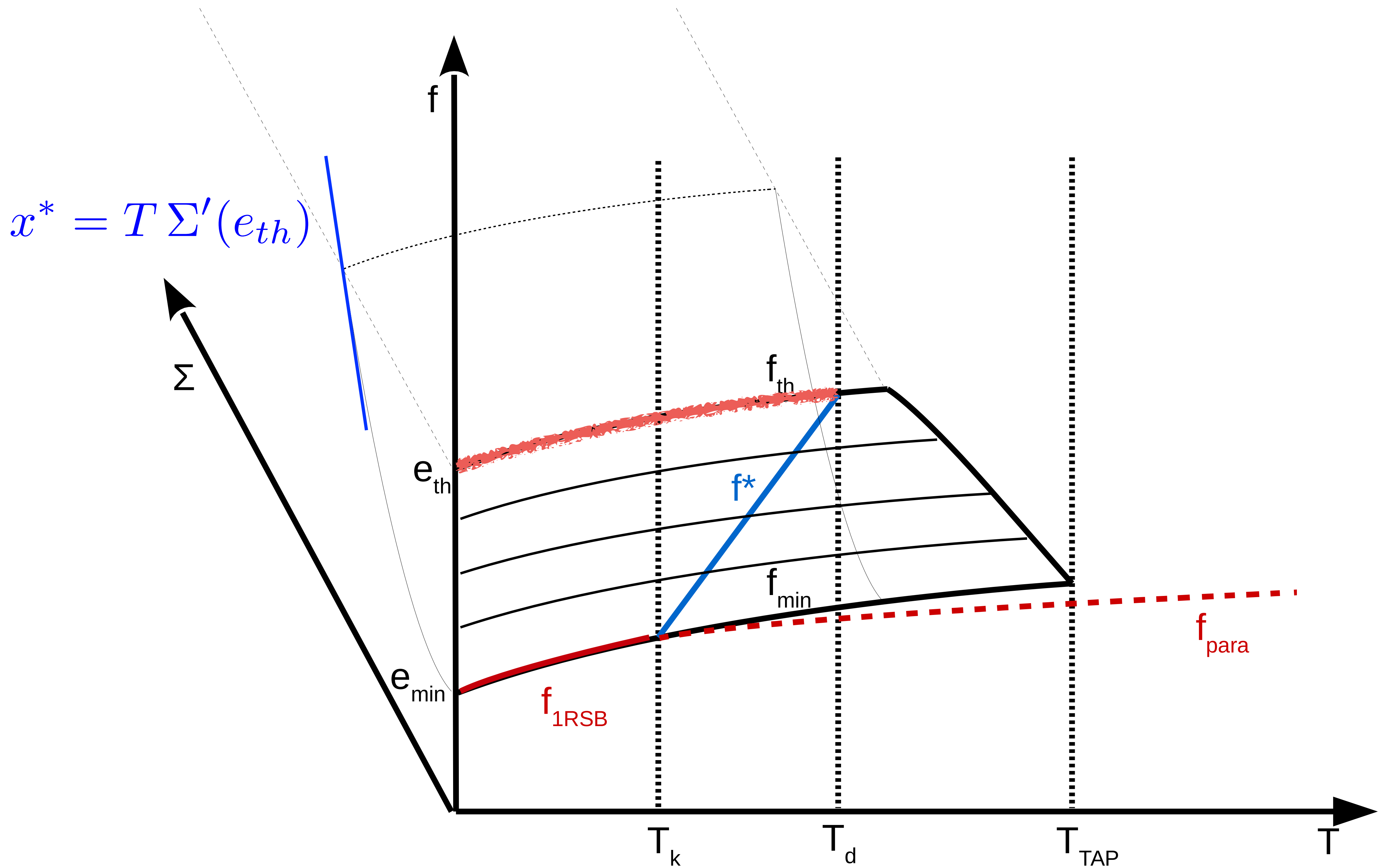

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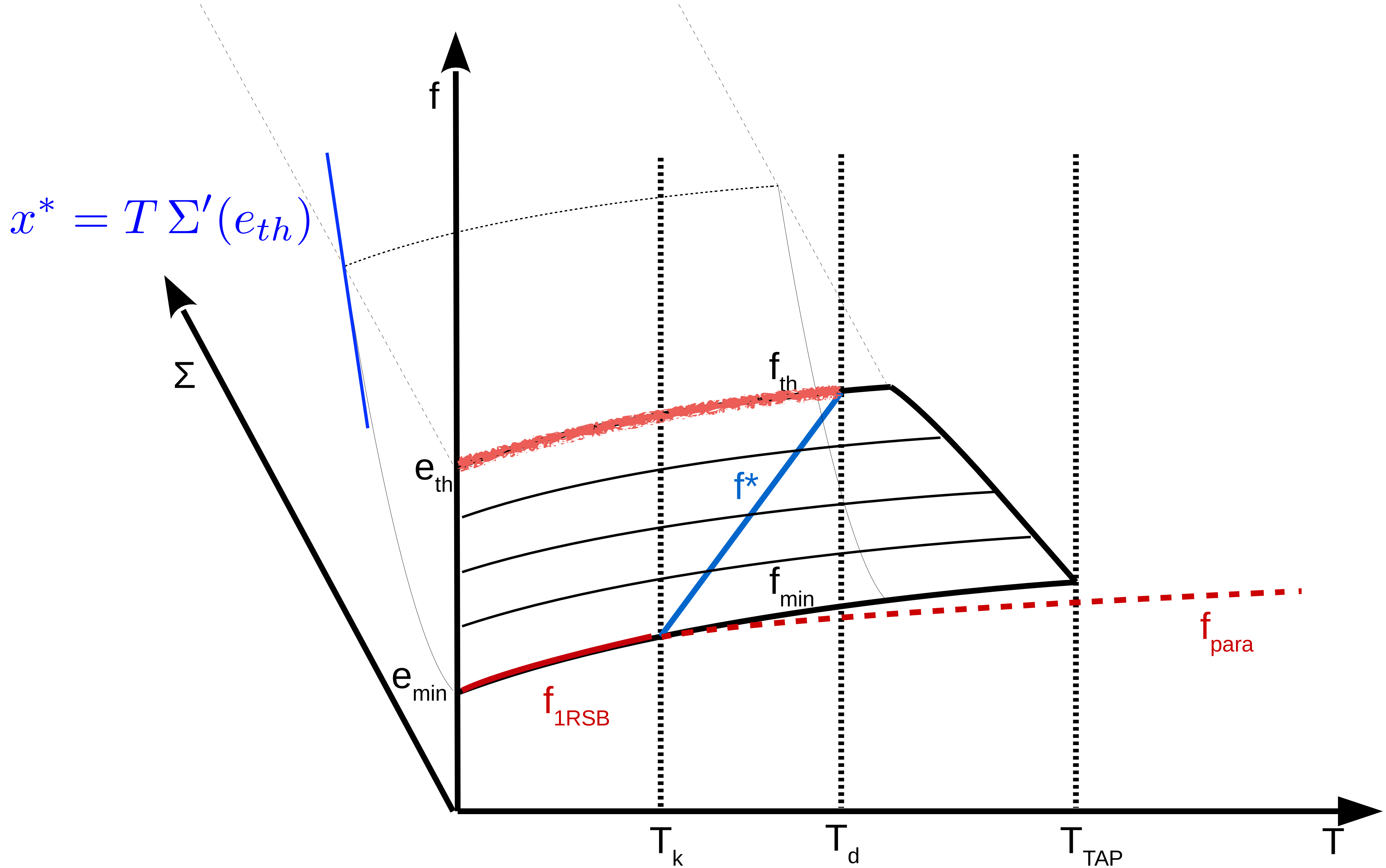


N.B. the mixed model has chaos in T

What happens for finite T' ?

- Simplest guess: all $T' > T_d$ are equivalent (ergodic phase)
- Asymptotic dynamics <-> FP potential saddle point

$$V(T|p,T') = \overline{\frac{-T}{Z(T')} \sum_{\underline{\sigma}} e^{-H(\underline{\sigma})/T'} \log\left[\sum_{\underline{\tau}} e^{-H(\underline{\tau})/T} \delta(pN - \underline{\sigma} \cdot \underline{\tau})\right]} - F(T)$$

$$p = \lim_{t\to\infty} C(t,0)$$

$$\mathcal{C}(\lambda) = \lim_{t\to\infty} C(t,\lambda t)$$

$$q_0 = \lim_{\lambda\to 0} \mathcal{C}(\lambda)$$

$$q_1 = \lim_{\lambda\to 1} \mathcal{C}(\lambda)$$

$$x = \mathcal{R}(\lambda)/\mathcal{C}'(\lambda)$$

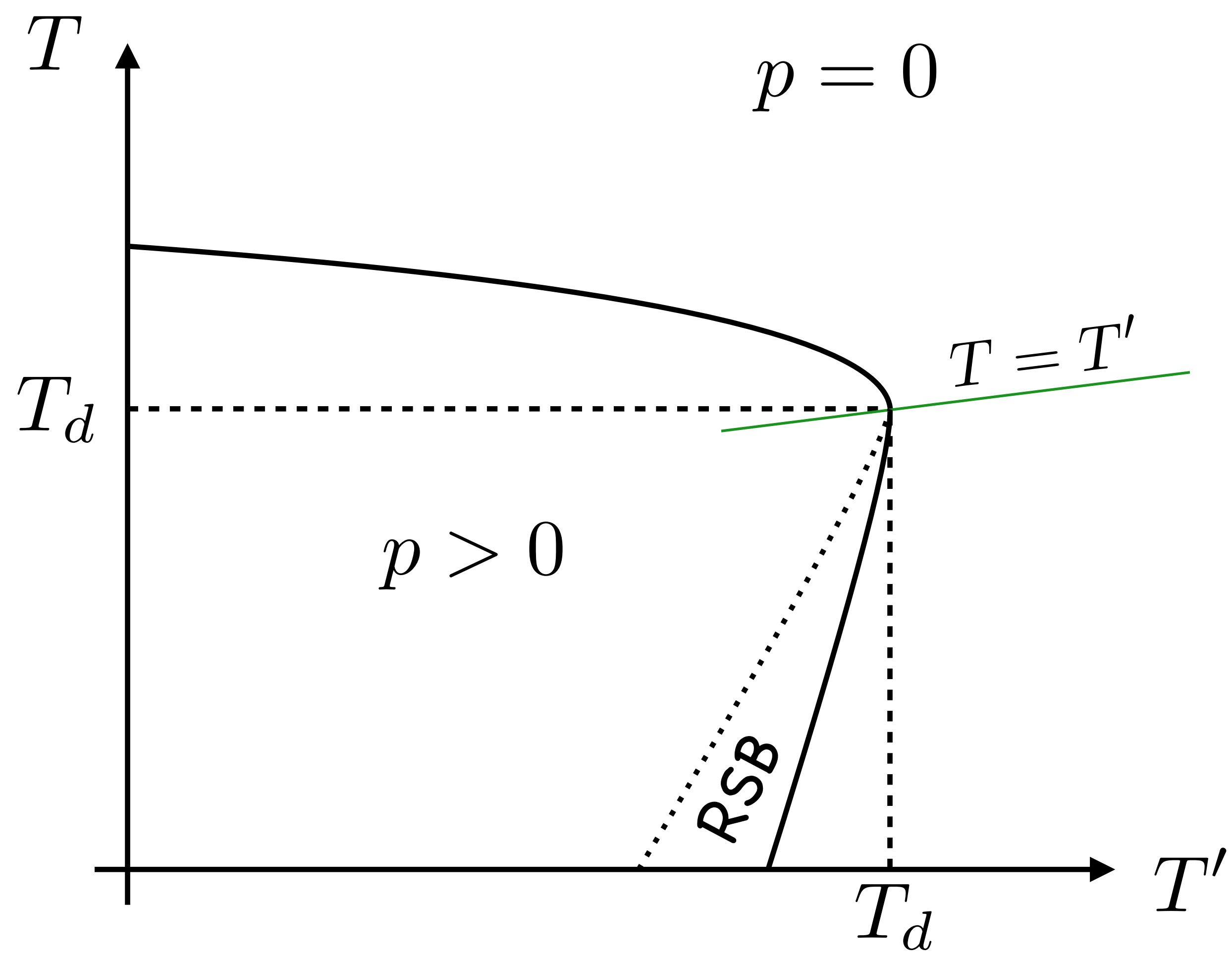
$$q_1 = q_m$$

$$0 = \partial_p V$$

$$0 = \partial_{q_0} V$$

$$0 = \partial_{q_1} V$$

What happens for finite T′ ?



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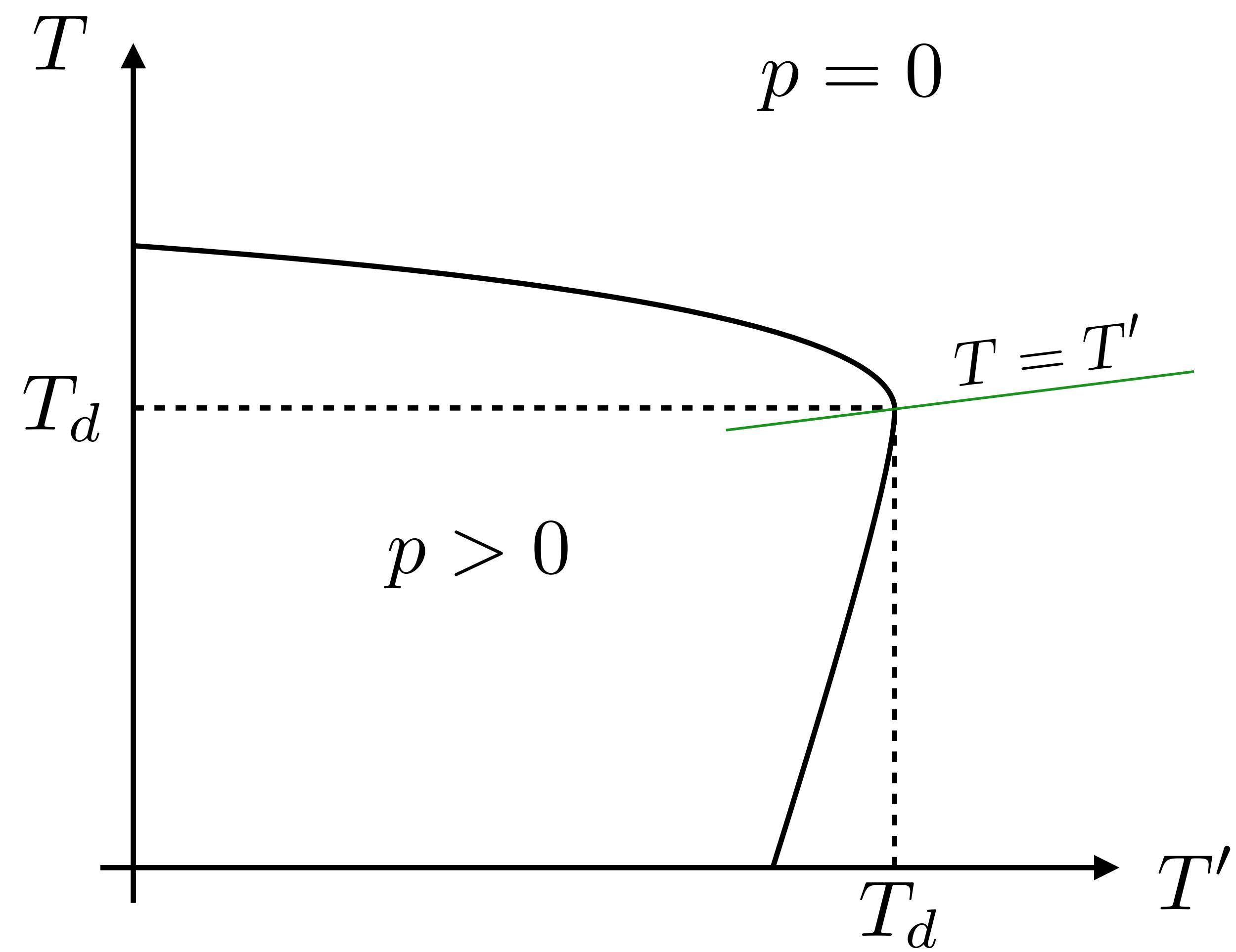

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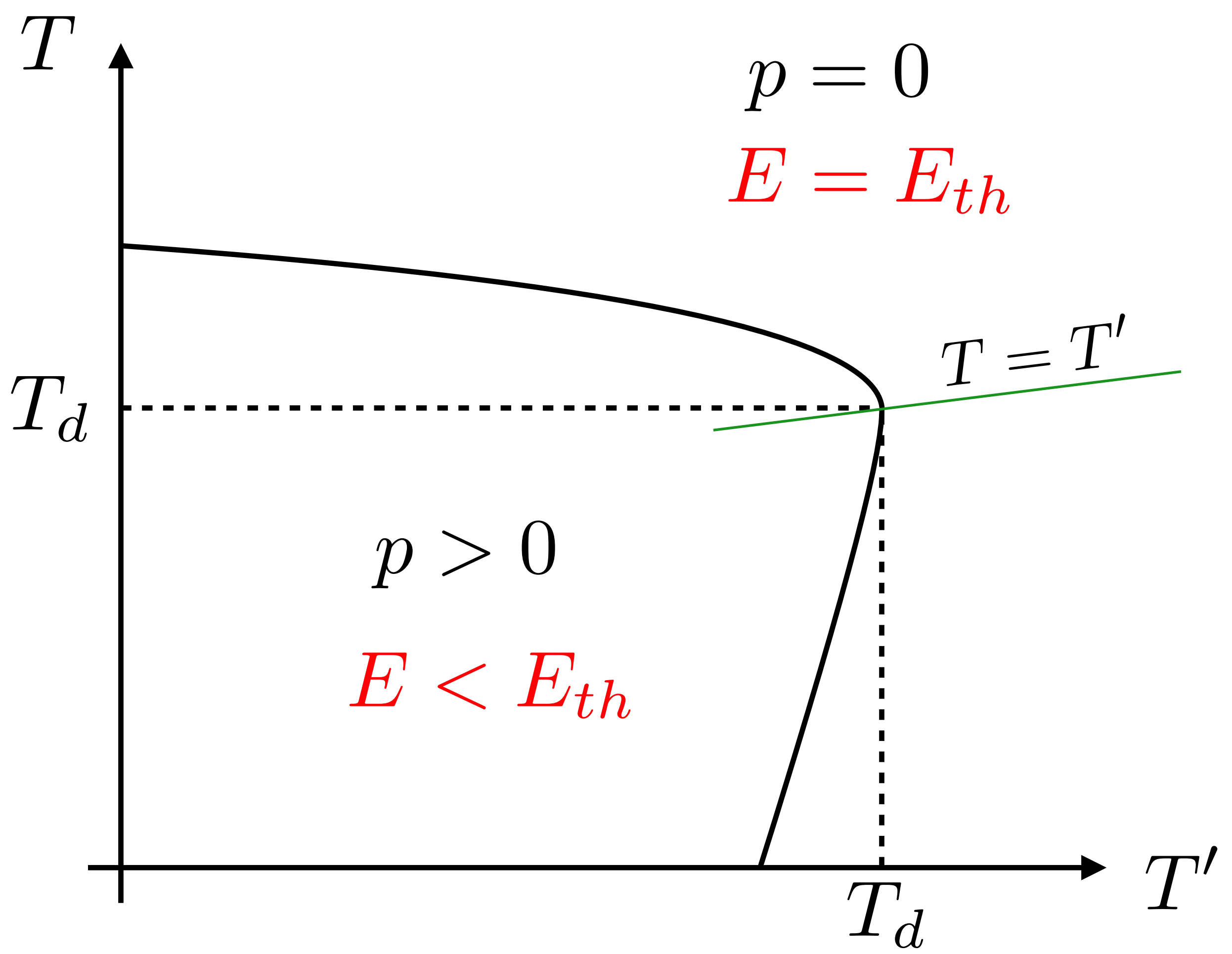

$$E = E_{th} - \beta' f(p) + \beta x f(q_0)$$

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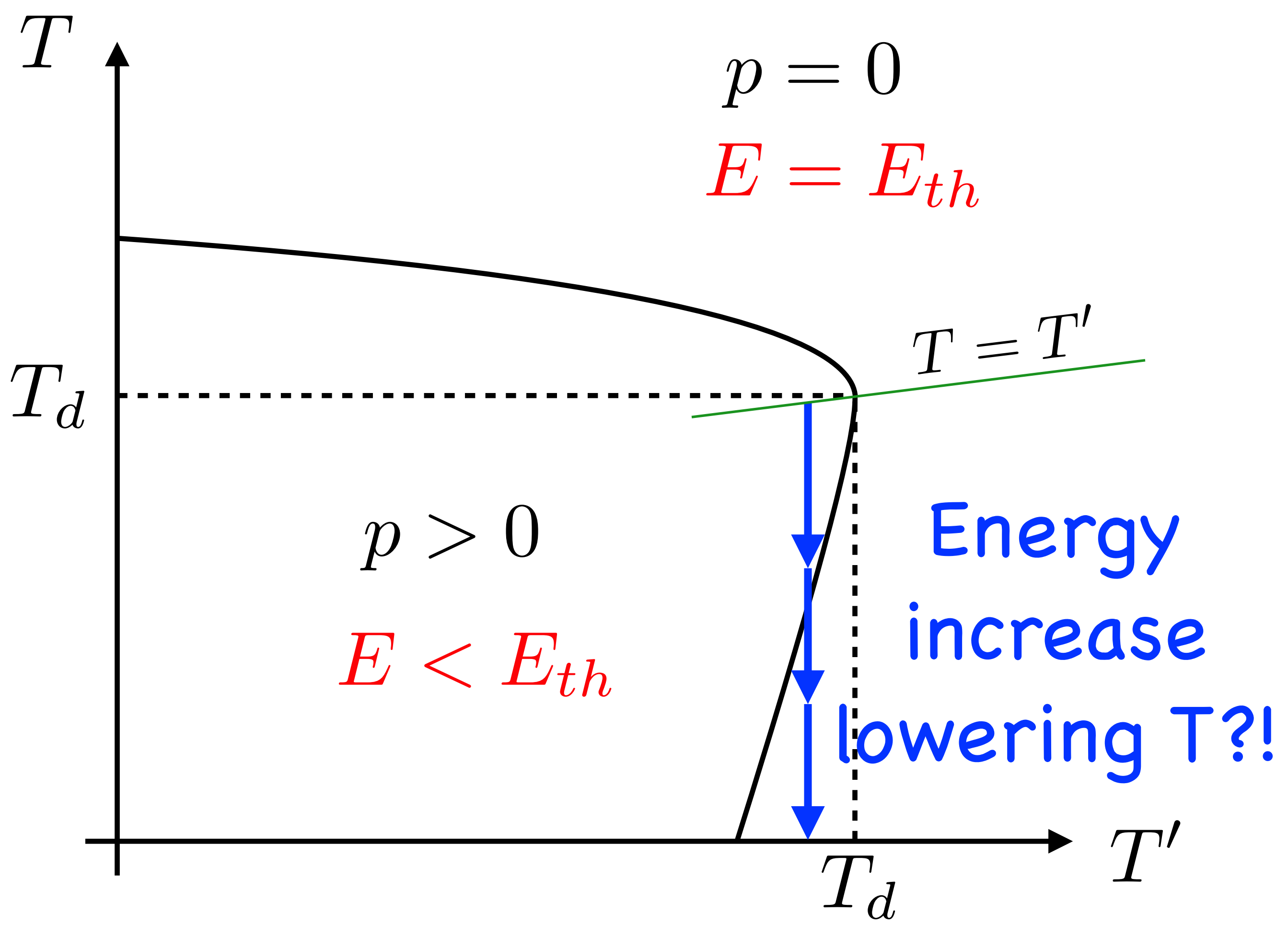

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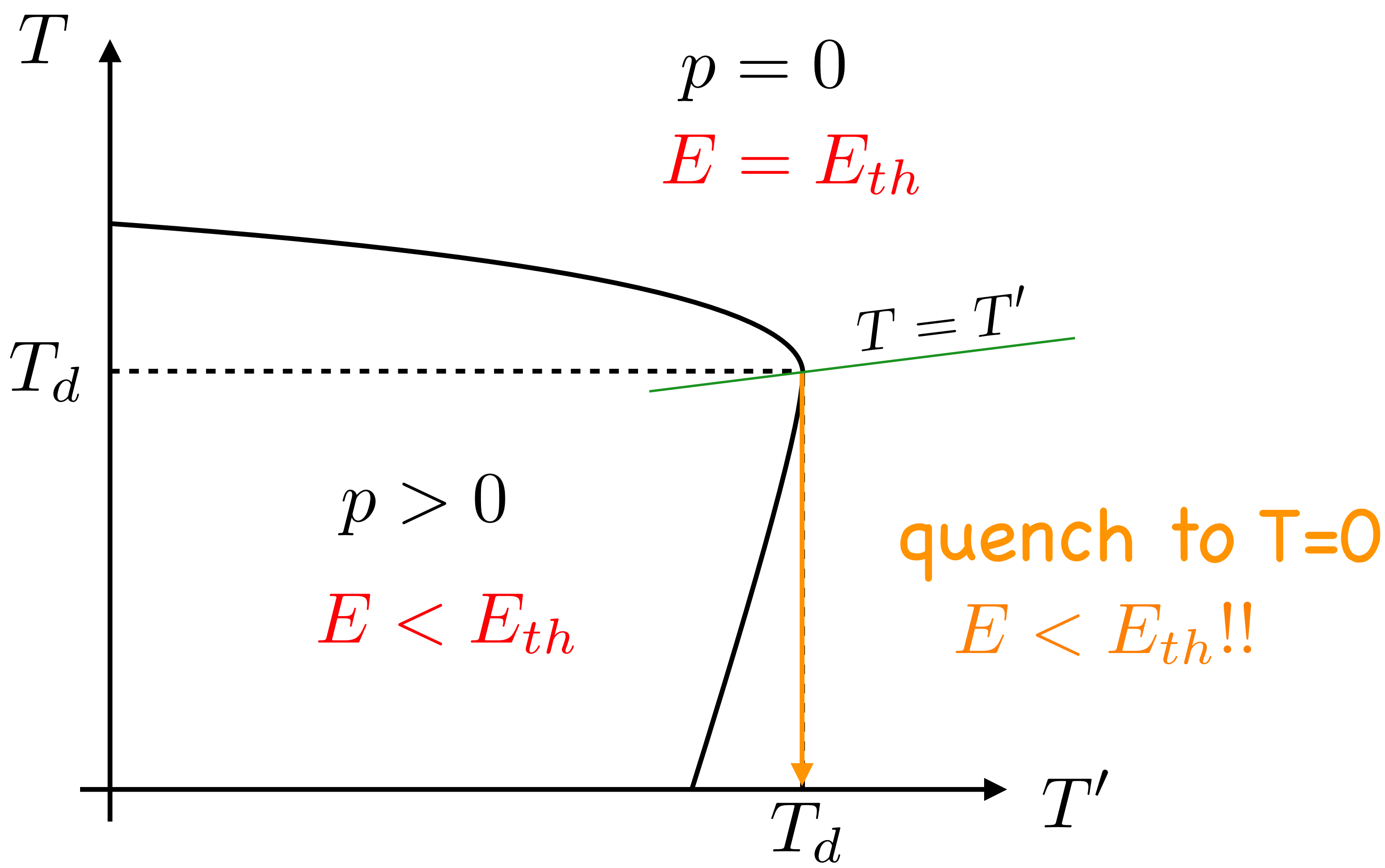

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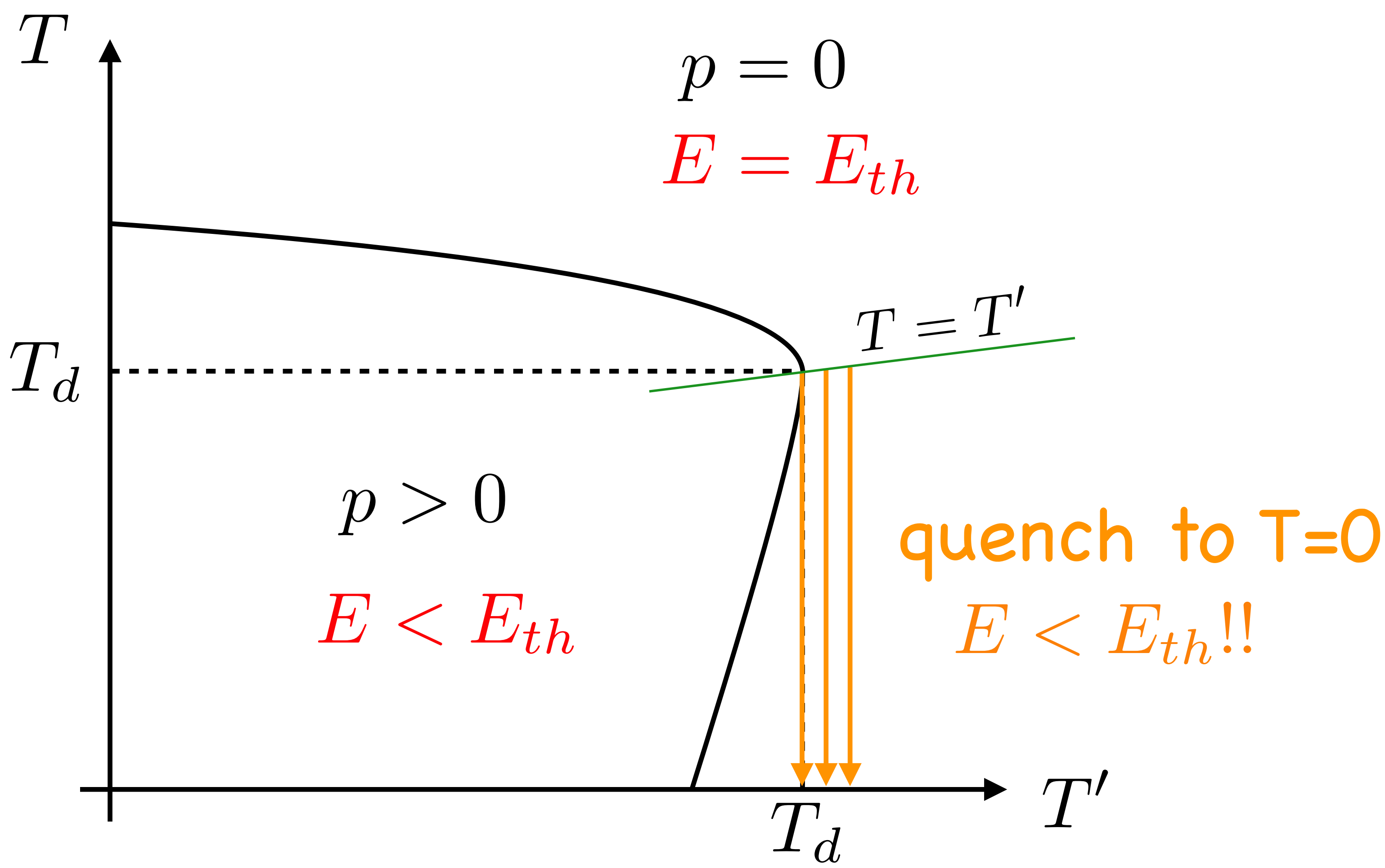

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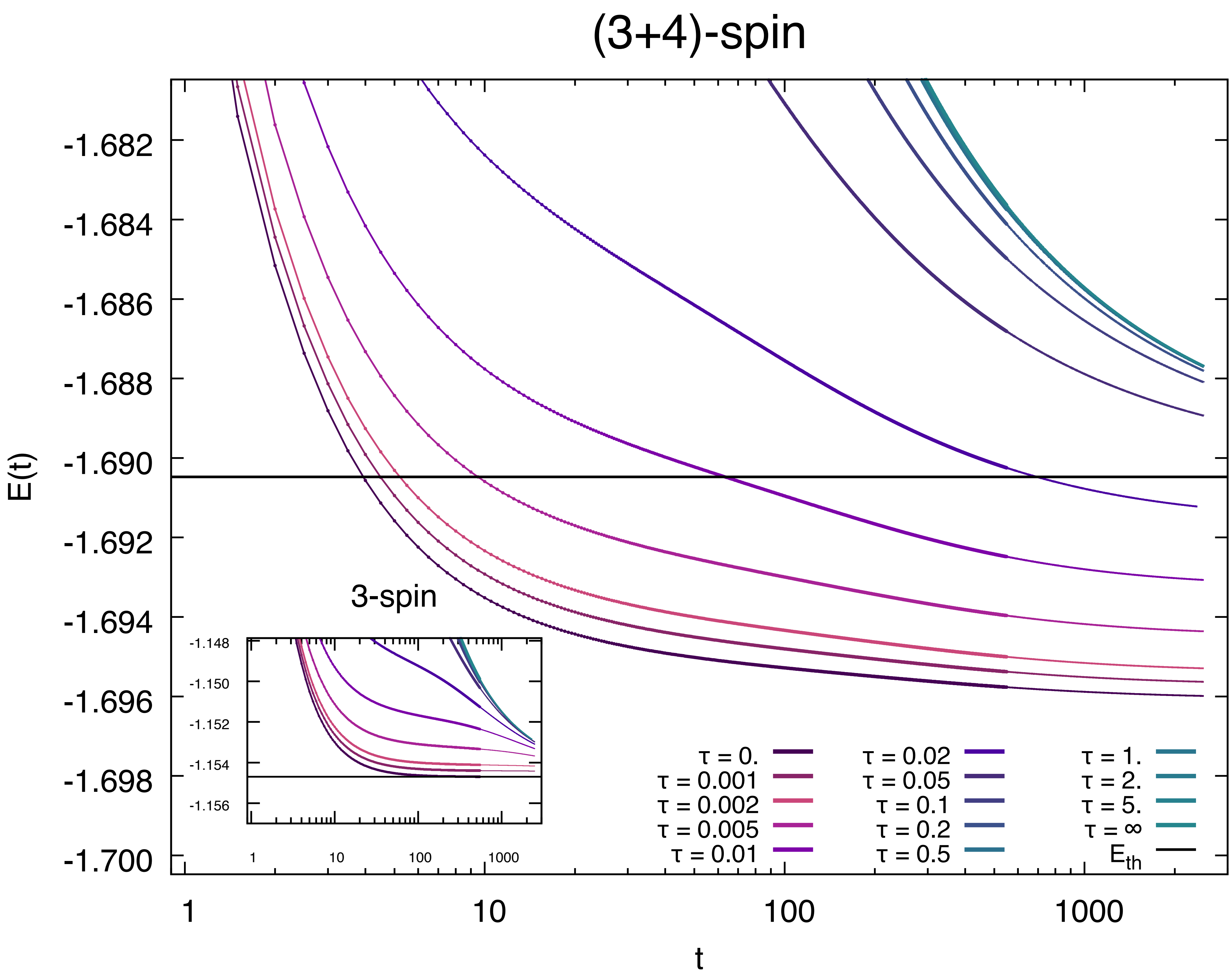
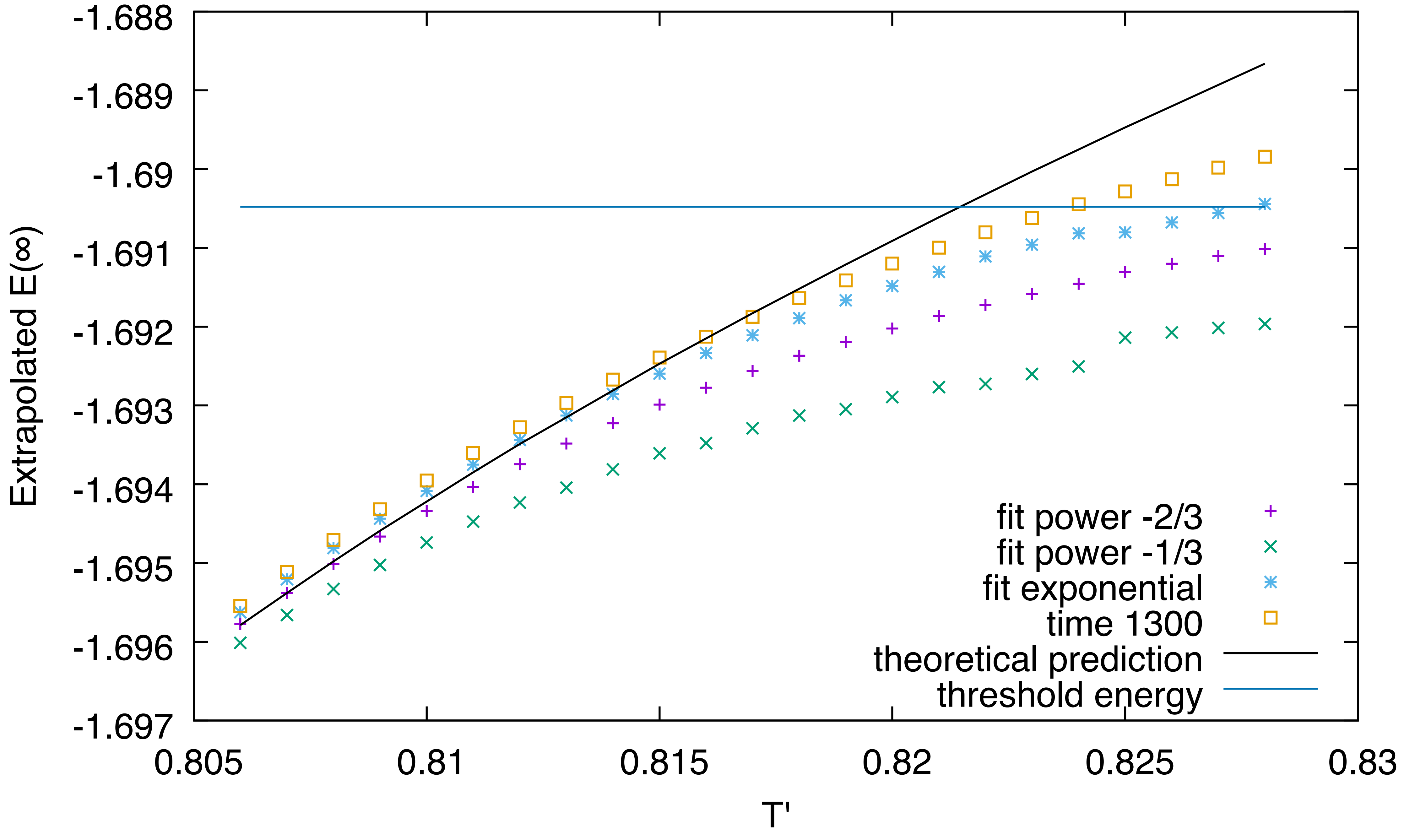
Quenches from T' to T=0



$$\tau = \frac{T' - T_d}{T_d}$$

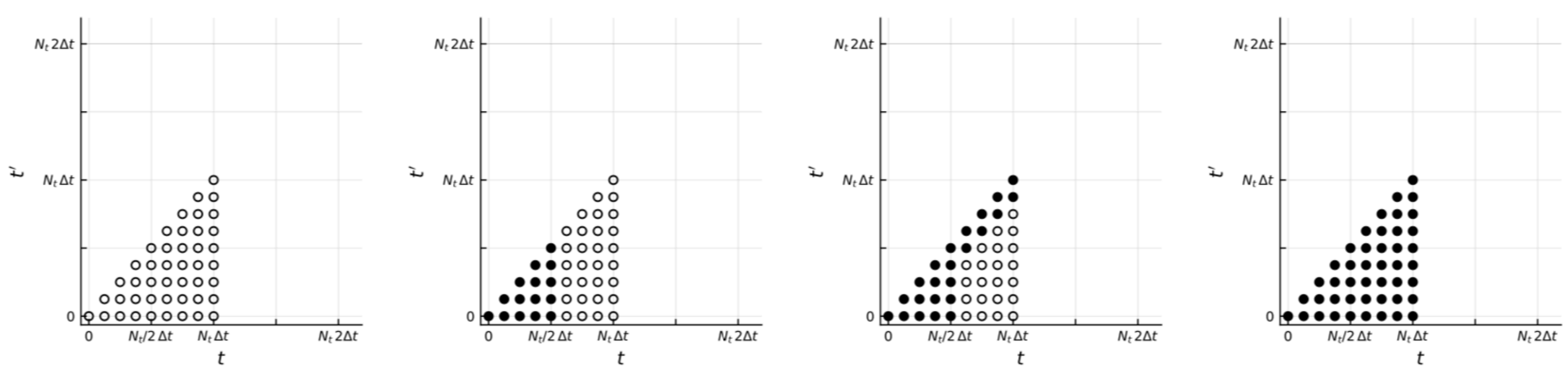
Quenches from T' to T=0



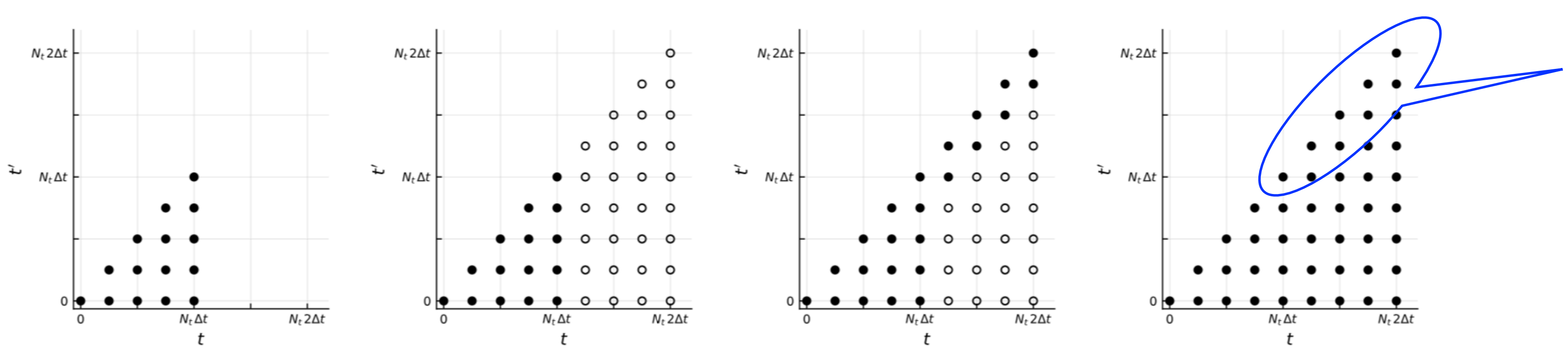
How to integrate the equations

- We use a <u>fixed</u> time step Δt
  Then we extrapolate in the limit $\Delta t \to 0$
  This is a safe procedure!

- A <u>variable</u> time step does not work for mixed models with discontinuous phase transition...



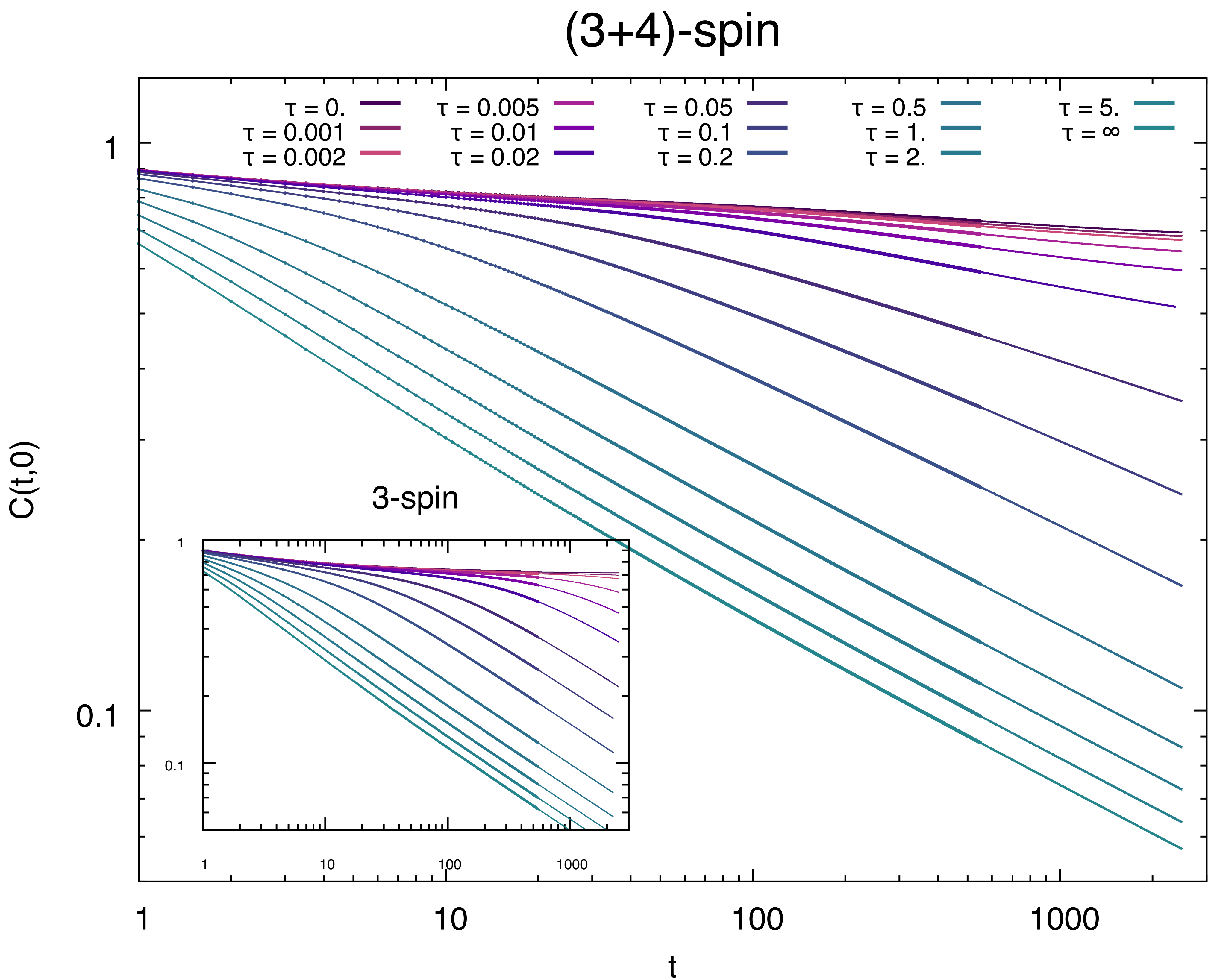
(a) Memory allocation; (b) step 1; (c) step 2; (d) step 3;



(e) step 4.1; (f) step 4.2; (g) step 2; (h) step 3.

numerical instabilities appear on the diagonal

Correlation with the initial configuration



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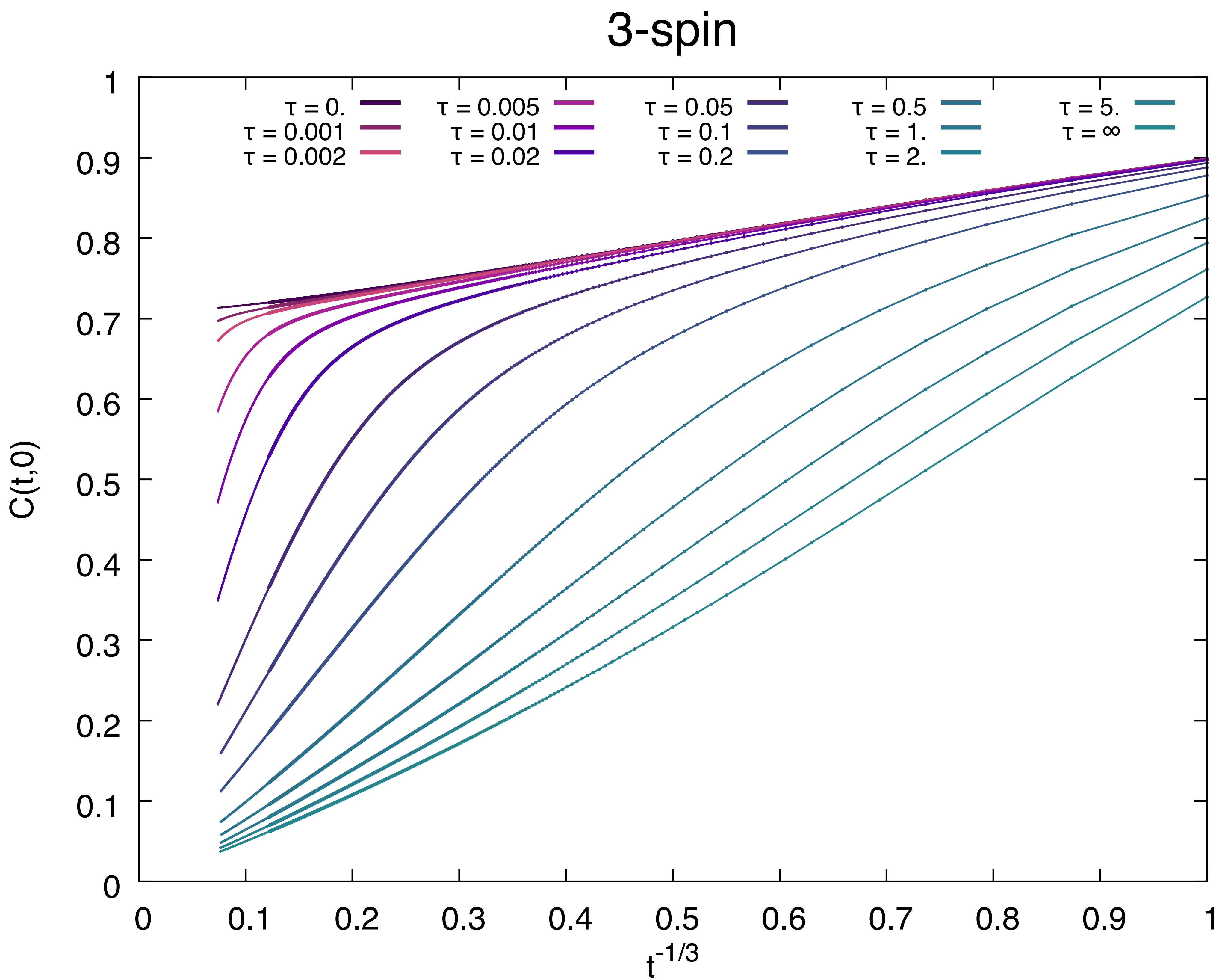

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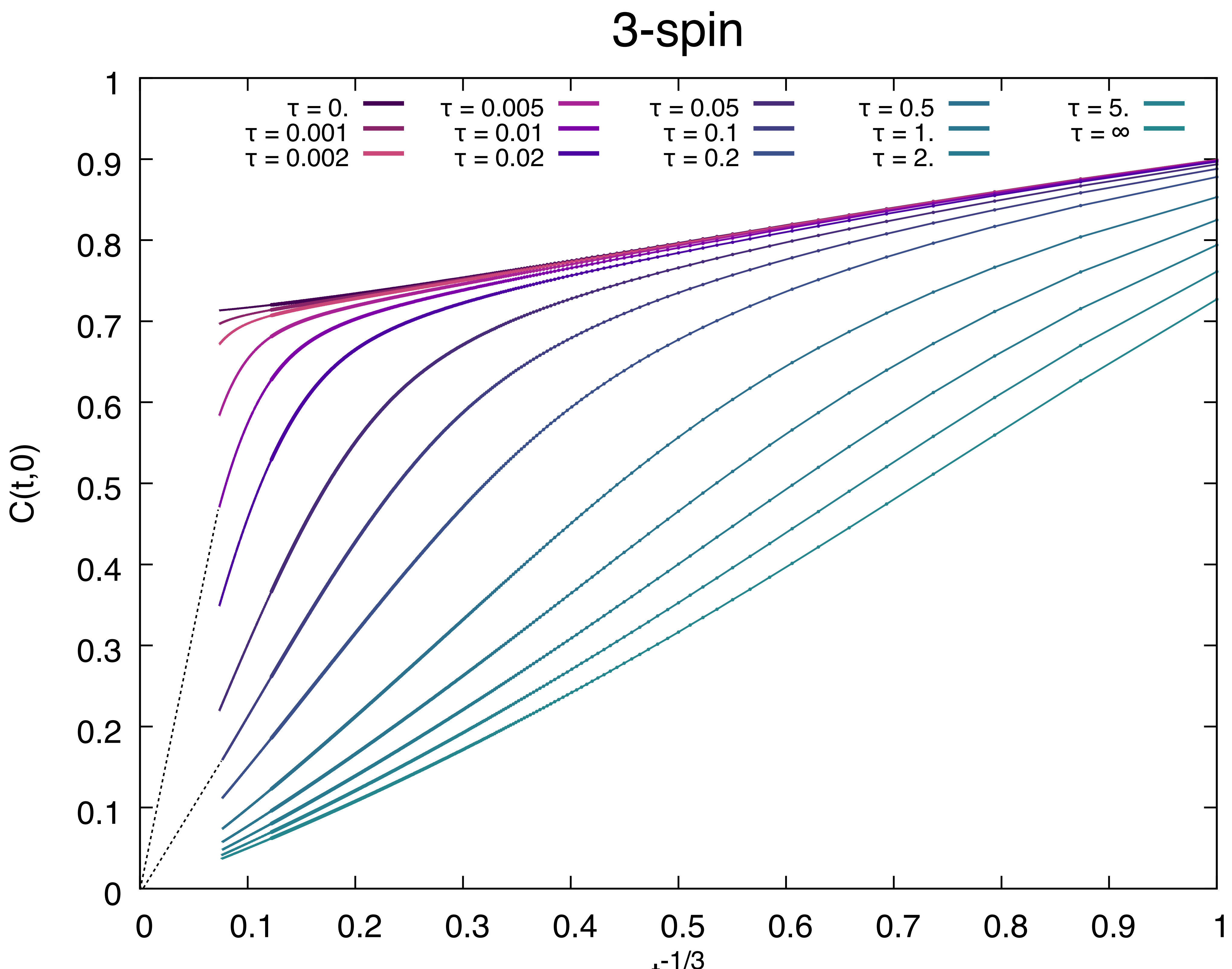

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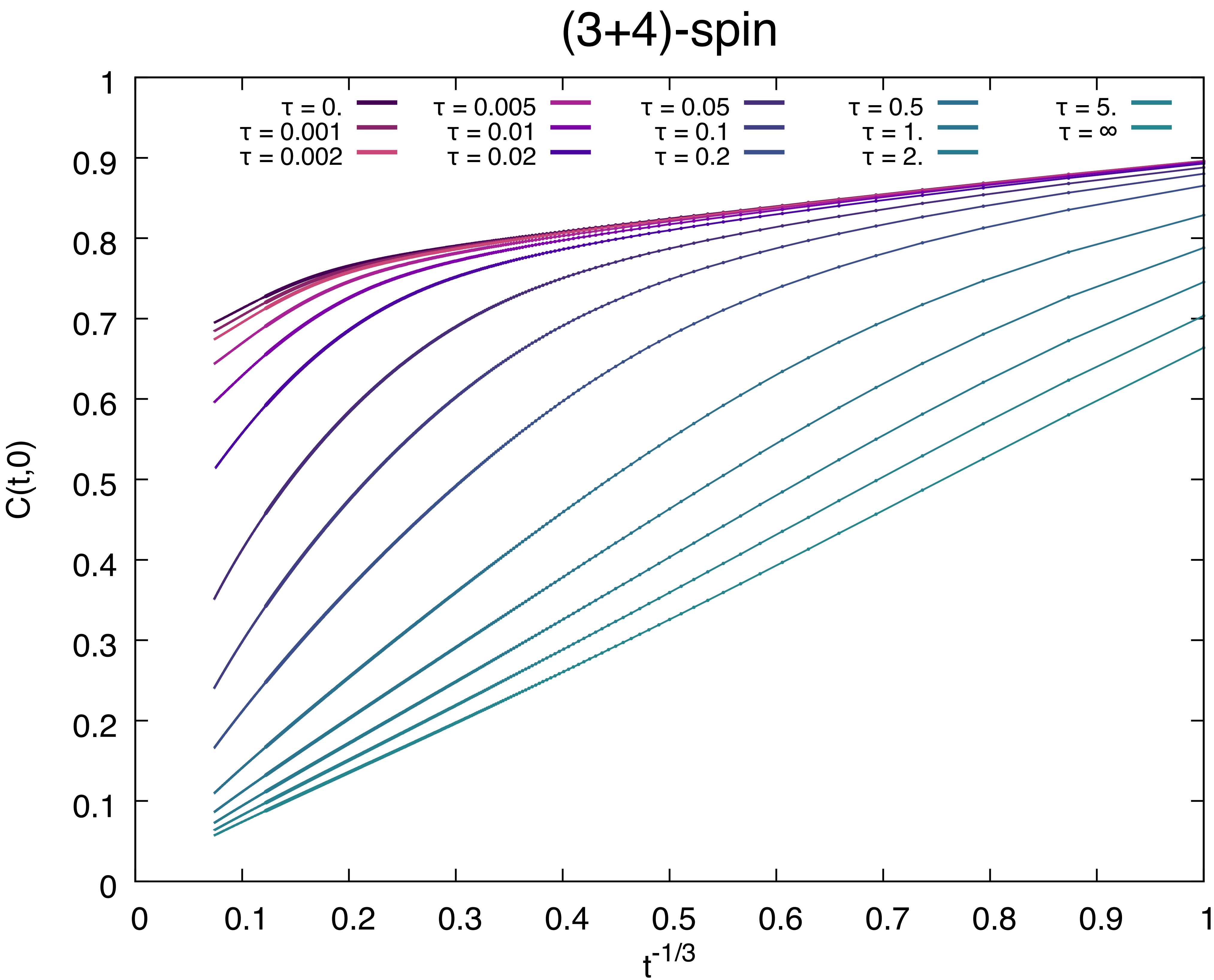

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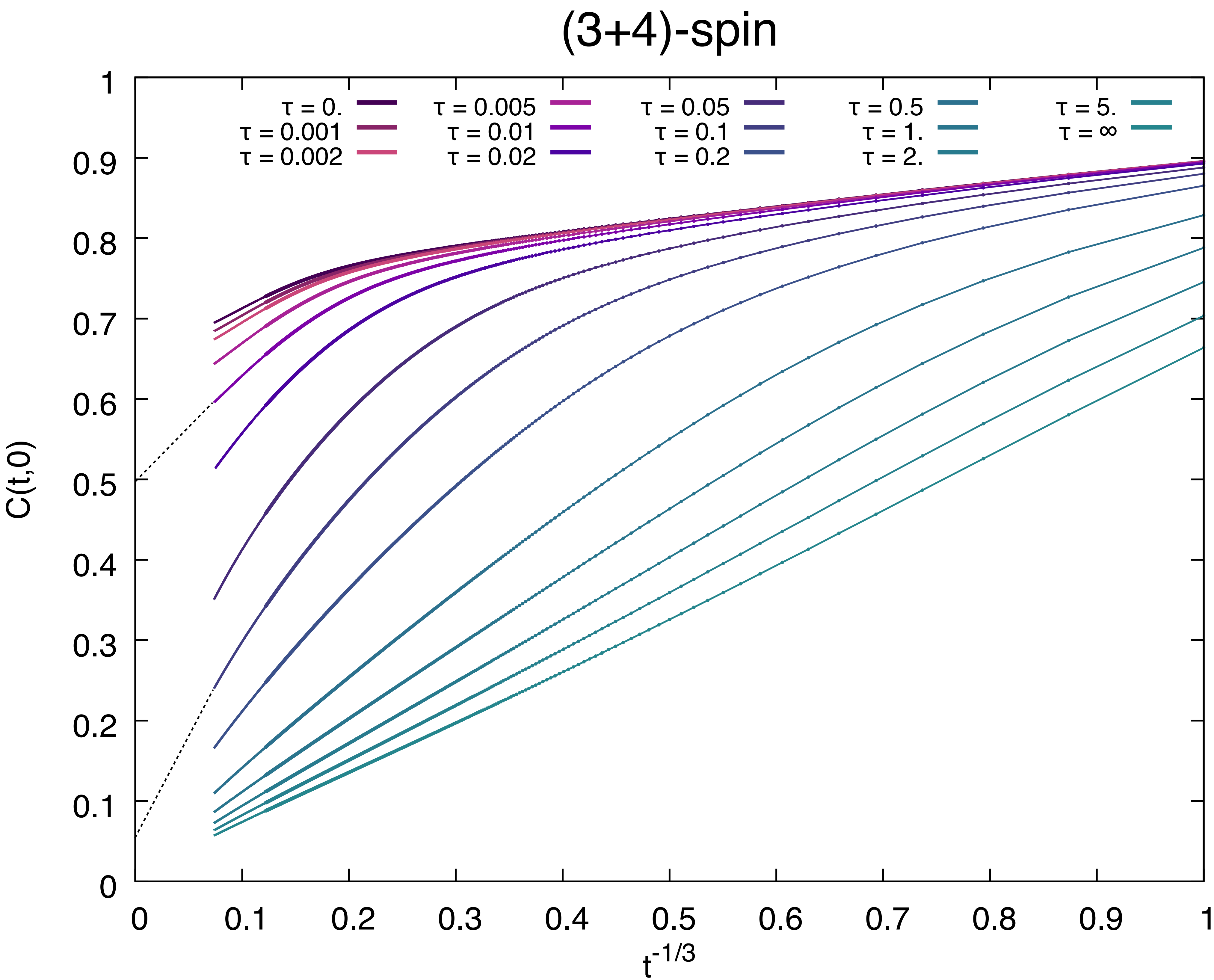
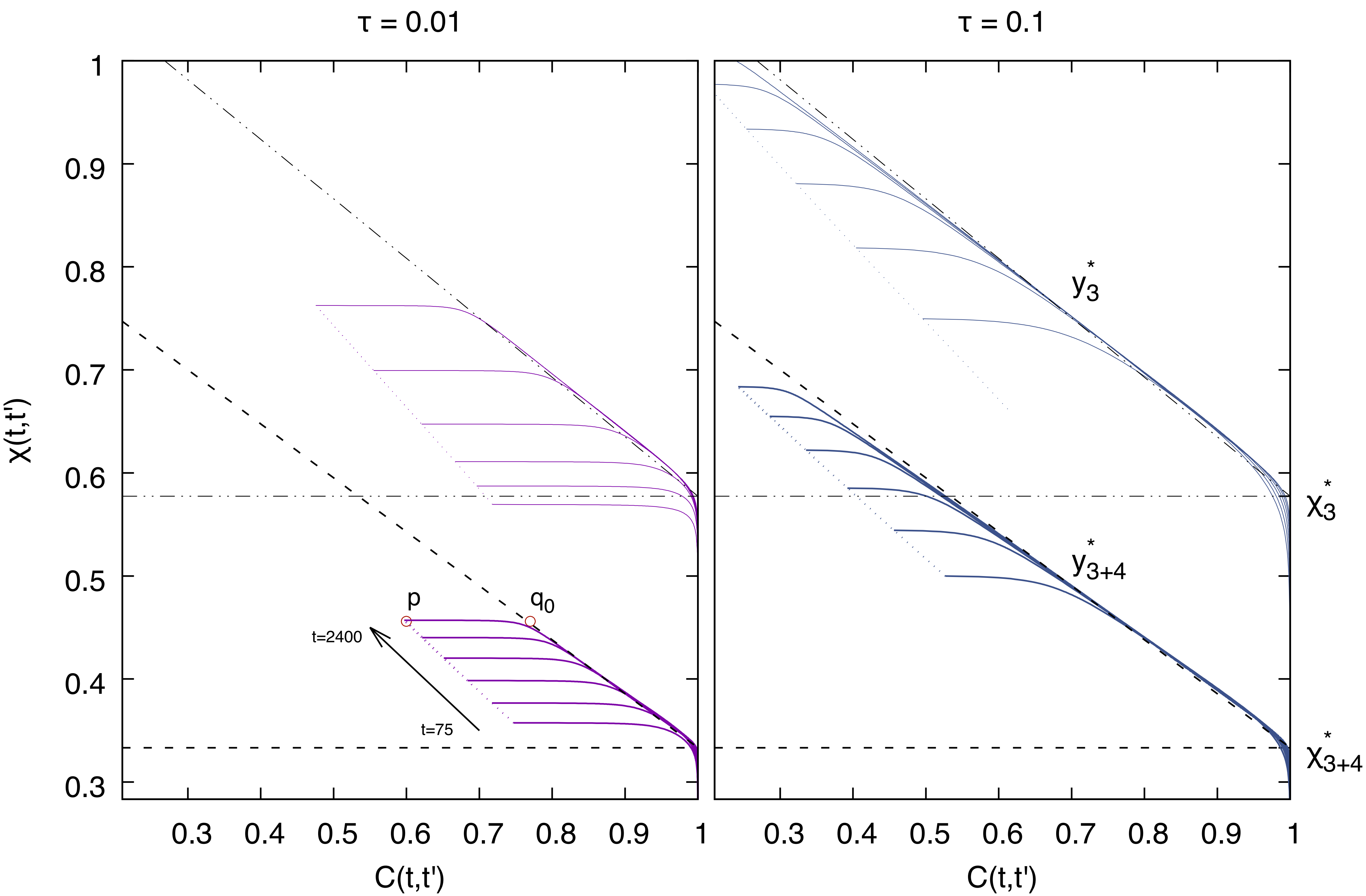


T=0 fluctuation-dissipation plot



T=0 fluctuation-dissipation plot

$$\chi(t,t') = \int_{t'}^{t} R(t,s)ds \quad \text{vs.} \quad C(t,t')$$

$$\chi^* = \lim_{T\to 0} \frac{1-q_m}{T} = \frac{1}{\sqrt{f''(1)}}$$

$$y^* = \lim_{T\to 0} \frac{x^*}{T} = \frac{\sqrt{f''(1)}}{f'(1)} - \frac{1}{\sqrt{f''(1)}}$$

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$$\frac{X[C]}{T} \simeq \begin{cases} y^* & q_0 < C < 1 \\ 0 & p < C < q_0 \end{cases}$$

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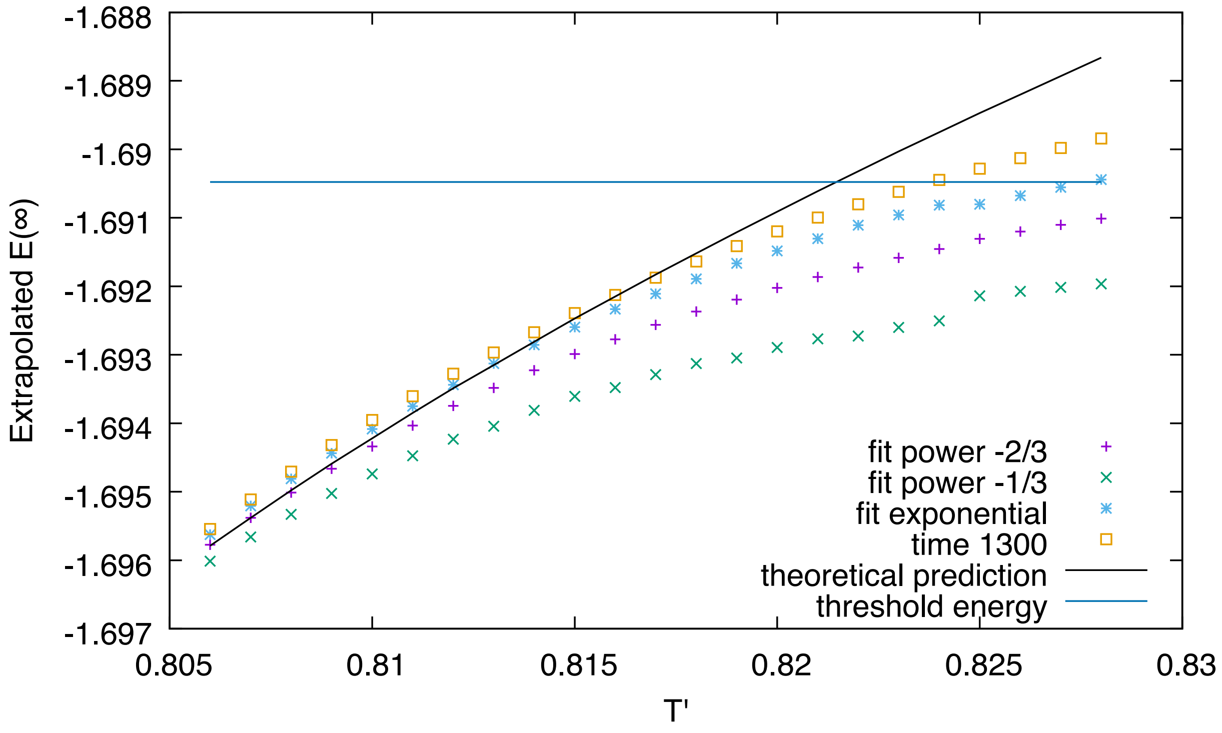
$0 < p < q_0 < 1$

aging in a confined space?

Phase transition in T'?



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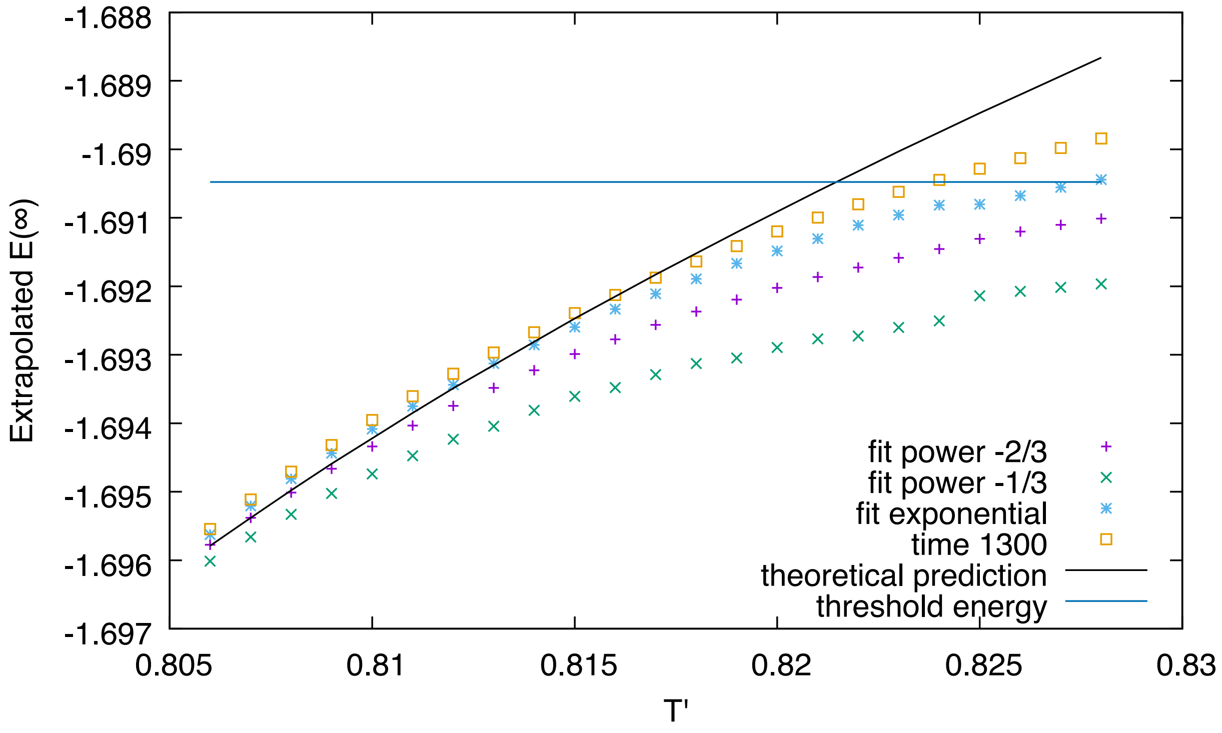
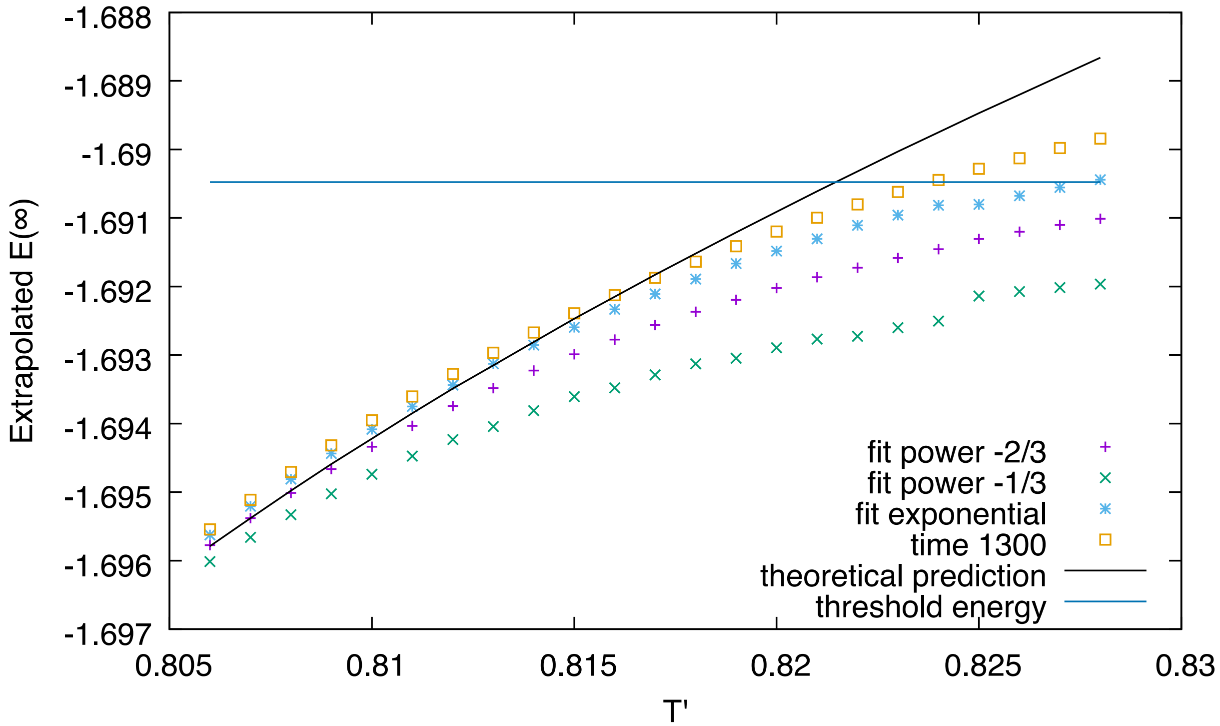


$$e(t) \simeq e_{th} + y^* f(q_0(t)) - \frac{f(p(t))}{T'}$$

$$q_0(t) \simeq \alpha\, p(t)$$

Phase transition in T'?



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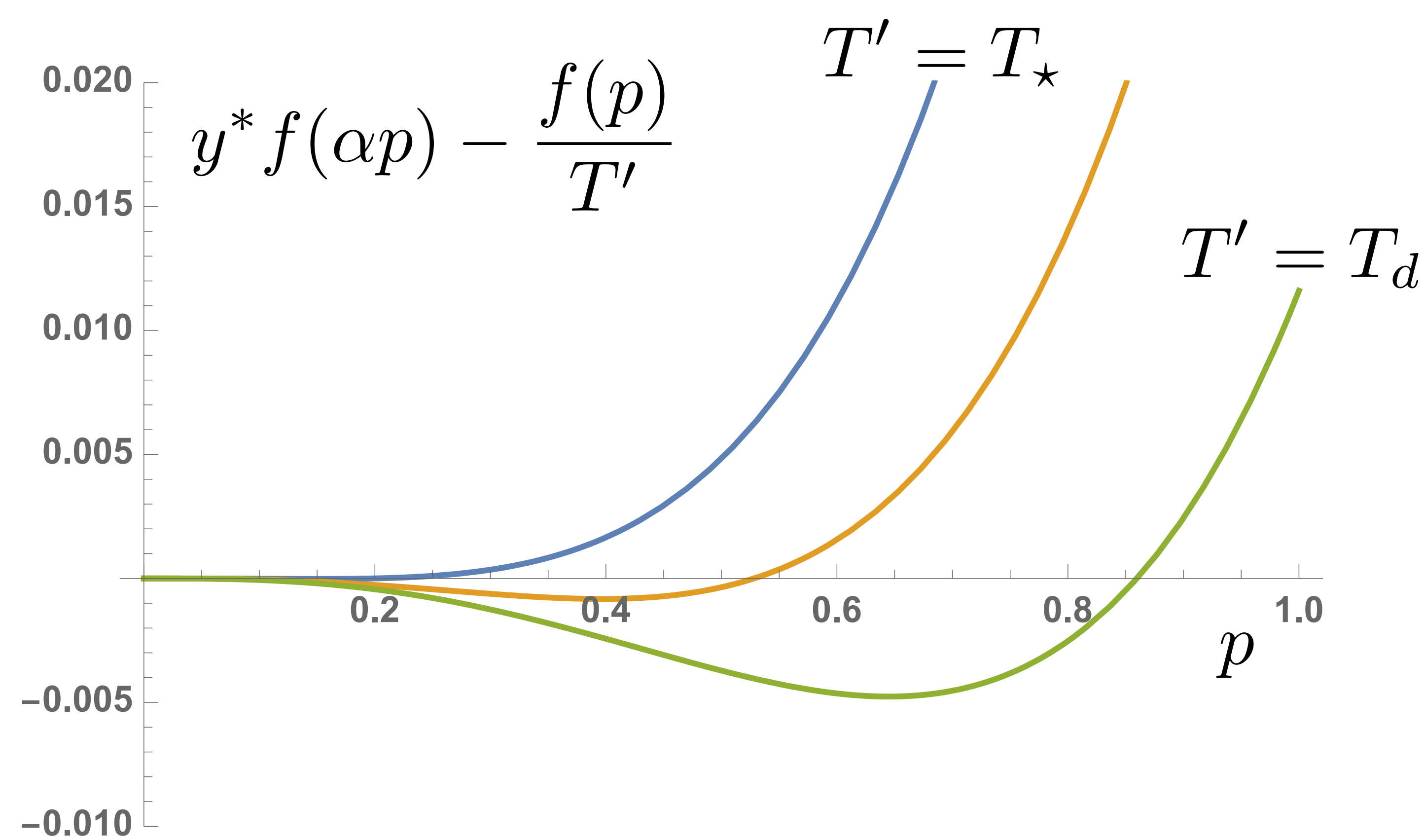
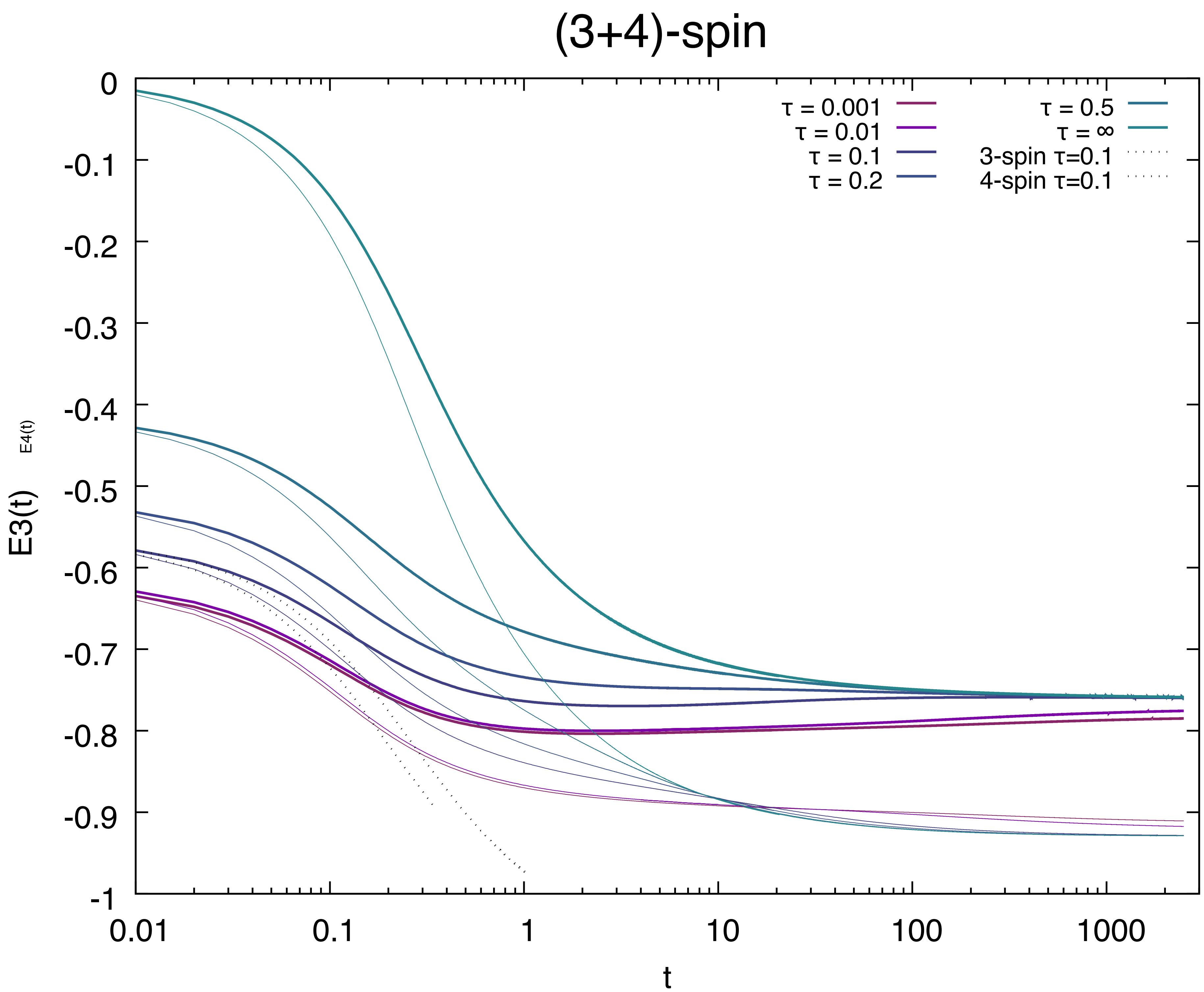
possibly a phase transition at

$$T_\star = \frac{1}{y^* \alpha^3}$$



How it goes below the threshold?



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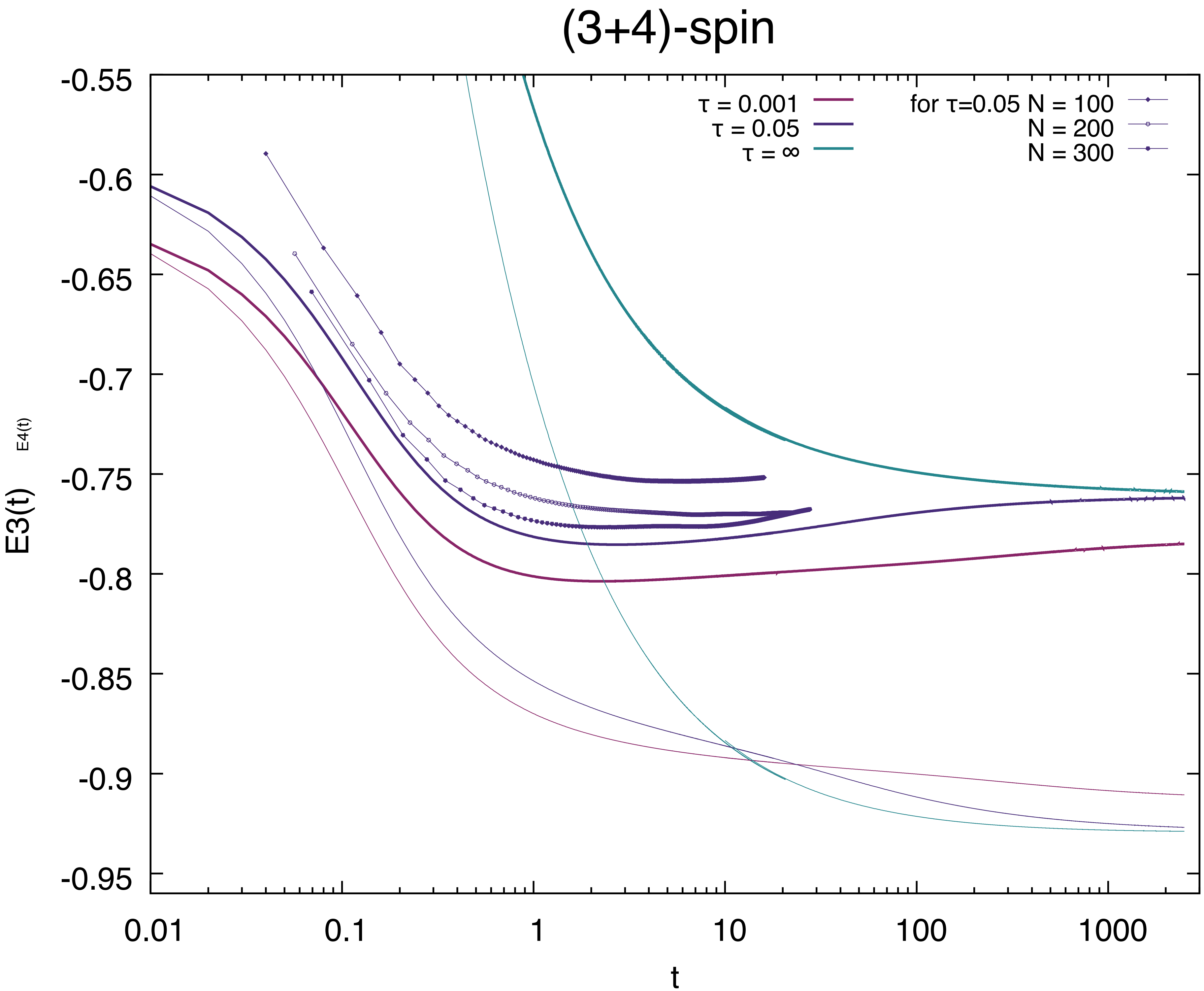

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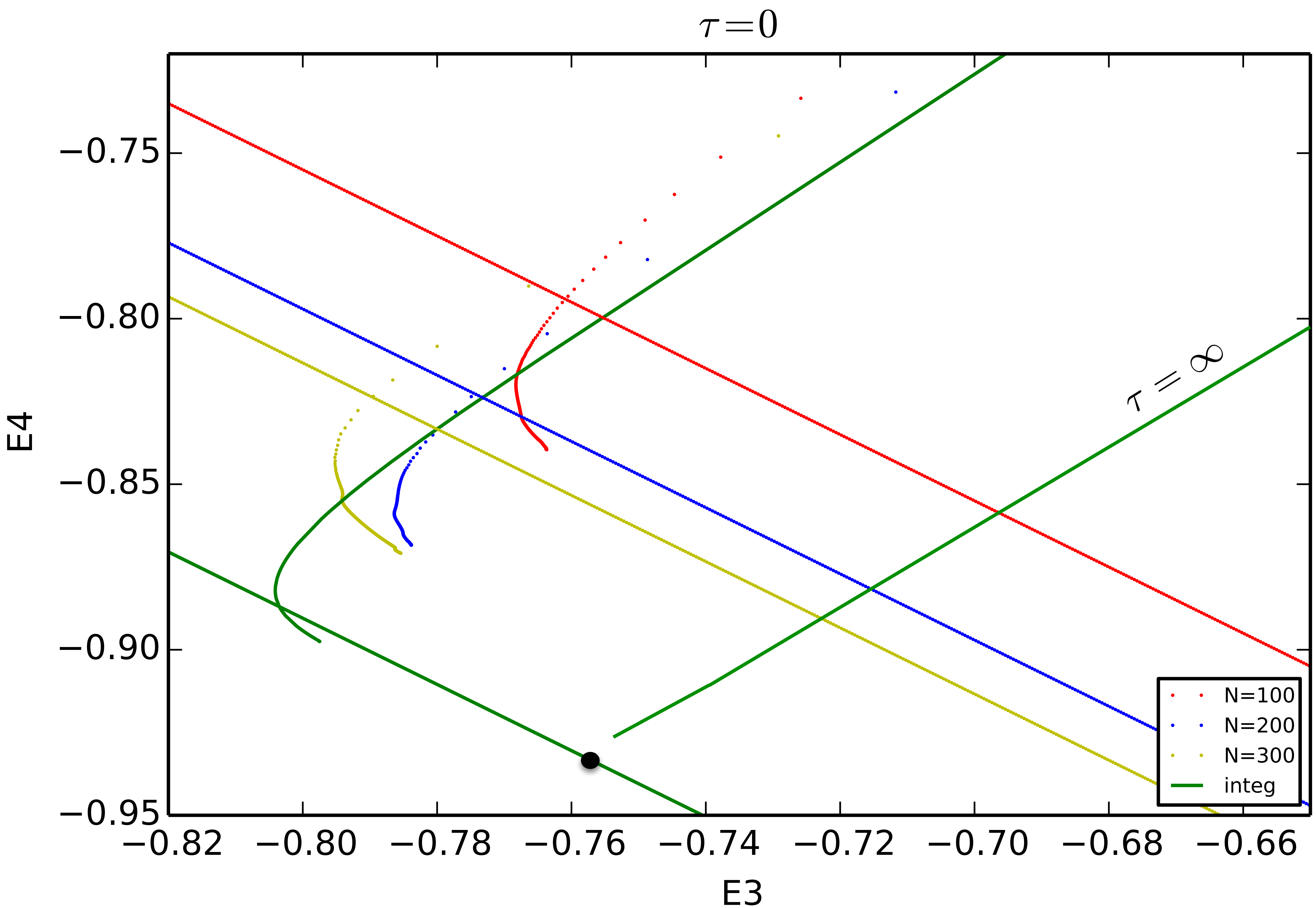


Can we predict the dynamics asymptotic via a static complexity computation?

- T=0 relaxation dynamics goes to local minima
- Count the mean logarithm of the number of local minima at fixed overlap p to an equilibrium configuration at T'

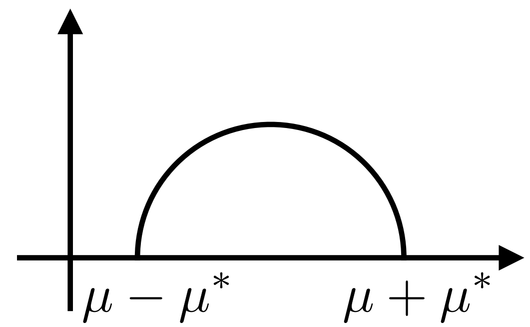
$$\Sigma(p,T',E,\mu) = \mathbb{E}_J \int \mathcal{D}\underline{\sigma}^0 \, e^{-H_J(\underline{\sigma}^0)/T'}$$

$$\log\left[\int \mathcal{D}\underline{\sigma}\, \delta\Big(\underline{\sigma}\cdot\underline{\sigma}^0 - pN\Big)\delta\Big(H_J(\underline{\sigma}) - E\Big)\delta\Big(\nabla H_J(\underline{\sigma}) + \mu\underline{\sigma}\Big)\Big|\det(\mathbb{H}(H_J(\underline{\sigma})) + \mu\mathbb{I})\Big|\right]$$

- Good candidates for describing the dynamics at large times are marginal stationary points of the energy function

Computing the complexity

- Hessian spectrum is a shifted semicircle law
  $\mu^* = 2\sqrt{f''(1)}$
  minima have $\mu > \mu^*$ and saddles have $\mu < \mu^*$



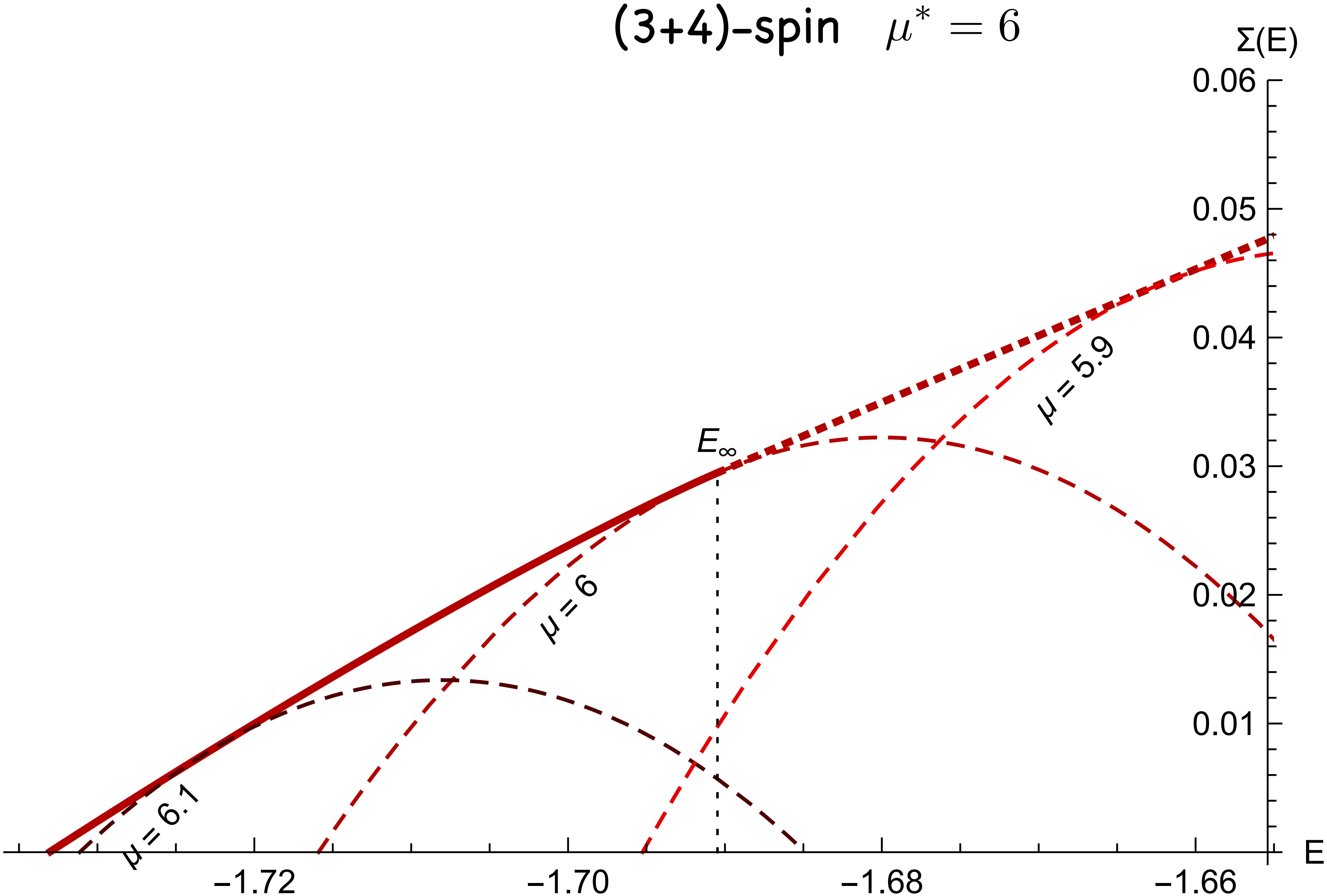
- Main difference between pure and mixed models

$$e = \sum_p c_p^2 \, e_p \qquad \mu = -\sum_p c_p^2 \, p \, e_p$$

pure model marginal states concentrate at $e_{th} = -\dfrac{\mu^*}{p}$

mixed model has marginal states at different energies!

Complexity of the mixed model



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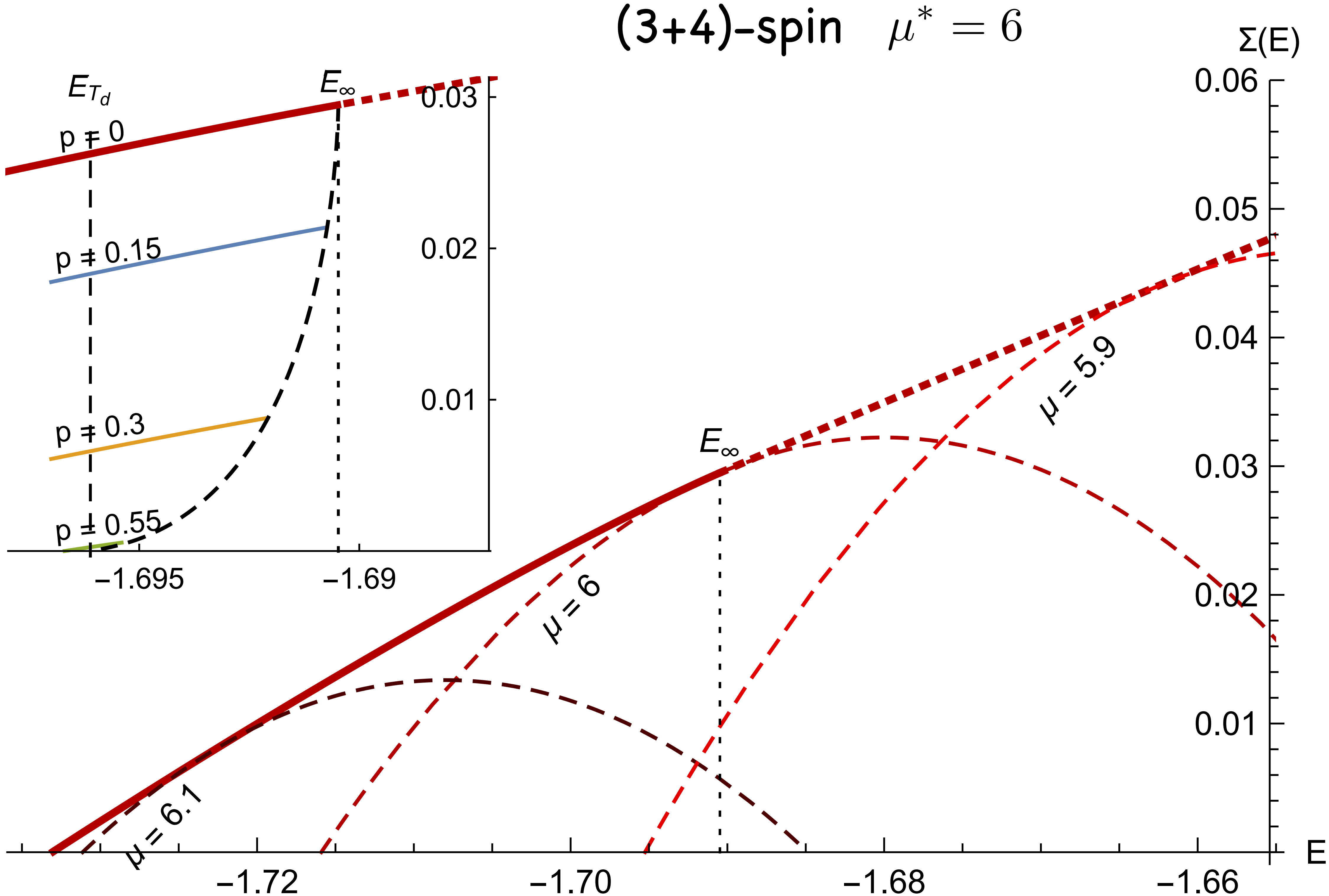

Satisfied with the constrained complexity?

- Explains well why marginals states closer to the initial configuration ($p>0$) have lower energies

- Predicts too low values for y and q_0

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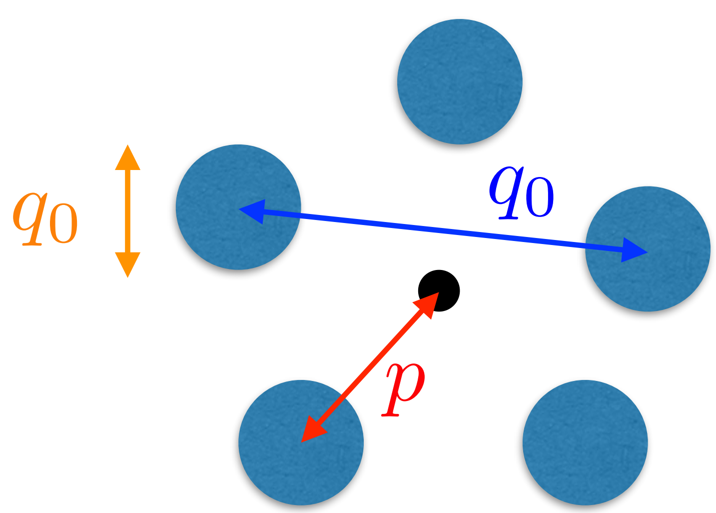
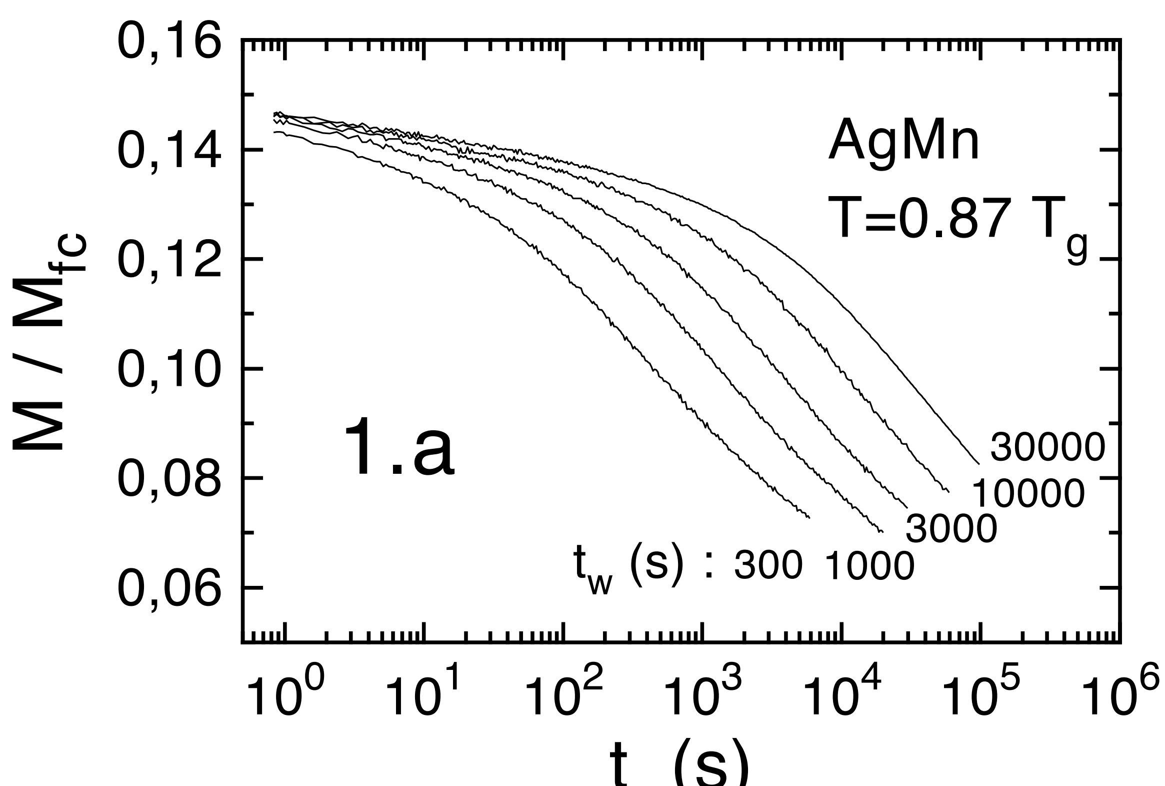
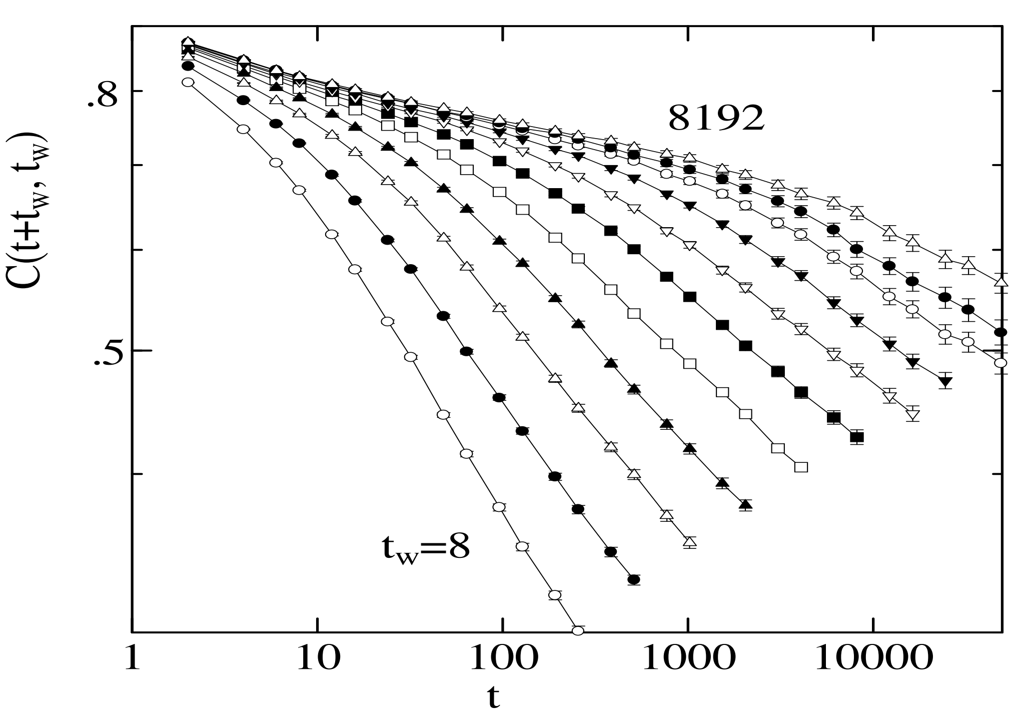


Possible explanation:
- dynamics gets trapped and makes aging in just one marginal manifold (large q_0)
- static complexity counts them all (small q_0)

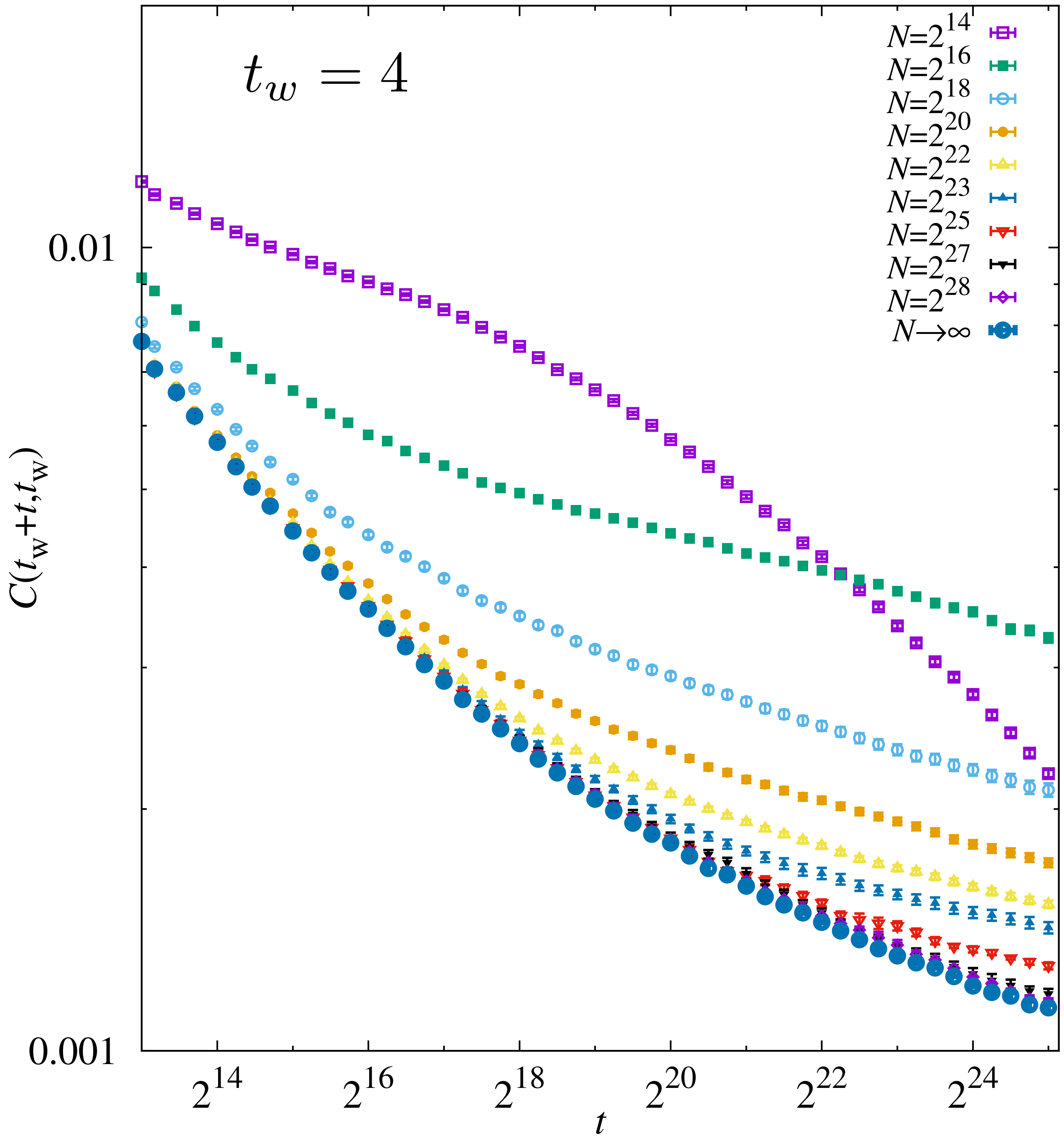
Ising SG on a sparse RRG

- Random 4-regular graph, $J_{ij} = \pm 1$
- Quench from T'=∞ to T=0.8 T_C
- Expected aging behavior $\lim_{t \to \infty} C(t, t_w) = 0 \quad \forall t_w$ based on previous experiments & numerics





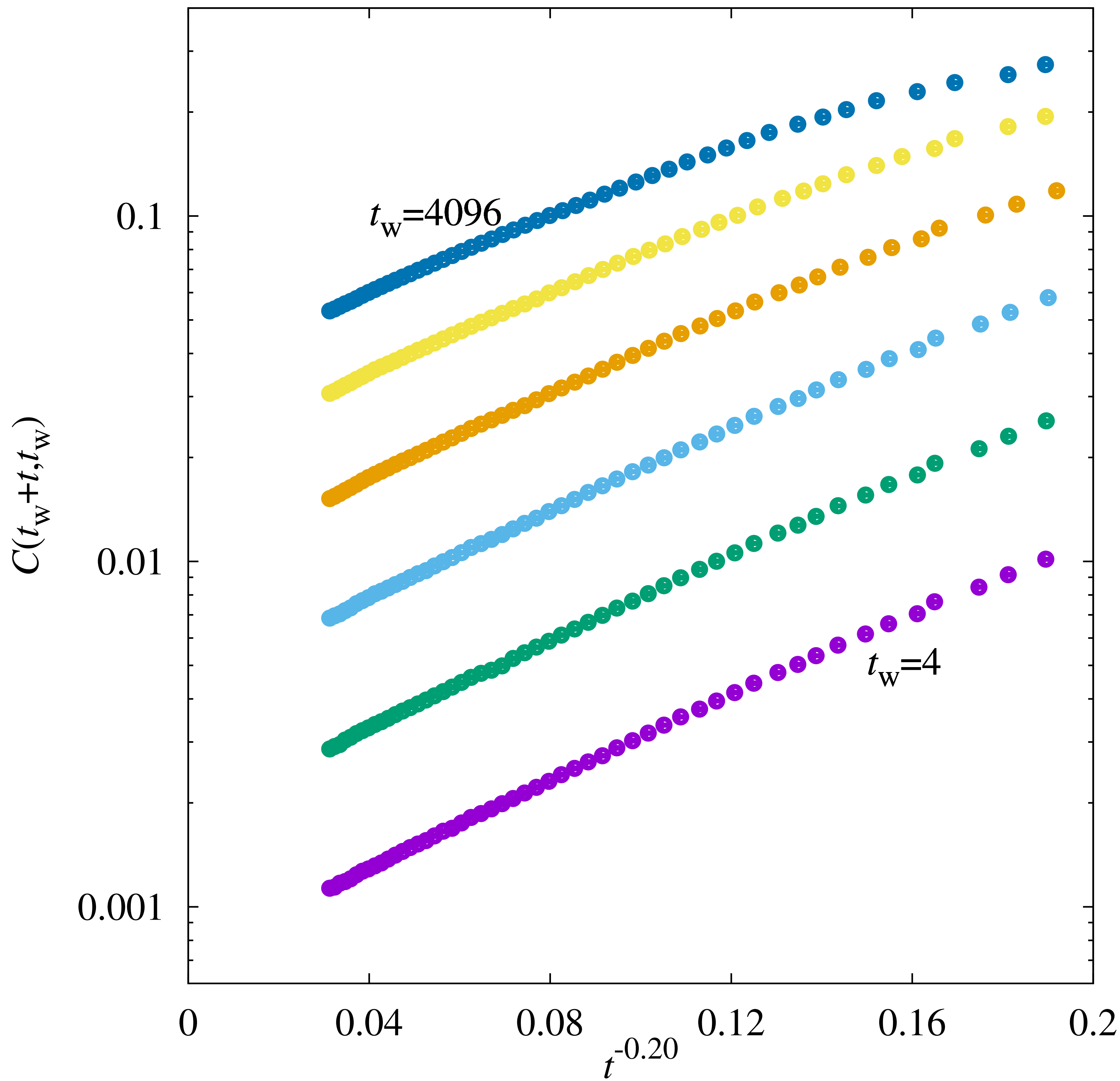
Finite size effects under control



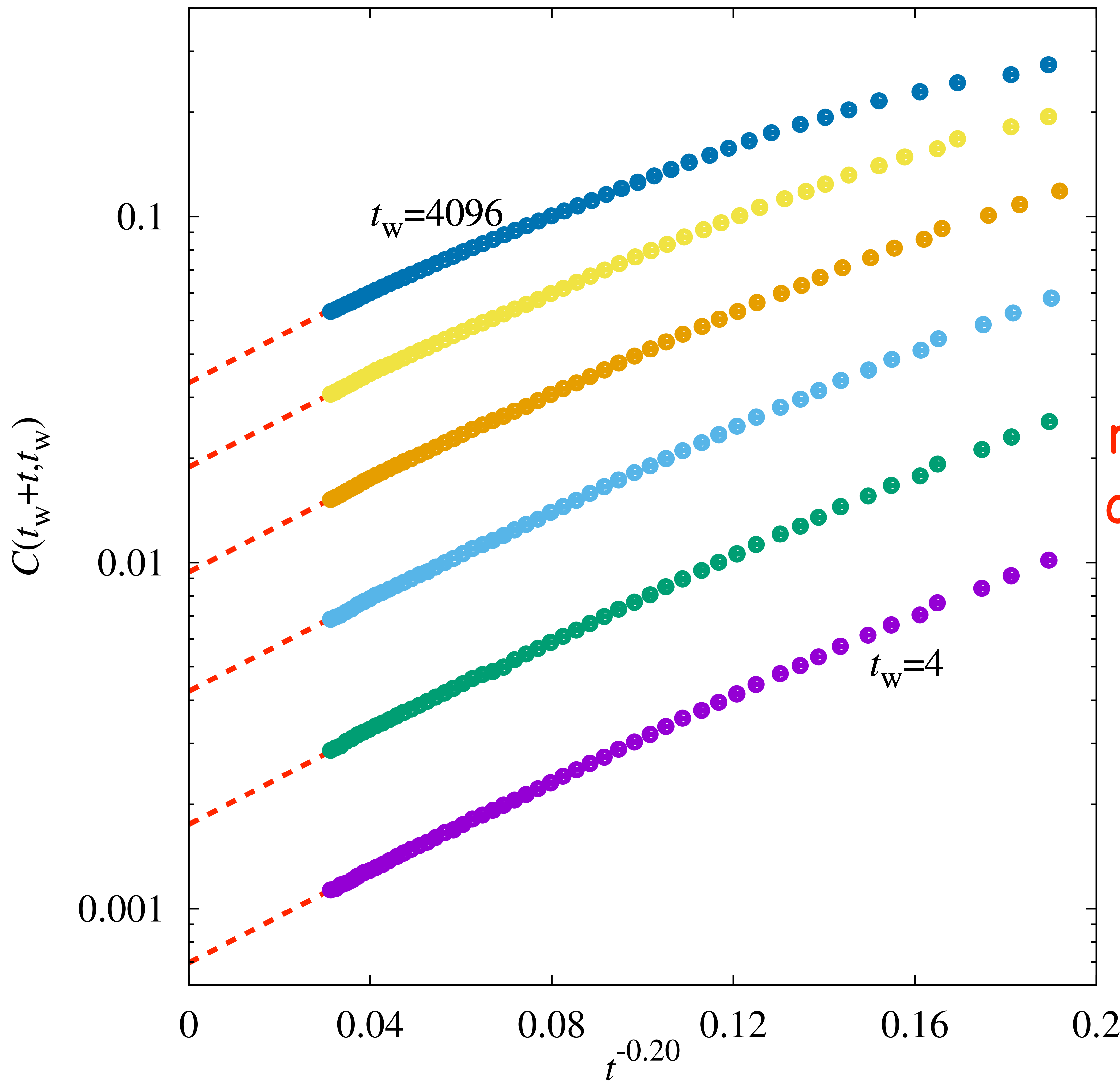
Huge sizes!
Safe extrapolation to the large N limit

We work in the regime of very long times (t>>1) and very small correlations (C<<1)

Non-zero large times limit



Non-zero large times limit



Aging in a
more and more
confined space?

Conclusions

- We studied aging in several mean-field prototypical models (spherical/Ising, fully-connected/sparse graphs, with continuous/discontinuous phase transitions)

- In general the configuration at initial or short time is **not forgotten** (strong ergodicity breaking) and long time dynamics takes place in a **restricted marginal manifold**

- The statics-dynamics connection should be rethought: predicting the long time behavior of Langevin dynamics is a very open problem