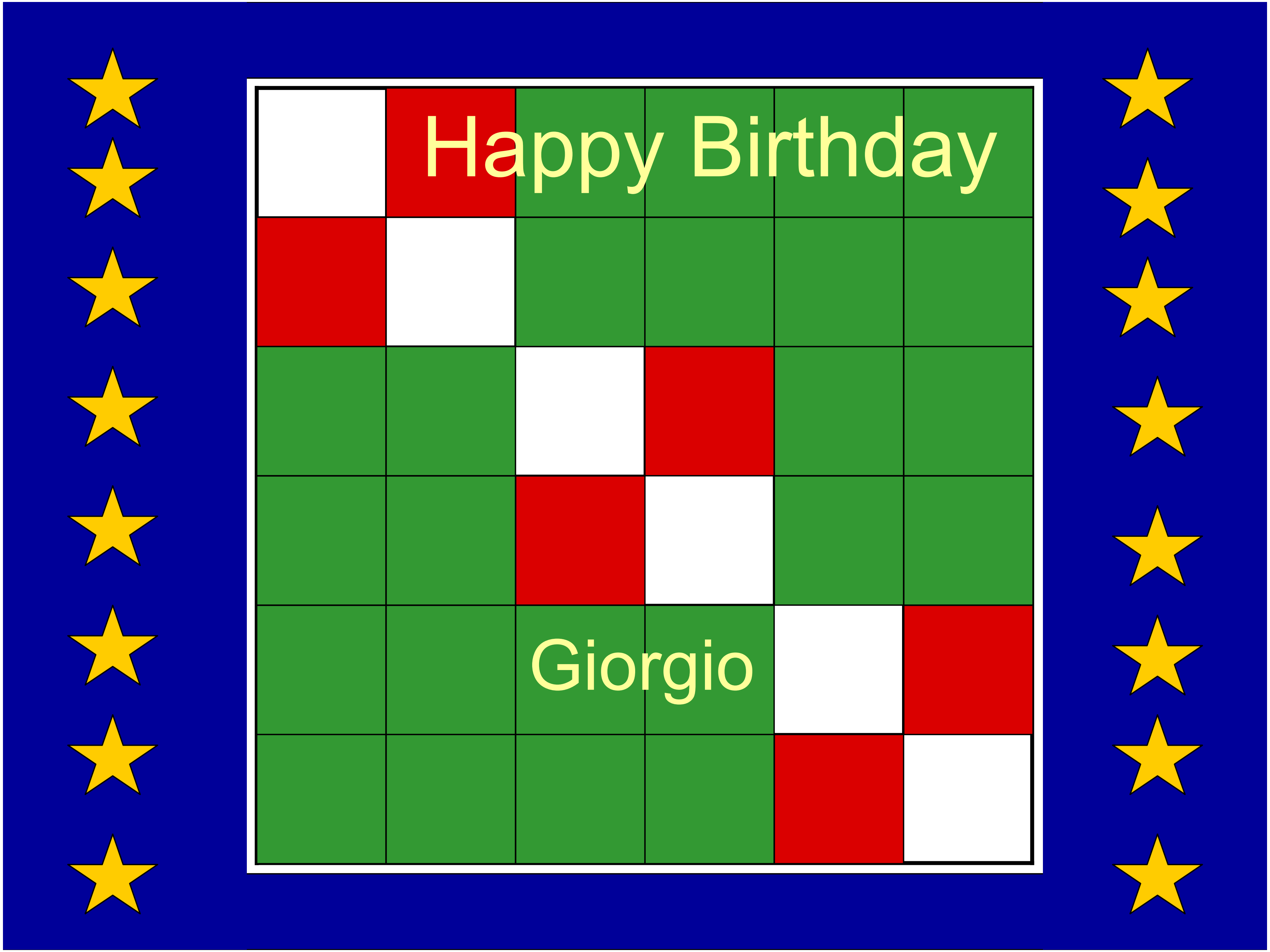

Happy Birthday
Giorgio

Reflections on Replica Symmetry Breaking and Spin Glasses

David Sherrington
University of Oxford

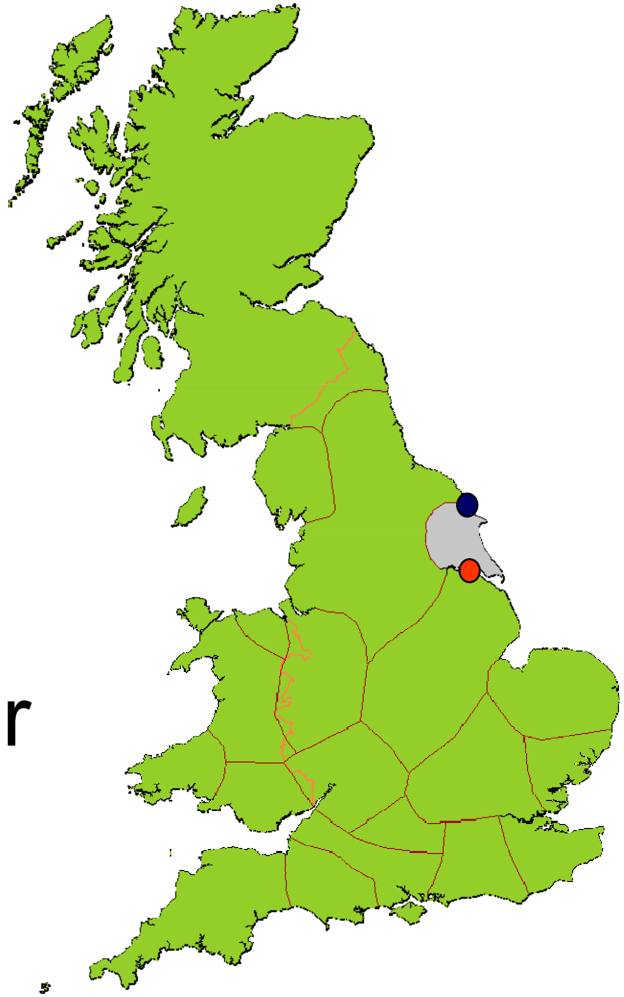
The Continental *Parisii*

- The *Parisii* were a people of Celtic Gaul, who lived along the banks of the Sequana (Seine), and on an island in the river known as *Lutetia*; the island is nowadays famous as the site of the Notre Dame Cathedral in the centre of Paris, the capital city of modern France.

- **The Celtic Tribes of Britain**

- **The Parisi**

- **Tribe:** Parisi/Parisii
  Capital: PETVARIA •
  Location: Brough on Humber
  Humberside



• Middlesbrough: where I come from

The n-m-agician

Replica Symmetry Breaking

RSB

RSJ

Rolled Steel ~~Joist~~
Beam

RSB

Support structures / Act as lever

ΔΟΣ ΜΟΙ ΠΟΥ ΣΤΩ ΚΑΙ ΚΙΝΩ ΤΗΝ ΓΗΝ

Give me a place to stand and I will move the Earth

(Archimedes quoted by Pappus of Alexandria, *Synagoge,* Book VIII, (340)

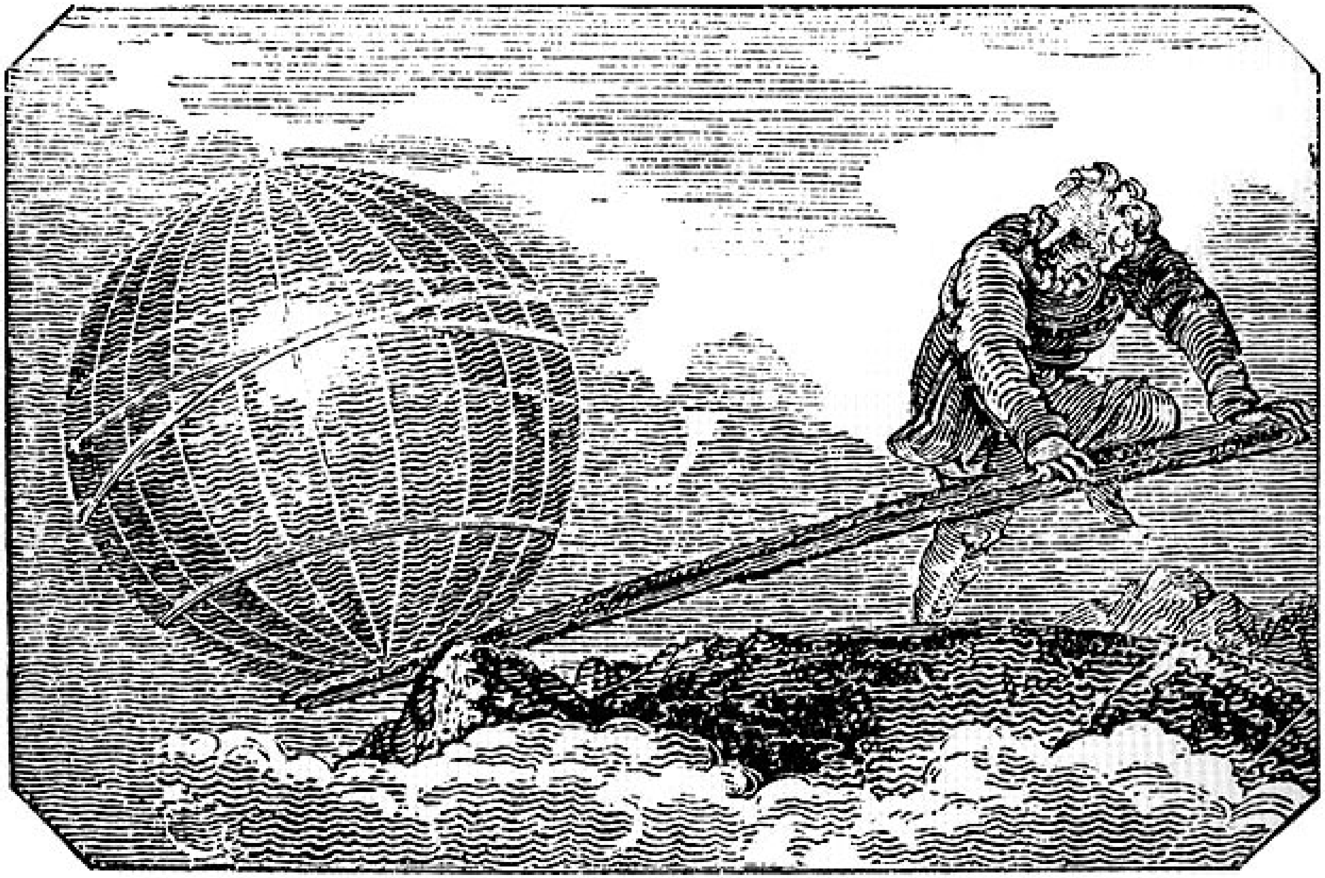


Fig adapted from Mechanic's Magazine (Knight & Lacey, London, 1824)

Giulio Parigi (1571-1635)

Stanzio delle Matematiche at the Galleria degli *Uffizi*, Firenze

Giorgio Parisi (1948-)

Dipartimento di Fisica, Universita di Roma “La Sapienza”

Spinning the world; thanks to Giorgio

Vatican

2005

SK model

$$H = -\sum_{(ij)} J_{ij}\sigma_i\sigma_j; \; \sigma_i = \pm 1$$

All pairs of "sites"

i.i.d $P(J_{ij})$

(i) Potentially soluble spin glass model

(ii) Hard optimization problem

(iii) Paradigm complex system

(iv) Challenges for statistical mechanics & probability

(v) Led Giorgio $\Rightarrow$ RSB

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All pairs of “sites”

(i) Potentially soluble spin glass model

(ii) Hard optimization problem

(iii) Paradigm complex system

(iv) Challenges for statistical mechanics & probability

(v) The problem that led Giorgio ➡ RSB

Replicas (EA)

- Philosophy
  - Interest in typical behaviour
    - Physical behaviour of alloys independent of precise atom positions, depends only on intensive generating rules
- Average out disorder
  - But for physical observables
    - Thermodynamics: $\ln Z = \ln \mathrm{Tr} \exp(-H/T)$
- Hard to average
  - Change to easier-to-average Z^n

$$\ln Z = Lim_{n\to 0}(Z^n - 1)/n \equiv Lim_{n\to 0}\partial Z^n / \partial n$$

Interpret as *n* replicas

Replica ordering

- Replica spins: $\sigma_i^\alpha ;\ \alpha = 1,..n$
- Average Z^n over J_{ij} :
  - Effective interaction between replicas
  - Order parameters: $q^{\alpha\beta} = N^{-1}\sum_i \left\langle \sigma_i^\alpha \sigma_i^\beta \right\rangle_{eff}$

SK: no site index on *q*

- Natural ansatz (EA/SK): $q^{\alpha\beta} = q;\ \alpha \neq \beta$
  - But not correct !
    - Some unphysical consequences
    - & instabilities in replica space

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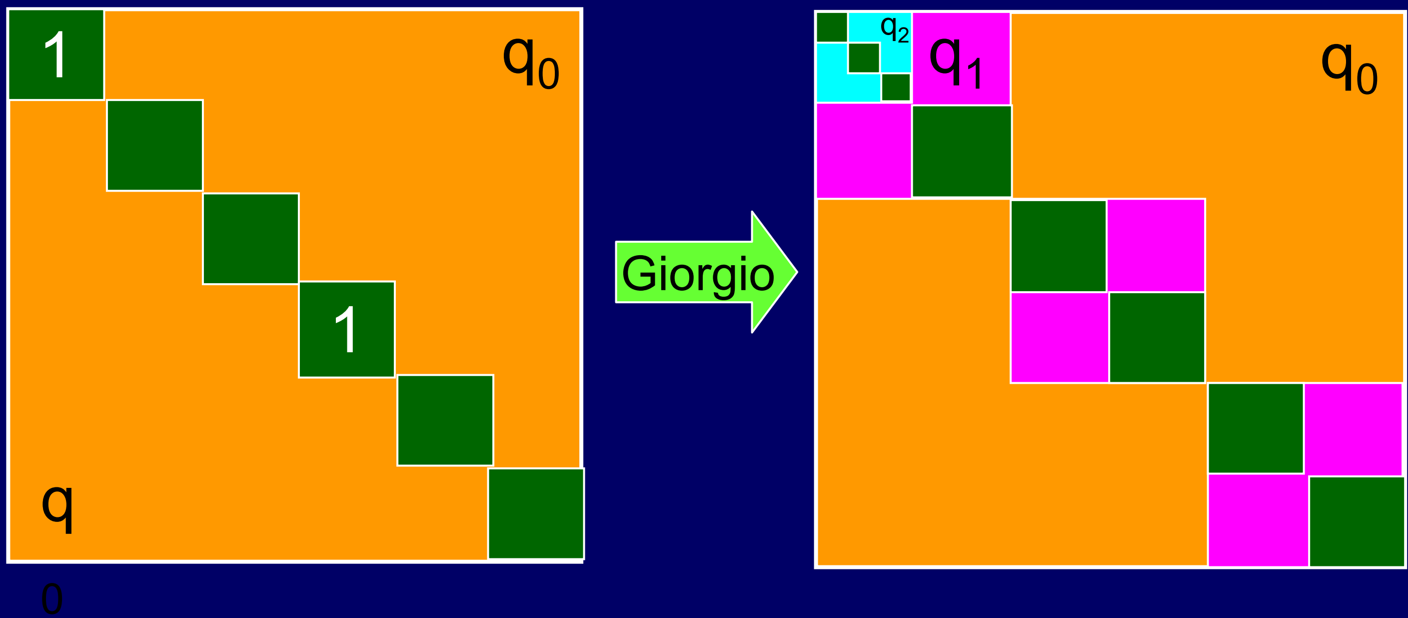
Saint George to the rescue

Dragon
Replicon

Replica Symmetry Breaking

$$q^{\alpha\beta} = N^{-1}\sum_i \left\langle S_i^{\alpha} S_i^{\beta} \right\rangle \;;\; \alpha,\beta = 1,..n,\; n \to 0$$

Replica Symmetric   Replica Symmetry Broken



$\to q(x)$

FRSB

But how did Giorgio come to this?

I do not know

A few people tried subdivision of

$\alpha = 1, \ldots n$; $n \rightarrow 0$

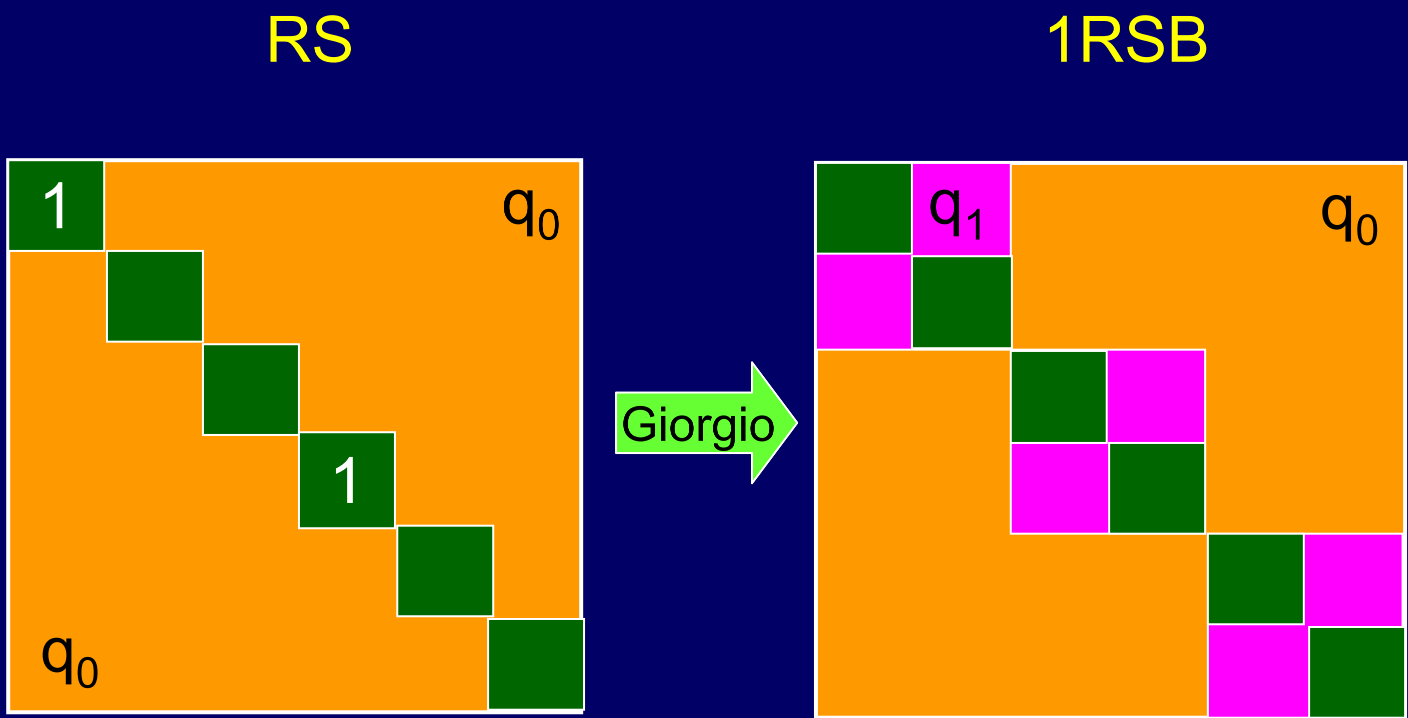
inserting m in interval 0 to 1

e.g. Bray & Moore:

$q=q_1$; $m<\alpha<\beta$, $q=q_2$; $\alpha < m < \beta$, $q=q_3$; $\alpha<\beta<m$

But found $m=\infty$

One step à la Giorgo



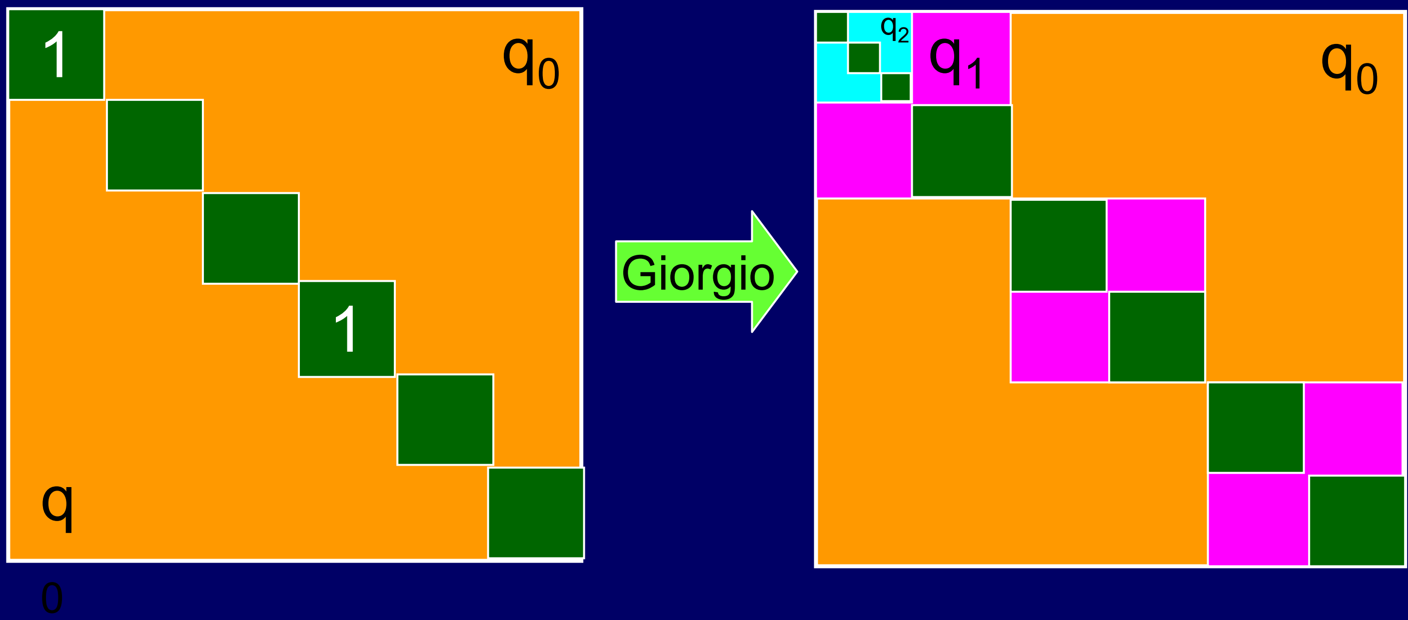
Better but not perfect. What else? Full hierarchy?

Full RSB (Giorgio)

$$q^{\alpha\beta} = N^{-1}\sum_i \langle S_i^\alpha S_i^\beta \rangle \ ; \alpha, \beta = 1,..n, \ n \to 0$$

Replica Symmetric

Replica Symmetry Broken



$\to q(x)$

FRSB

Free energy

$$f = Sup_{\{q,m\}} Lim_{K\to\infty} \left\{ -\frac{\beta}{2}(1-2q_1) + \frac{\beta}{4}\sum_{i=1}^{K+1} q_i^2 (m_i - m_{i-1}) - \widetilde{f} \right\}$$

$$\widetilde{f} = \frac{T}{m_K} \int_{K+1}^{G} \ln\left[\int_{K}^{GE} .. \int_{1}^{GE} (2\cosh(\beta h_1))^{m_1} \right]$$

where

$$\int_i^G g(h_i) \equiv \int_{-\infty}^{\infty} \frac{dh_i}{\sqrt{2\pi\Delta q_i}} \exp\left(-\frac{(h_i - h_{i+1})^2}{2\Delta q_i} \right) g(h_i), \quad \int_i^{GE} g(h_i) \equiv \int_i^G g(h_i)^{r_i - 1};$$

$$\Delta q_i = q_i - q_{i+1}, \quad \Delta q_{K+1} = q_{K+1}, \quad r_{i-1} = \frac{m_i}{m_{i-1}}$$

Initially pretty mysterious

- Analytic continuation of n
- Lim $N \to \infty$, $n \to 0$ reversed
- Normal minima $\to$ maxima (for q)

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Now rigorously proven

(Talagrand 05, Guerra)

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- Analytic continuation of n
- Lim $N \to \infty$, $n \to 0$ reversed
- Normal minima $\to$ maxima (for q)

Now rigorously proven

(Talagrand 05, Guerra)

Physical meaning?

Again, provided by Giorgio (83)

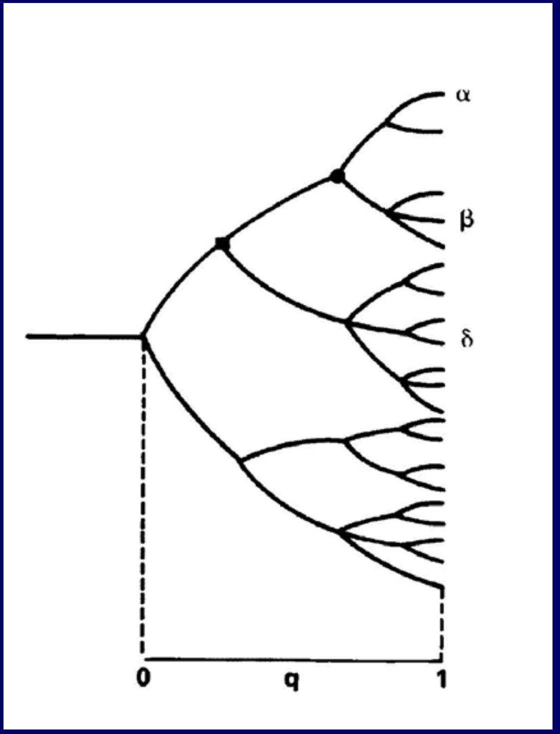
Overlap distribution

$$q^{SS'} = N^{-1} \sum_i \langle S_i \rangle_S \langle S_i \rangle_{S'}$$

$$\overline{P(q)} = \overline{\sum_{S,S'} \widetilde{P}_S \widetilde{P}_{S'} \delta(q - q^{SS'})}$$

$$= \int_0^1 dx \delta(q - q(x)) = dx / dq$$

$$q(x);\ 0 \le x \le 1 \quad \sim \quad Lim_{K \to \infty} \{q, m\}$$



Ultrametric hierarchy

Later

Dynamical relevance of q(x),P(q)

Fluctuation-dissipation

$$dR/dC = X(C) = \int_0^C \overline{P(q)}\ dq$$

C = correlation function

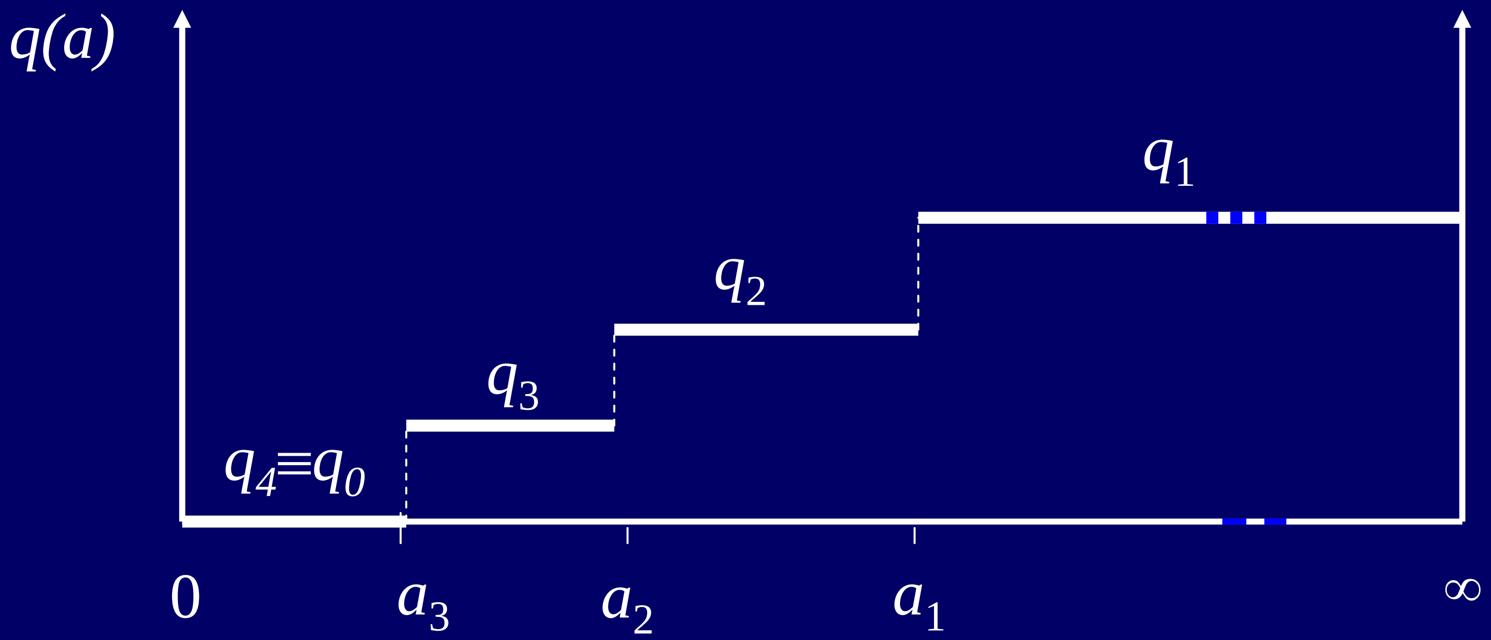
R = response function

$T \to 0$ numerical extremization of f

Oppermann, Schmidt & S

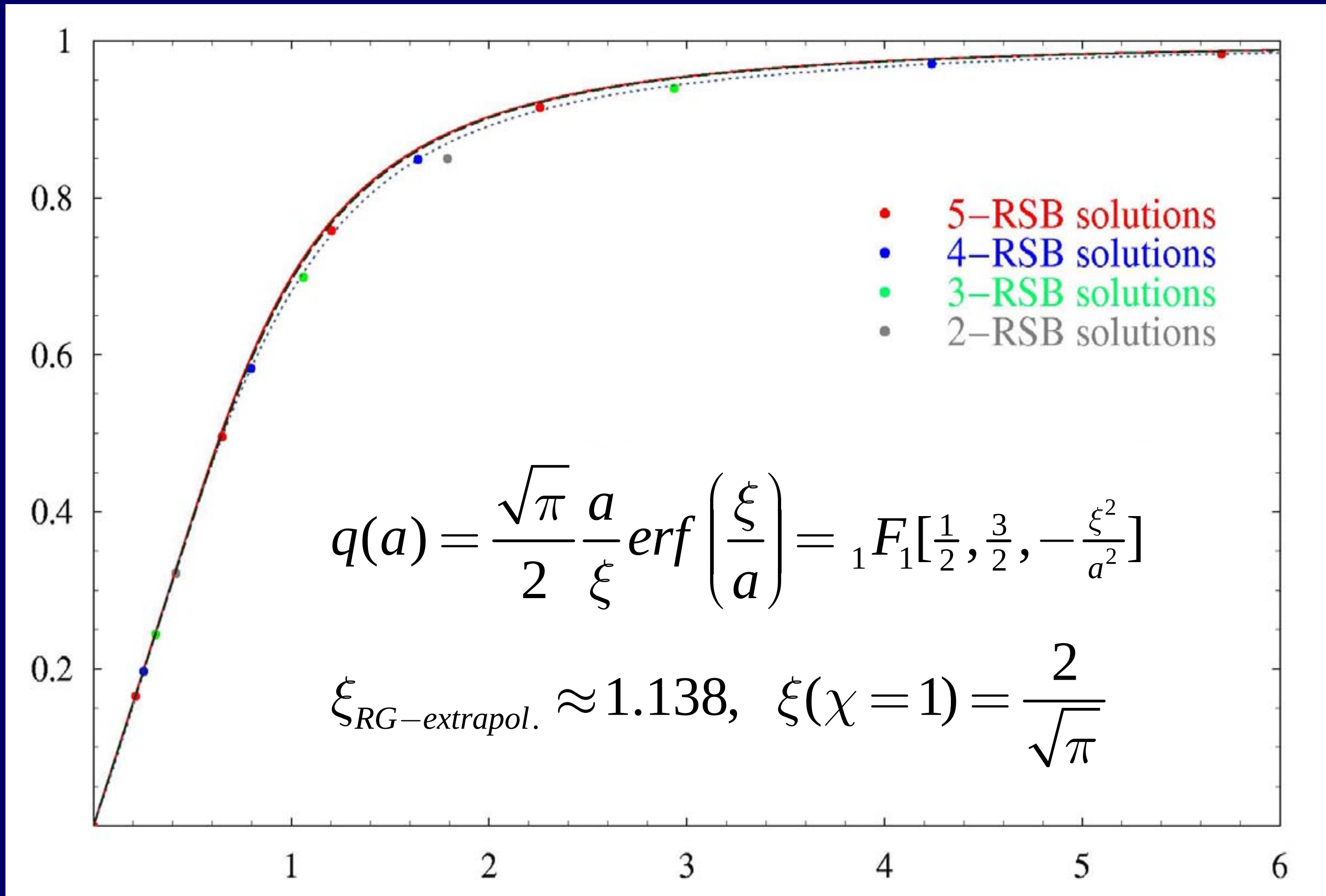
- For $T \to 0,\ x_i^K \to a_i^K T$
- Hence study $q(a) = Lim_{T \to 0} q_{Parisi}(aT)$
  - via sequence of discrete RSB orders K
  - numerically very accurate
  - now up to K=200,…….
  - extrapolation to K=∞

Stepped function



Note: not uniform steps

Simple low RSB fit

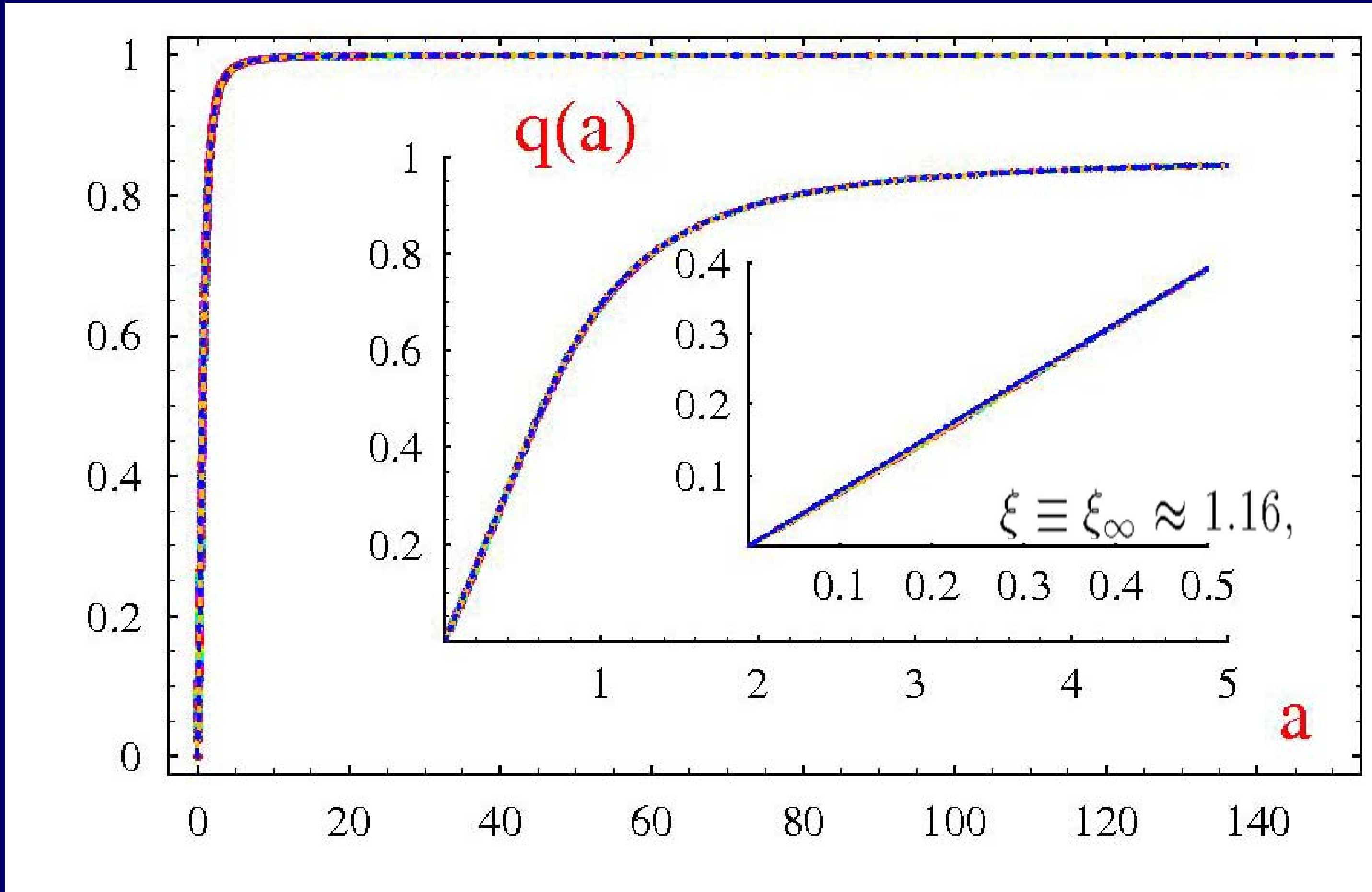


$$q(a) = \frac{\sqrt{\pi}}{2}\frac{a}{\xi} erf\left(\frac{\xi}{a}\right) = {}_1F_1[\tfrac{1}{2}, \tfrac{3}{2}, -\tfrac{\xi^2}{a^2}]$$

$$\xi_{RG-extrapol.} \approx 1.138, \quad \xi(\chi = 1) = \frac{2}{\sqrt{\pi}}$$

No step at a=0. Limiting "correlation length" ξ in RSB space.

42 RSB

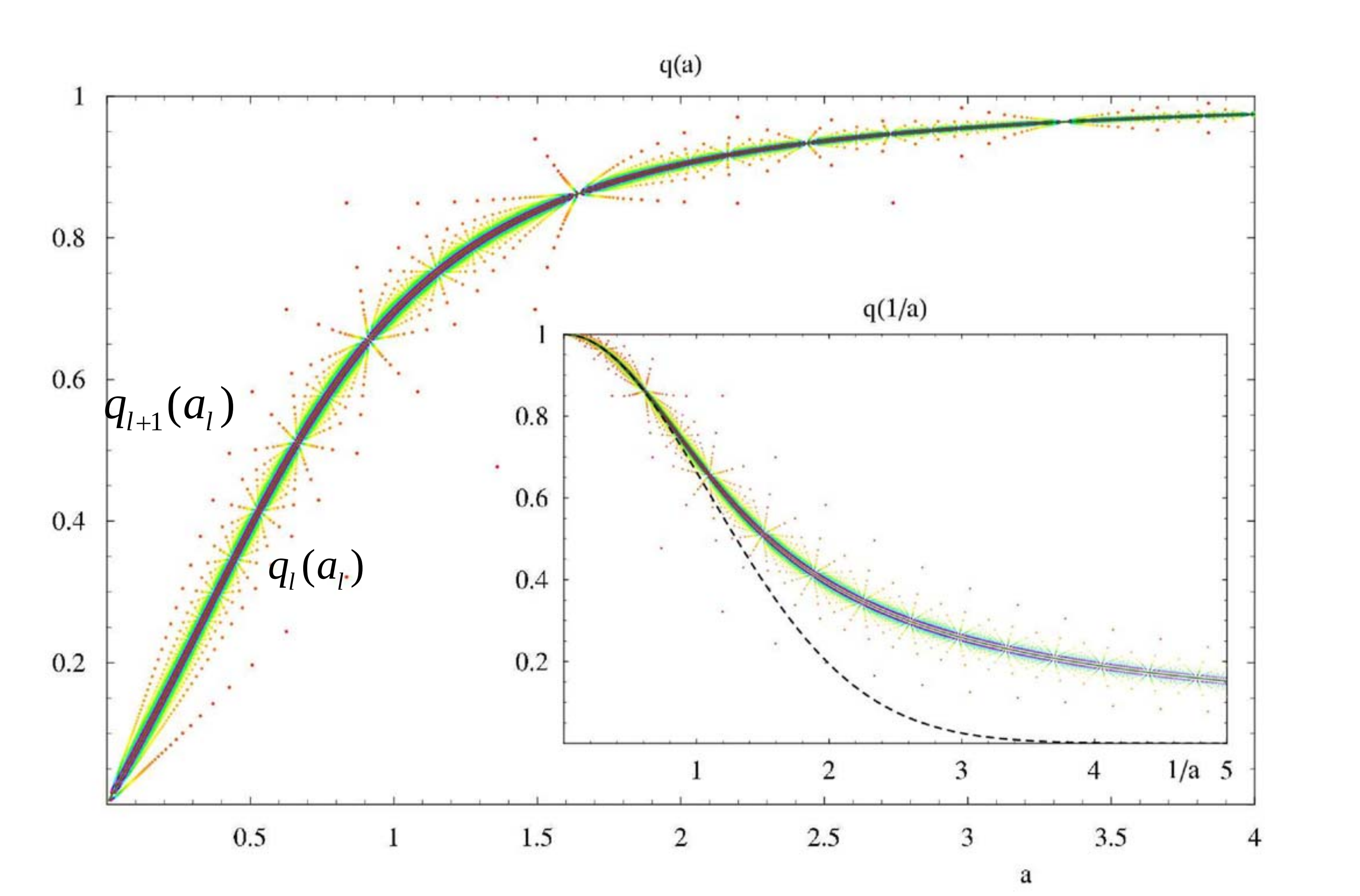


$_1F_1(\alpha, \gamma, -\xi^2/(a^2+w))a/\sqrt{a^2+w}$ with $\{\alpha = \frac{1}{2}, \gamma = \frac{3}{2}, w = O(10^{-6})\}$ blue

$\{\alpha \approx 0.53, \gamma \approx 1.71, w \approx 0.02\}$ orange

Flows in K-space

Oppermann & Schmidt (08)



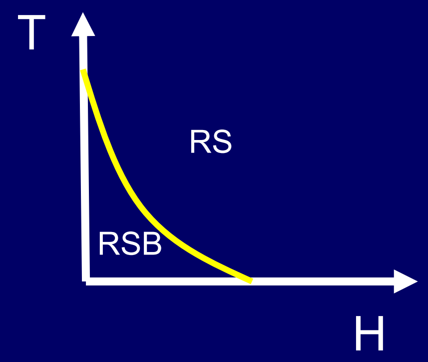
To 200 RSB

Parisi-Toulouse scaling

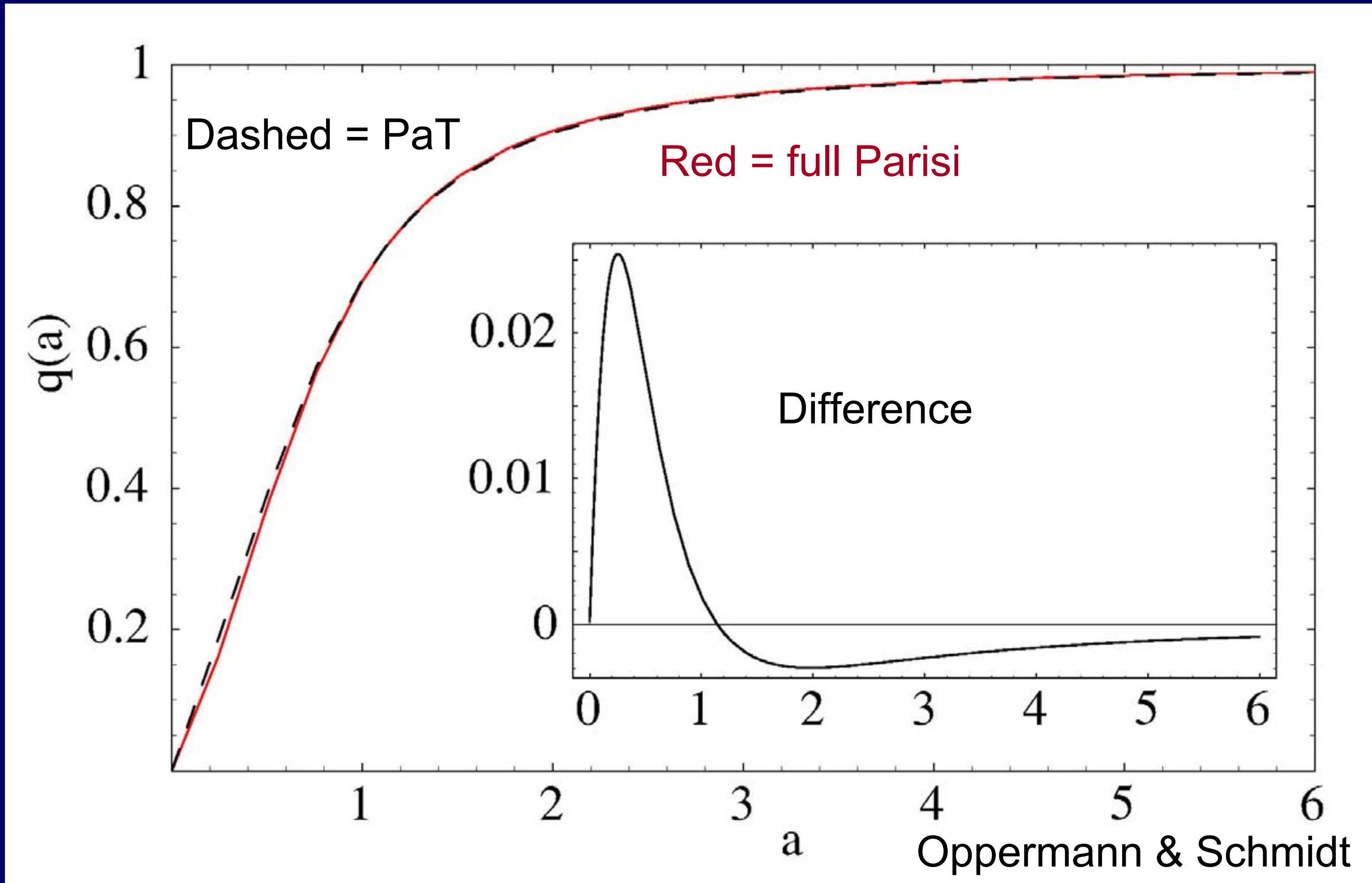
Hypothesis: gives q(x) from RS results on Almeida-Thouless line

$$q(x,T,H) = q_{AT}(T_{AT}(H),H);\ 0 < x < x_1$$
$$q(x,T,H) = f(x/T);\ x_1 < x < x_2$$
$$q(x,T,H) = q_{AT}(T);\ x_2 < x < 1$$

$$M(H,T) = M(H)$$



Parisi vs. PaT at $T=0$



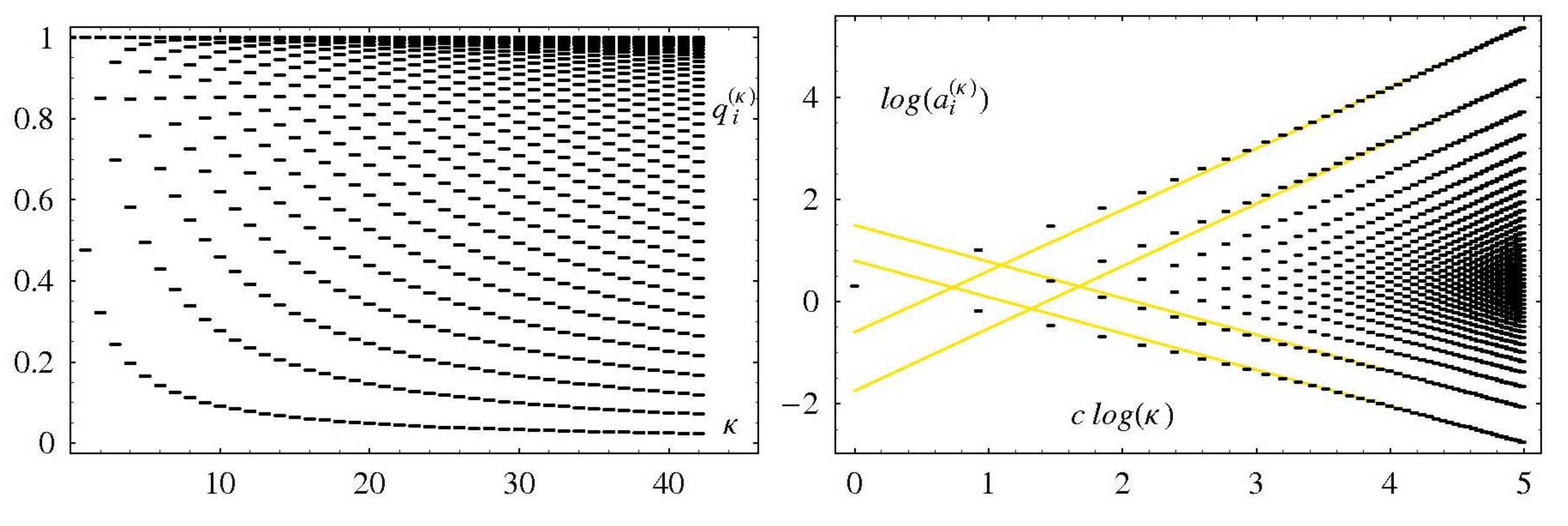
PaT amazingly good but not exact c.f. Crisanti & Rizzo

SKPaT

Villa Farnesina, May 1981

Spectra 1

q_i^K spectra versus RSB order   *a*-spectra versus log K

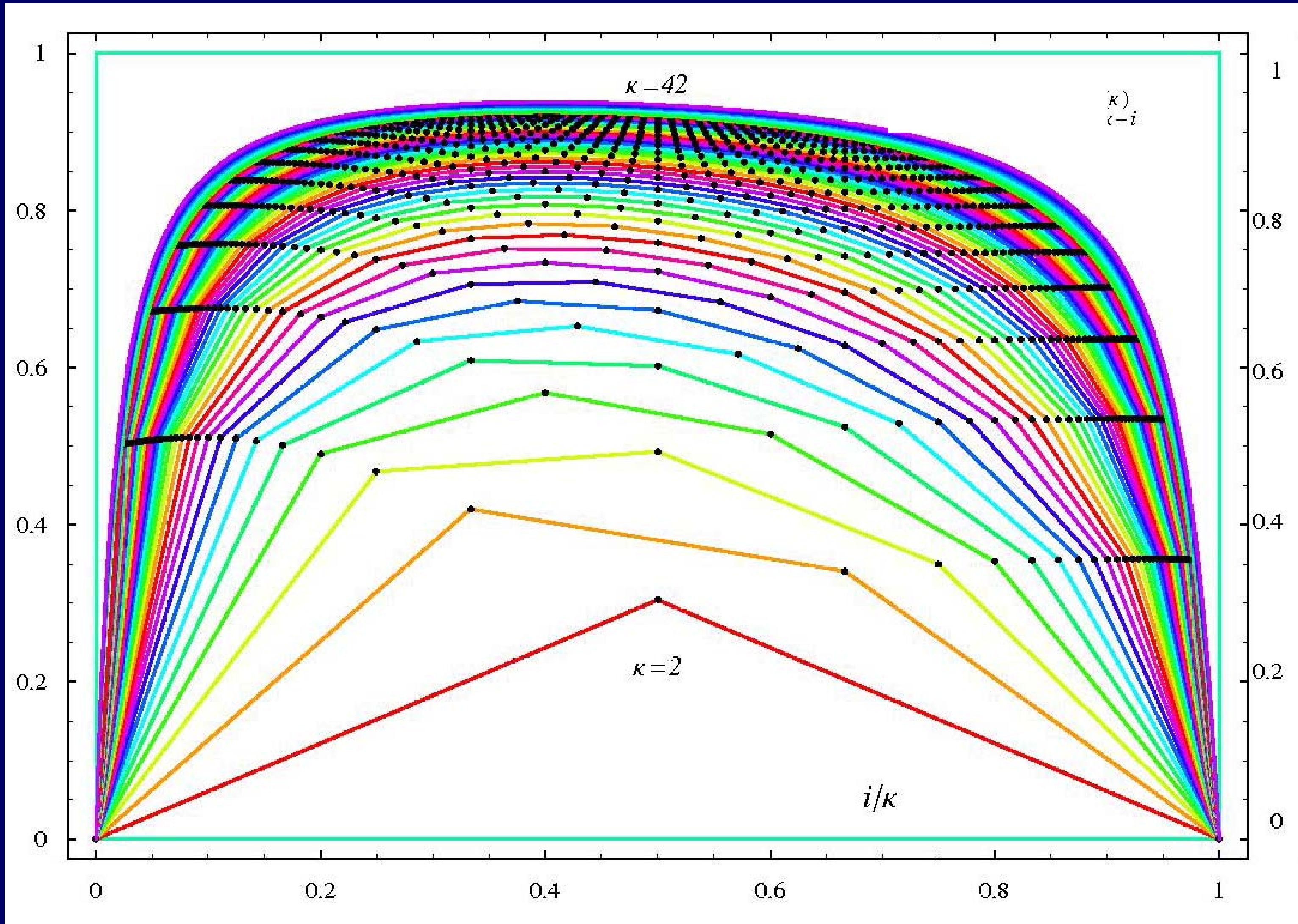


Lines have slope c~4/3

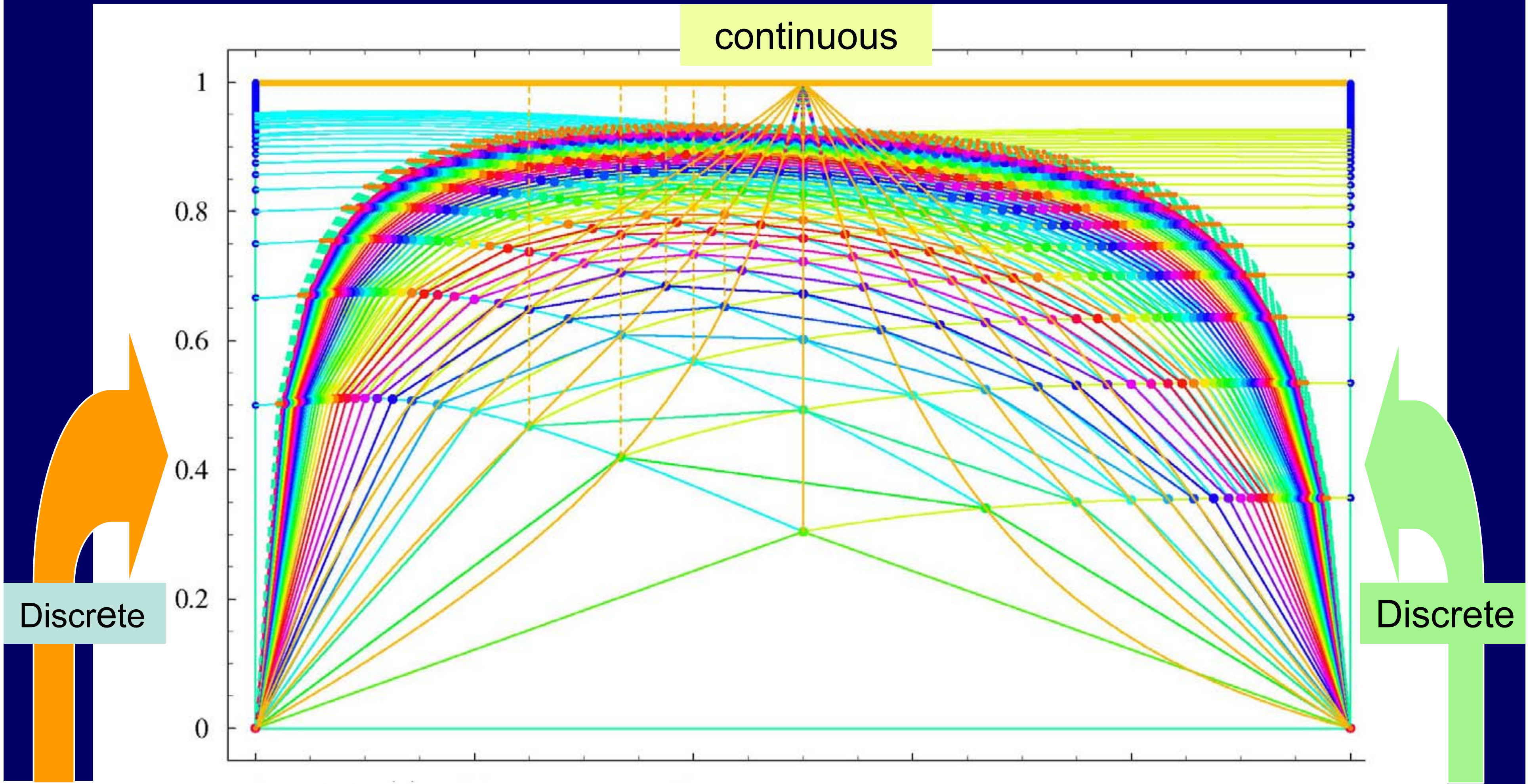
a-distribution not uniform

Bunching near $a \sim O(K^0) \sim \xi$

Ratio spectra: $r^{(\kappa)}_{\kappa-i} \equiv a^{(\kappa)}_{\kappa-i+1} / a^{(\kappa)}_{\kappa-i}$



Pade fits → limiting spectra



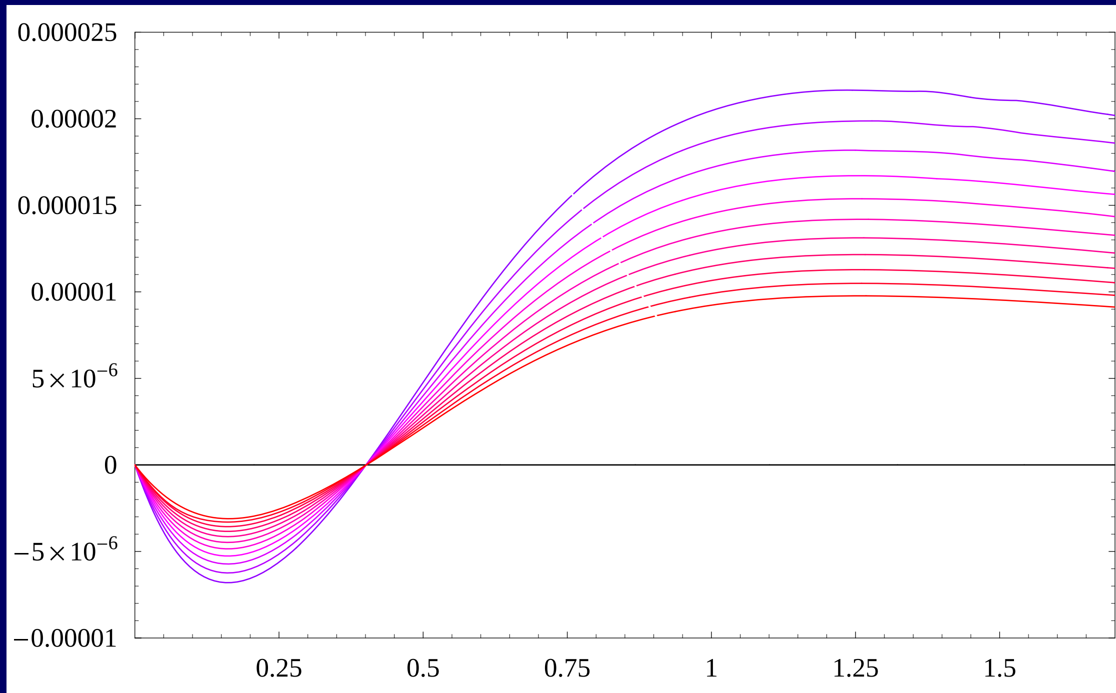
$$\tilde{r}_i \equiv lim_{\kappa\to\infty} r^{(\kappa)}_{\kappa-i} = 1 - 1/(i+1)$$

$$r_i = 1/\sqrt{1 + \tfrac{6}{5}/((i+\tfrac{1}{2})^{2/5} - \tfrac{3}{4})^2)}$$

More curiosities in q

- Invariance points: e.g. $q^K(a)$

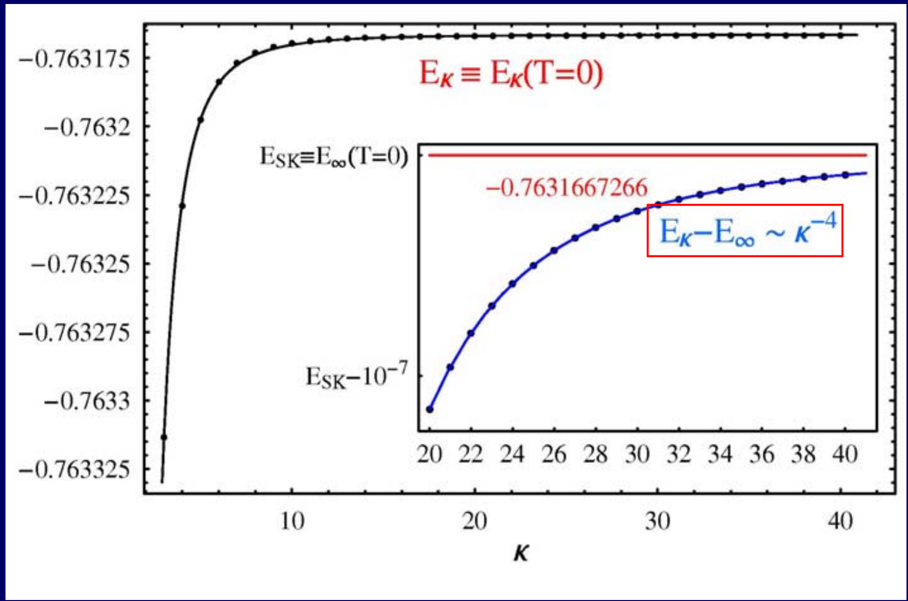
$q^{K+1}(a)-q^K(a)$

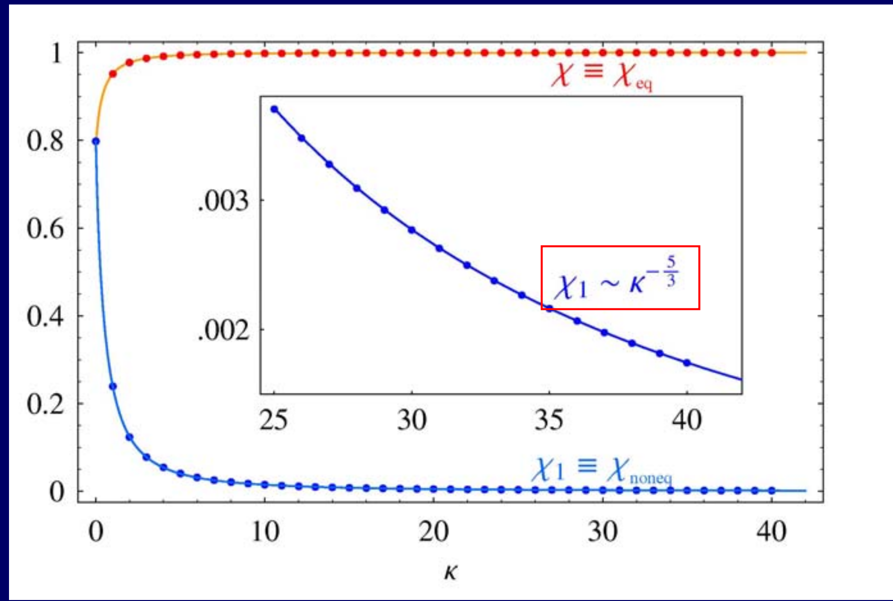


a

? Analogies with finite-size scaling → location of critical points
Meaning/ significance ?

Energy(K) Susceptibilities(K)





$$E - E_\infty \sim K^{-4}$$

42 RSB

$$\chi_{SS} \sim K^{-5/3}$$

Now available to 200 RSB
(Oppermann & Schmidt)

Correlation lengths in RSB order

Oppermann & Schmidt '08

Scaling functions

$$f(K/\xi)$$

$$\xi^1(a=\infty, H=0, T) \sim T^{-\nu_T}; \quad \nu_\mathrm{T} = 3/5$$

$$\xi^1(a=0, H, T=0) \sim T^{-\nu_H}; \quad \nu_\mathrm{H} = 2/3$$

Finite N: K(N) ?

- Aspelmeier, Billoire, Marinari, Moore ('08)

Finite-N SK → Can truncate RSB at $K \sim N^{1/6}$

- Finite-N simulations of E_{GS} (e.g. Boettcher '05)

$$e_0 = -0.7632 + 0.70 N^{-2/3}$$

- Finite-K RSB (Oppermann et al. '07,'08)

$$e_0 = -0.7632 - 0.0467 K^{-4}$$

(*c.f.* Parisi, Janic & Klic)

i.e. suggest also $K \sim N^{1/6}$

But deviations of opposite sign

∴ need terms beyond mean field

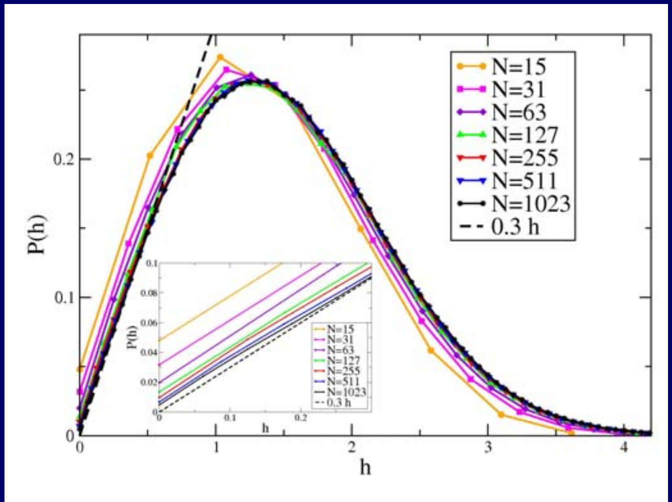
& self-consistent self-energy corrections to propagators

Also P(h,T=0)

Finite-N sim^n

$p(h)$ raised at low h

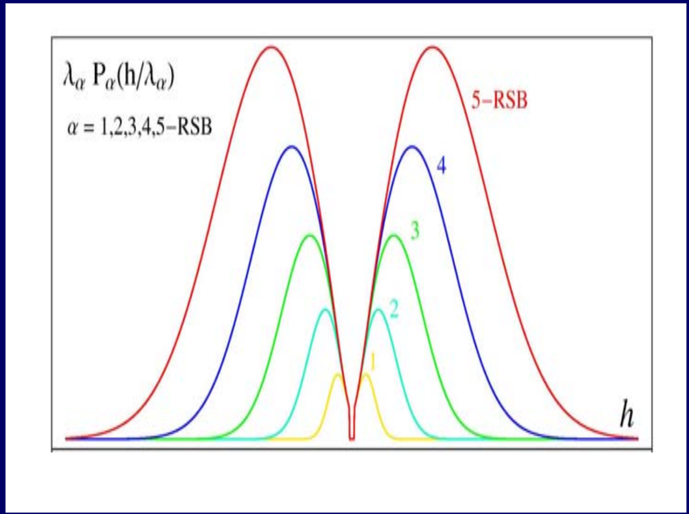
- $p(h=0) \sim N^{-1/2}$
  (Boettcher et al. '08)



Finite-K RSB

$p(h)$ hole at low h

- width ~ $K^{-5/3}$
  (Oppermann et al. '07)



Again, different deviations. Need to go beyond m.f.t.

Questions

- Physically, why is PaT so good?

  without being perfect?
- Beyond m.f.t. – finite N?
- Extensions of FRSB to dilute random s.g. (e.g. Viana-Bray)? $q^{\alpha\beta}, q^{\alpha\beta\gamma}, \ldots\ldots q^{\alpha\beta\ldots\ldots\ldots\omega}$
  - or corresponding overlap distribution $P(\{q\})$
  - beyond 1RSB and cavity field distributions

Note: need for Full RSB?

- Many systems: first transition to 1RSB
  - Lack the symmetry of definiteness of SK
    - p(>2) -spin glass
    - Potts
    - Quadrupolar
    - Dynamically self-disordered
      - Structural glasses
- But normally a later transition to FRSB

Nonsymmetric spin glass

p > 2

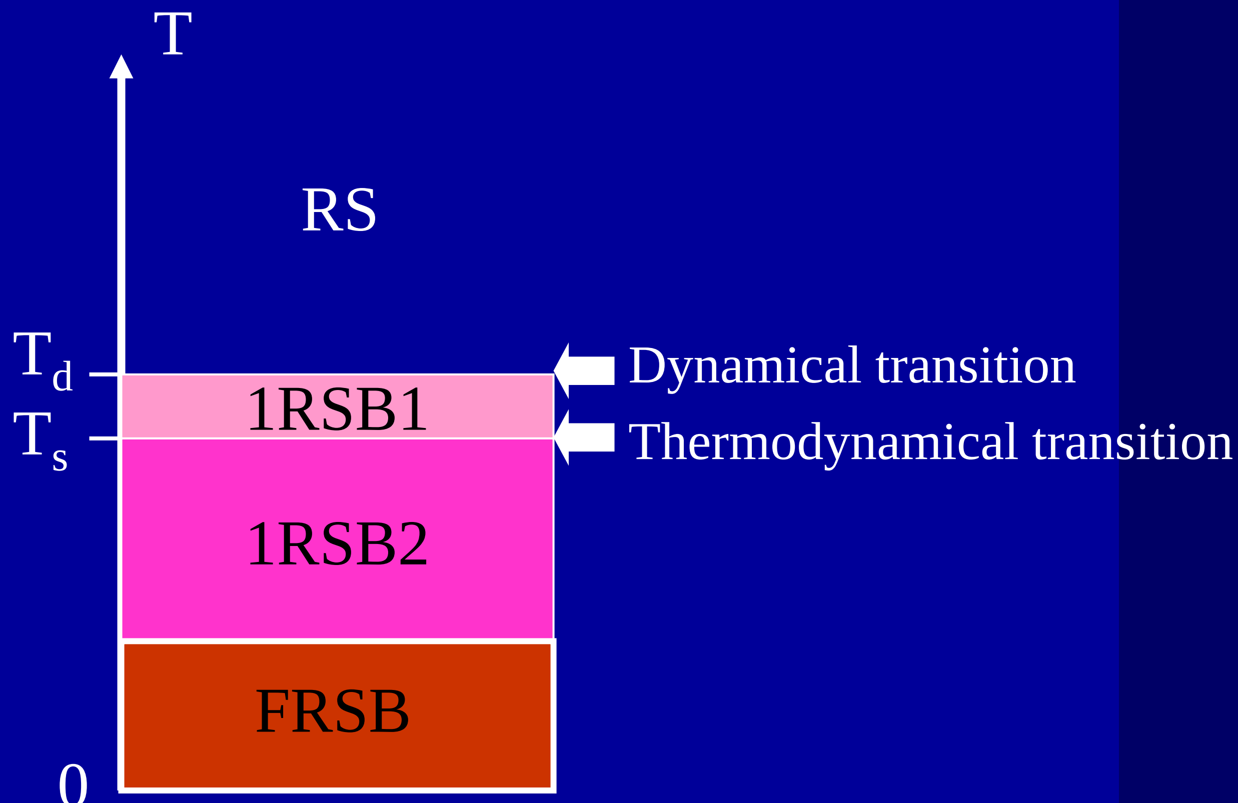
Potts

Quadrupoles

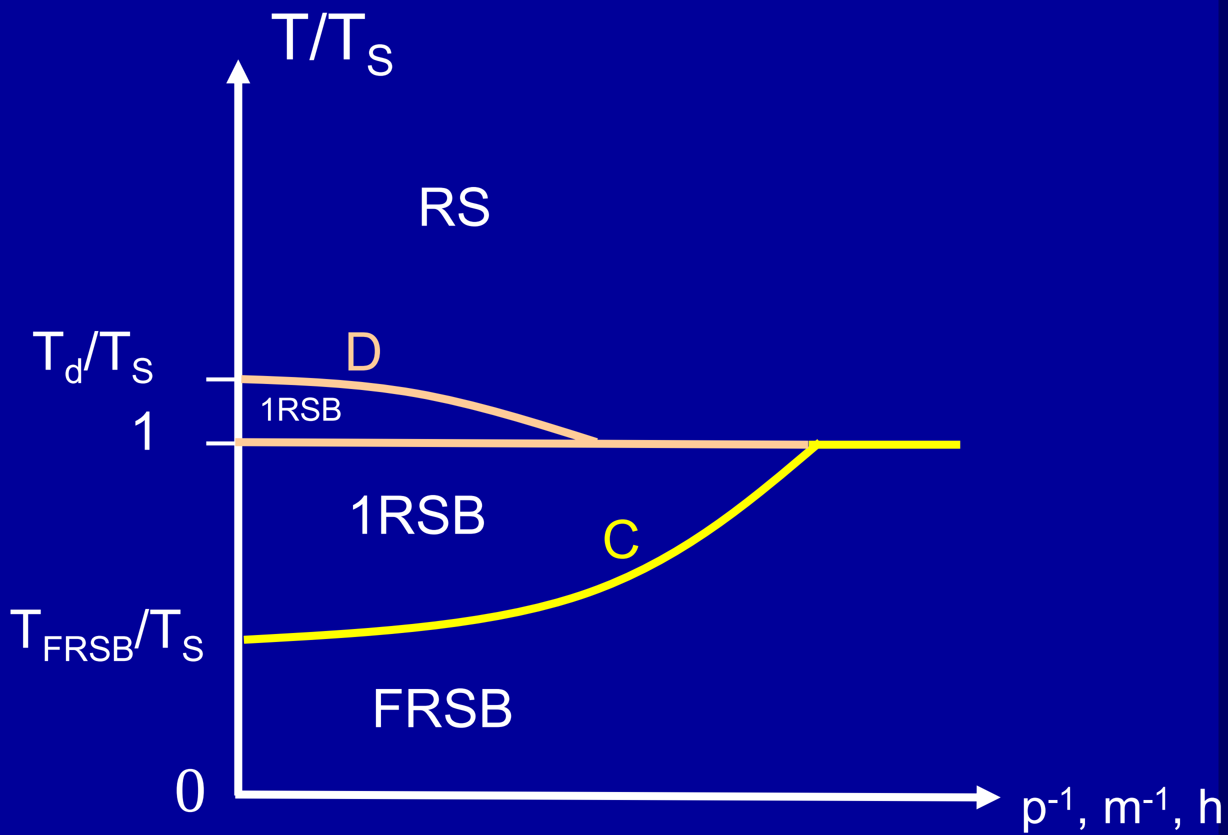
↓

Structural glasses

Protein folding

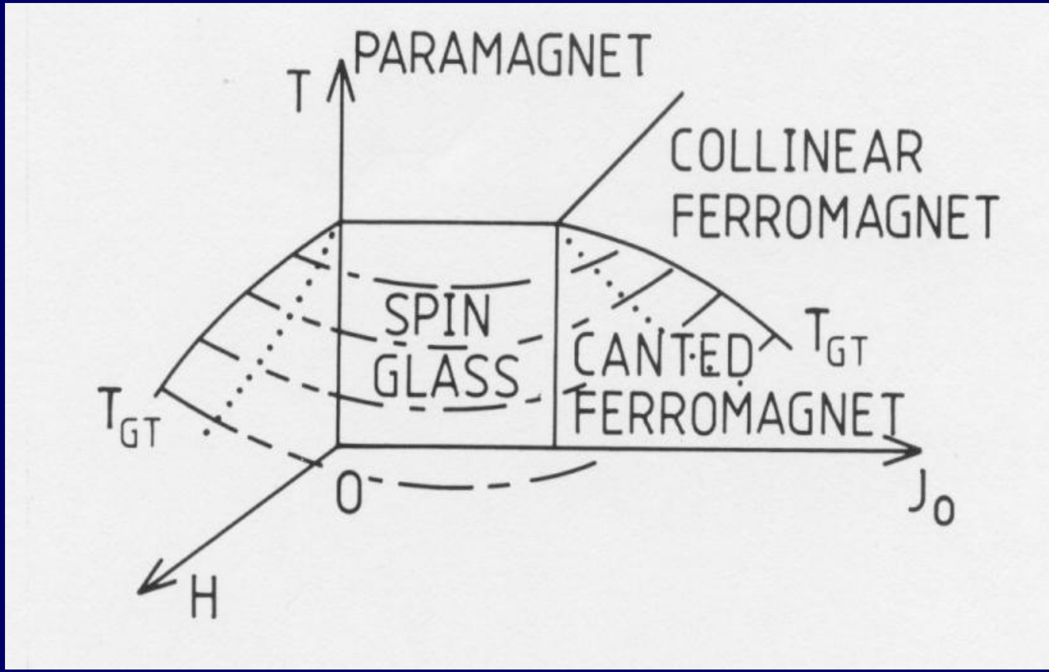


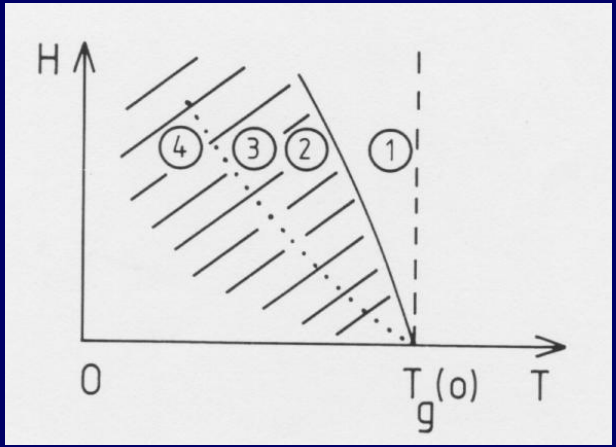
Generic phase transitions



Potts, quadrupolar, p-spin in field

Vector spin glass





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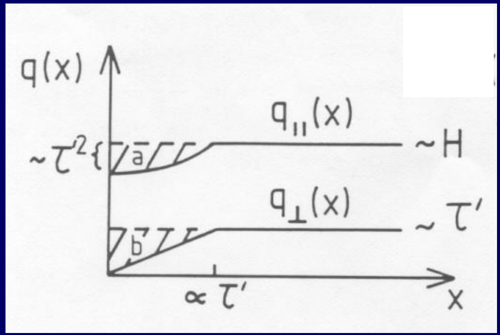
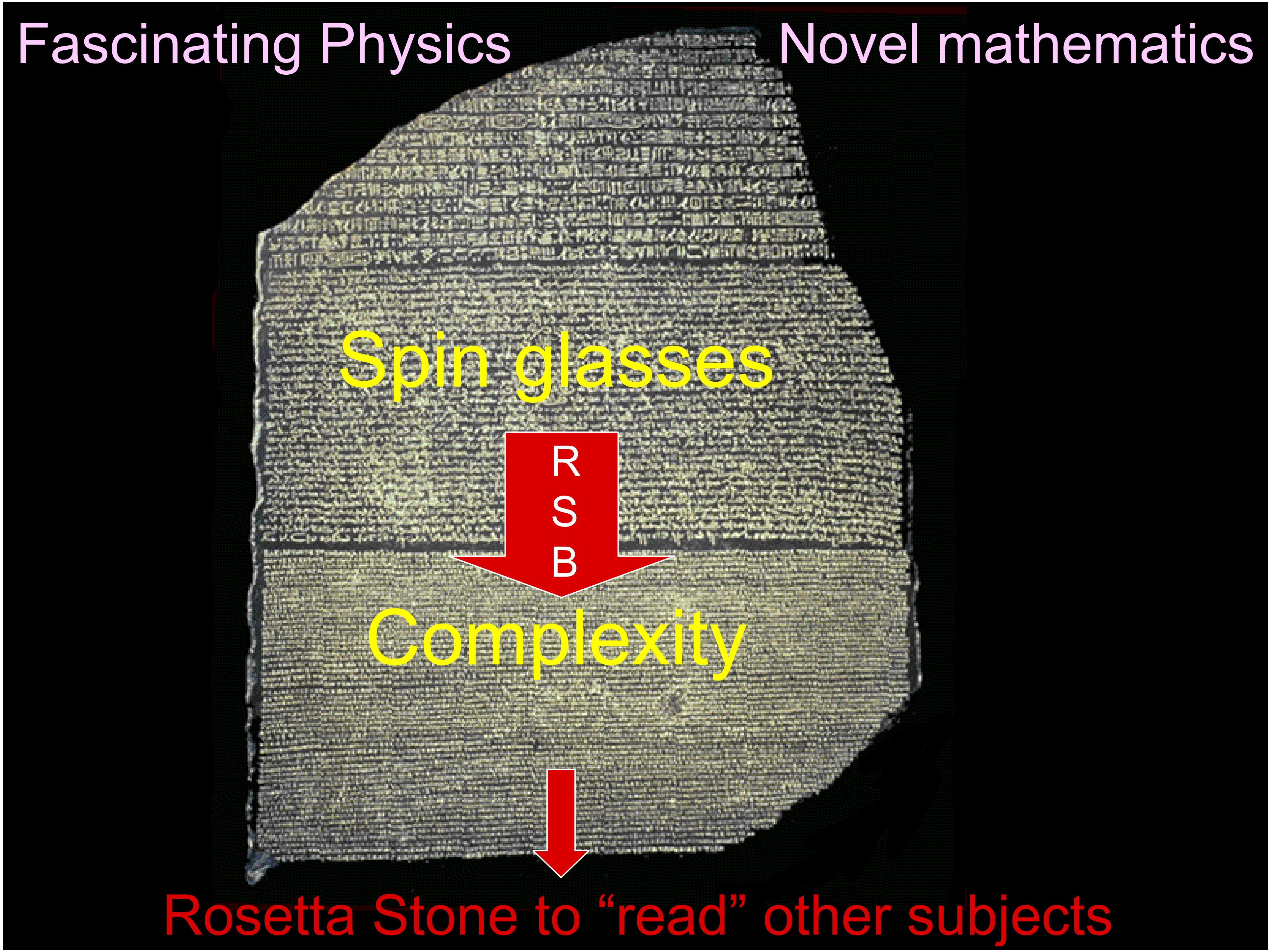


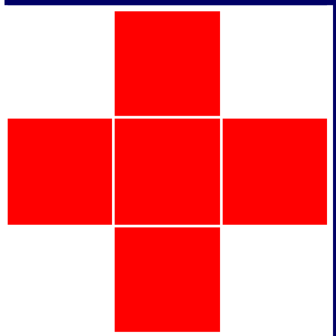
Aside: How did I get to SK?

- PhD functional integrals in m.b.t. – Sam Edwards (63-66)
  - Had introduced auxiliary fields by Hubbard-Stratovich
  - Knew ∞-range & *N*-scaling $\rightarrow$ m.f.t. & solubility for fm.
- Joined IC group where expts on spin glasses (69-)
  - Magnetic impurities; isolated $\rightarrow$ collective
  - Statistical clustering in transition metal alloys (73)
    - Analogies with Anderson localization $\rightarrow$ spin glass: <u>Rh</u>Co
- Sam tried his new EA ideas on me (74)
  - Clearly very interesting, but many unusual ansätze
  - Wanted to find simpler way and exact model to check
  - Hence $\rightarrow$ infinite-ranged model $\rightarrow$ SK


Fascinating Physics
Novel mathematics
Spin glasses
R
S
B
Complexity
Rosetta Stone to “read” other subjects

San Giorgio

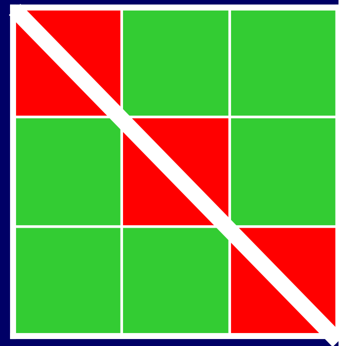
Patron or honoured



- Belgium
- Brazil
- Bulgaria
- England
- Georgia
- Greece
- India
- Italy
- Lebanon
- Malta
- North Ossetia
- Palestine
- Portugal
- Serbia
- Spain
- United States

Giorgio-San

Seijin & sensei



- Europe, ESF,EC
- Belgium
- Brazil
- China
- Czech Republic
- Denmark
- France
- Germany
- Greece
- Hungary
- Italy
- Japan
- Spain
- Switzerland
- United Kingdom
- United States…………….

Giorgio Primo

2003 La Massa Giorgio Primo
93 Points Wine Spectator

Dark and powerful
with intense aromas of tar, berry and raisins.
Full-bodied, with loads of fruit and big, velvety tannins.
Full-throttle. Huge wine.

Discussions at Ascona

Photos by
Erwin Bolthausen

Happy birthday