Contents

Chapter 16

Plane Waves as exact solutions of Einstein's equation

16.1 What kind of solutions are gravitational waves?

16.1.1 Linearized plane wave

We have seen that if we consider a small perturbation of a flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1 \tag{16.1}$$

$h_{\mu\nu}$ satisfies the Einstein equations in vacuum which, by a suitable choice of the gauge, can be reduced to

$$\begin{cases} \Box_F h_{\mu\nu} = 0 \\ h^{\mu}{}_{\nu,\mu} = \frac{1}{2} h^{\mu}{}_{\mu,\nu} \end{cases} \tag{16.2}$$

These equations admit plane waves solutions of the form

$$h_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha} \tag{16.3}$$

provided $\vec{k}$ is a null vector, and satisfies the further constraint of being orthogonal to the polarization tensor $\mathbf{A}$, i.e.

$$k_\alpha k^\alpha = 0 \qquad \text{and} \qquad A_{\mu\nu} k^\nu = 0 \tag{16.4}$$

We have seen that in the TT-gauge the metric of a linearized plane gravitational wave propagating along the x-direction can be written as

$$ds^2 = -c^2 dt^2 + dx^2 + (1 + h_{yy}) dy^2 + 2h_{yz} dy dz + (1 - h_{yy}) dz^2, \tag{16.5}$$

where the metric functions depend on $(t - \frac{x}{c})$ only. The component of the Riemann tensor for such a metric are

$$R_{ambn} = \frac{1}{2} \left[h_{an,mb} + h_{mb,an} - h_{mn,ba} - h_{ab,mn} \right] + O(h^2), \tag{16.6}$$

where and $h_{mn,ab} = -k_a k_b h_{mn}$. It is easy to show that:

$$R_{ambn} k^n = 0, \tag{16.7}$$

which means the wave vector is an eigenvector of the Riemann tensor.

In the TT-gauge the only non vanishing components of the Riemann tensor are:

$$R_{\alpha 0\beta 0} = -\frac{1}{2c^2}\frac{d^2 h_{\alpha\beta}}{dt^2}.$$

16.2 Plane waves: exact solutions

We want to find an exact solution of Einstein's equations in vacuum, which satisfies the following conditions:

$$1)\quad R_{ambn}k^n = 0; \qquad\qquad 2)\quad k_\alpha k^\alpha = 0 \tag{16.8}$$

We also impose that

$$3)\qquad k_{\alpha;\beta} = 0, \tag{16.9}$$

i.e. that the "rays" are parallel. Condition 3) implies 1), since

$$k_{m;s;q} - k_{m;q;s} = R^{\alpha}_{msq}k_\alpha.$$

The vector $\vec{k}$ identifies a congruence of worldlines $x^\mu(\lambda)$ so that $k^\mu = \frac{dx^\mu}{d\lambda}$. The condition $k_{\alpha;\beta} = 0$ implies that

$$k^{\alpha}{}_{;\beta}k^\beta = 0 \qquad \to \qquad \left(\frac{\partial k^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta}k^\mu\right)\frac{dx^\beta}{d\lambda} = 0 \tag{16.10}$$

i.e.

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^{\alpha}{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{16.11}$$

which means that k^α is a geodesic vector.

How can we choose the coordinates in such a way that the metric is as simple as possible? We note that

$$k_{\alpha;\beta} - k_{\beta;\alpha} = k_{\alpha,\beta} - \Gamma^{\mu}_{\alpha\beta} k_{\mu} - k_{\beta,\alpha} + \Gamma^{\mu}_{\alpha\beta} k_{\mu} = k_{\alpha,\beta} - k_{\beta,\alpha}; \tag{16.12}$$

it follows that, since $k_{\alpha;\beta} = 0$,

$$k_{\alpha,\beta} - k_{\beta,\alpha} = 0 \tag{16.13}$$

which means that k_{α} is the gradient of a scalar function $u(x^{\mu})$

$$k_{\alpha} = u(x^{\mu})_{,\alpha} \tag{16.14}$$

and consequently $\vec{k}$ is **hypersurface orthogonal**. Then, we can choose the coordinate line x^0 as coincident with the worldline to which $\vec{k}$ is tangent, and since $\vec{k}$ is a null vector, this will be a null coordinate; thus we put $x^0 = u$, and in this coordinate system, (u, x^1, x^2, x^3), the components of $\vec{k}$ are $k_{\alpha} = (1, 0, 0, 0)$.

As k^{μ} is a null vector $g^{\alpha\beta} k_{\alpha} k_{\beta} = 0$. Consequently , if we choose the first tetrad vector as coincident with $\vec{k}$ we find that

$$g^{00} = \vec{e}^{(0)} \cdot \vec{e}^{(0)} = \vec{k} \cdot \vec{k} = 0. \tag{16.15}$$

Now we can make a coordinate transformation, $x^{\alpha'}(x^{\alpha})$, $\alpha = 1, 2, 3$ (three degrees of freedom) and put

$$g^{01} = g^{02} = 0, \quad g^{03} = 1. \tag{16.16}$$

It follows that, being $g^{0\alpha}g_{\alpha\beta} = \delta^0_\beta$,

$$g^{0\alpha}g_{\alpha 0} = g^{00}g_{00} + g^{01}g_{10} + g^{02}g_{20} + g^{03}g_{30} = g_{30} = 1 \rightarrow \mathbf{g_{30} = 1}$$

$$g^{0\alpha}g_{\alpha 1} = g^{00}g_{01} + g^{01}g_{11} + g^{02}g_{21} + g^{03}g_{31} = g_{31} = 0 \rightarrow \mathbf{g_{31} = 0}$$

Similarly

$$\mathbf{g_{32} = g_{33} = 0}$$

Moreover, since $k_{\alpha;\beta} = 0$, it follows that

$$g_{\alpha\beta,3} = 0$$

Thus the metric is independent of the coordinate x^3.

Proof:

$$k_{\alpha;\beta} = 0 \qquad \rightarrow \qquad k_{\alpha,\beta} - \Gamma^\mu_{\alpha\beta}k_\mu = 0.$$

Since $k_\alpha = (1, 0, 0, 0)$, then

$$k_{\alpha,\beta} = 0 \qquad \rightarrow \qquad \Gamma^\mu_{\alpha\beta}k_\mu = 0. \tag{16.17}$$

$$\Gamma^\mu_{\alpha\beta}k_\mu = \frac{1}{2}g^{\sigma\mu}\left(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}\right)k_\mu = \frac{1}{2}\left(g_{\alpha 3,\beta} + g_{\beta 3,\alpha} - g_{\alpha\beta,3}\right) = 0,$$

and consequently

$$g_{\alpha\beta,3} = 0 \qquad \text{q.e.d..}$$

Thus, at this point the situation is this:

$$g_{31} = g_{32} = g_{33} = 0; \quad g_{30} = 1; \quad g_{\alpha\beta,3} = 0$$

$$g_{00}, g_{01}, g_{02} \neq 0;\ g_{11}, g_{12}, g_{13} \neq 0.$$

Let us now define

$$g_{00} = 2m_0(u, x^1, x^2) \qquad g_{01} = m_1(u, x^1, x^2) \qquad g_{02} = m_2(u, x^1, x^2) \tag{16.18}$$

and set $x^3 = v$ and $x^1 = x,\ x^2 = y$. The metric becomes

$$ds^2 = 2m_0 du^2 + 2dudv + 2m_1 dudx + 2m_2 dudy + g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2. \tag{16.19}$$

The 2-metric

$$g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$$

can be put in the form

$$p^2(u, x, y)\left(dx^2 + dy^2\right)$$

through a coordinate transformation involving only x and y, therefore the metric becomes

$$ds^2 = 2m_0 du^2 + 2dudv + 2m_1 dudx + 2m_2 dudy + p^2\left(dx^2 + dy^2\right)$$

$$g_{\alpha\beta} = \begin{array}{c} \begin{matrix} u & x & y & v \end{matrix} \\ \begin{pmatrix} 2m_0 & m_1 & m_2 & 1 \\ 0 & p^2 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}. \tag{16.20}$$

If we now calculate the the components of the Einstein equations, we find that $R_{xx} = 0$, and $R_{yy} = 0$ reduce to the same equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln p = 0 \tag{16.21}$$

the solution of which gives

$$\ln p = Ref(x + iy) \tag{16.22}$$

where f is an arbitrary, analytic function of x and y.

At this point we can make a rotation in the (x, y)-plane in order to set $p = 1$.

We now compute the Einstein equations $R_{ux} = R_{uy} = 0$. We find:

$$2R_{ux} = (m_{2,x} - m_{1,y})_{,y} = 0,$$

$$2R_{uy} = (m_{2,x} - m_{1,y})_{,x} = 0,$$

therefore we can set

$$(m_{2,x} - m_{1,y}) = F'(u).$$

It can be shown that with the following coordinate transformation

$$\begin{cases} \overline{v} = x^3 + \int m_1 dx - \frac{1}{2}F'(u)x\ y \\ \overline{u} = F(u) \\ \overline{x} = xcos(u) + ysin(u) \\ \overline{y} = -xsin(u) + ycos(u) \\ \text{and} \\ \hat{v} = \overline{v}F'^{-1}(u) \end{cases} \tag{16.23}$$

the metric can be cast in the form

$$ds^2 = \left[2d\overline{u}d\hat{v} + d\overline{x}^2 + d\overline{y}^2\right] + H(\overline{u},\overline{x},\overline{y})d\overline{u}^2, \tag{16.24}$$

where the function $H(\overline{u},\overline{x},\overline{y})$ satisfies the equation

$$\left(\frac{\partial^2}{\partial\overline{x}^2} + \frac{\partial^2}{\partial\overline{y}^2}\right) H(\overline{u},\overline{x},\overline{y}) = 0 \tag{16.25}$$

which comes from $R_{\overline{u}\overline{u}} = 0$. It should be stressed that eq. (16.25) is a **linear equation**, but $H(\overline{u},\overline{x},\overline{y})$ can now be arbitrarily large.
The simplest solution is

$$H = (\overline{x}^2 - \overline{y}^2)h(\overline{u}). \tag{16.26}$$

It should be noted that

- the part between square brackets in the metric (16.24) is the flat line-element, indeed

  $$-c^2dt^2 + dz^2 + dx^2 + dy^2 \quad \rightarrow \quad 2dudv + dx^2 + dy^2$$

  where $u = \frac{1}{\sqrt{2}}(z - ct)$, $v = \frac{1}{\sqrt{2}}(z + ct)$. Thus eq. (16.24) represent a flat spacetime plus "something traveling along the u-direction".

- Since eq.(16.25) is linear, two independent solutions of this equation can be superimposed. It follows that waves that propagate in the same direction do not interact.

Since a solution of eq. (16.25) regular in the whole $\overline{xy}$ plane does not exist, H always owns a singularity. However, the following change of coordinates

$$x = \overline{x}a(u), \qquad v = \overline{v} - \frac{1}{2}a'a\overline{x}^2 - \frac{1}{2}b'b\overline{y}^2,$$
$$y = \overline{y}b(u), \quad \text{with} \quad h(u) = \frac{a''}{a} = -\frac{b''}{b},$$

allows to cast the metric in a singularity-free form

$$ds^2 = 2d\overline{u}d\overline{v} + a^2(u)dx^2 + b^2(u)dy^2, \tag{16.27}$$

where a and b are arbitrary functions of u which satisfy the constraint imposed by Einstein's equations

$$a''b + ab'' = 0 \tag{16.28}$$

If we now put $a = 1 + \frac{\alpha}{2}$, $b = 1 - \frac{\alpha}{2}$ and linearize assuming α small, we find

$$ds^2 = 2d\overline{u}d\overline{v} + (1+\alpha)dx^2 + (1-\alpha)dy^2$$

which is the metric of a weak plane-gravitational wave solutions of Einstein's equations which describes the perturbations of a flat spacetime discussed in chapter 11.

In conclusion, the metric of an arbitrarily strong gravitational wave, exact solution of Einstein's equations in vacuum, can be written as

$$ds^2 = \left[2d\overline{u}d\overline{v} + d\overline{x}^2 + d\overline{y}^2\right] + H(\overline{u}, \overline{x}, \overline{y})d\overline{u}^2, \tag{16.29}$$

with

$$H = (\overline{x}^2 - \overline{y}^2) h(\overline{u}) \tag{16.30}$$

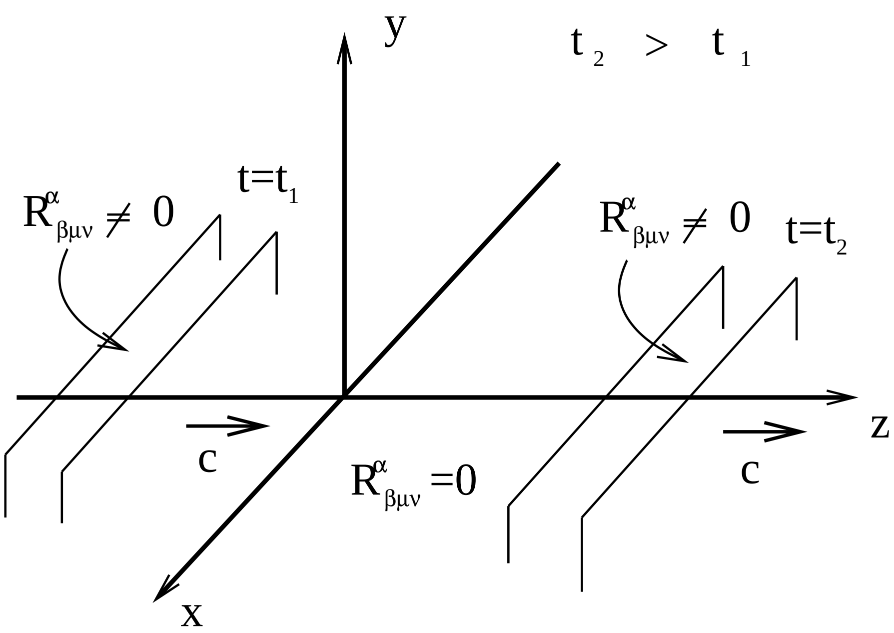
or as

$$ds^2 = 2d\overline{u}d\overline{v} + a^2(u)dx^2 + b^2(u)dy^2, \tag{16.31}$$

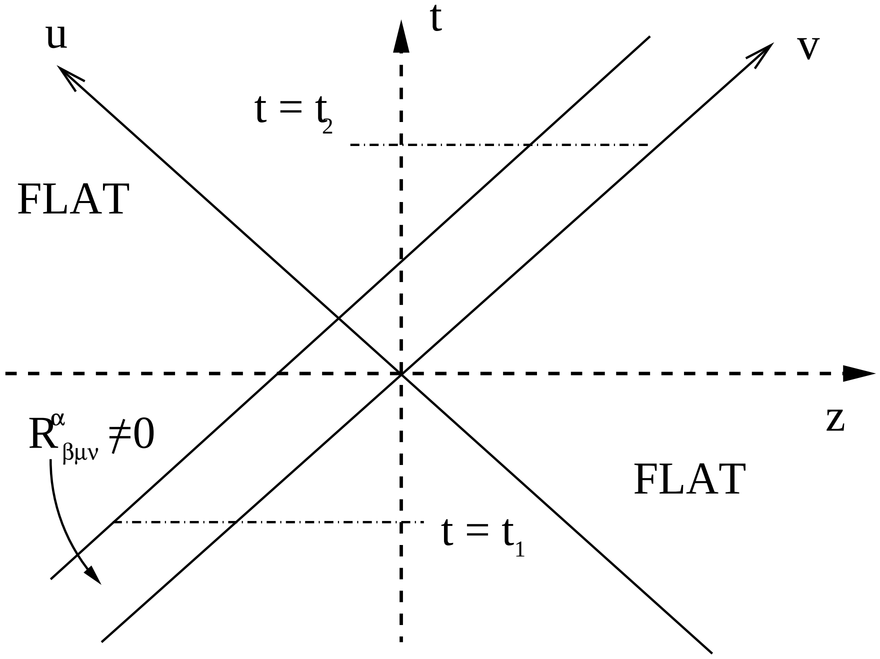
with

$$a''b + ab'' = 0.$$

We define a “sandwich wave” as a region of spacetime confined between two parallel planes, where the Riemann tensor is non vanishing, and which propagates in flat spacetime at the speed of light.



The same wave can be represented in terms of the u-v coordinates, as a projection on the (u,v)-plane as follows



16.2.1 Null Fields

Let us consider a plane electromagnetic wave, whose vector potential is $A_\mu = Re\left(p_\mu e^{ik_\alpha x^\alpha}\right)$, where

$$p_\mu k^\mu = 0 \tag{16.32}$$

$$k_\mu k^\mu = 0.$$

The electromagnetic tensor is

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} = Re\left([p_\mu k_\nu - p_\nu k_\mu] e^{ik_\alpha x^\alpha}\right)$$

For a plane electromagnetic wave the two invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$ vanish; indeed

$$F_{\mu\nu}F^{\mu\nu} = H^2 - E^2 = Re\left([p_\mu k_\nu - p_\nu k_\mu][p^\mu k^\nu - p^\nu k^\mu] e^{2ik_\alpha x^\alpha}\right) =$$
$$Re\left([p_\mu p^\mu k_\nu k^\nu - p_\nu k^\nu k_\mu p^\mu + \cdots] e^{2ik_\alpha x^\alpha}\right) = 0,$$

and

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = F_{\mu\nu}\cdot\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = \vec{H}\cdot\vec{E} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} Re\left([p_\mu k_\nu - p_\nu k_\mu][p^\alpha k^\beta - p^\beta k^\alpha] e^{2ik_\alpha x^\alpha}\right) = 0.$$

Such null fields represent electromagnetic fields far from the sources.

The stress-energy tensor of the null field can be found from the action

$$S_E = -\frac{1}{4}\int d^4x \sqrt{-g} F_{\mu\nu}F^{\mu\nu}.$$

$T_{\mu\nu}$ is the functional derivative of S_E with respect to $g_{\mu\nu}$.
Let us assume that $g_{\mu\nu}$ suffers an infinitesimal variation

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

The action S_E is not stationary with respect to this variation, because the dynamical variables are x^μ and A^μ , while $g_{\mu\nu}$ is an external field.

Therefore δS_E will be a linear functional of $\delta g_{\mu\nu}$:

$$\delta S_E = \frac{1}{2}\int d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}$$

The coefficients of the linear combination are the components of the stress-energy tensor, which can be shown to be

$$T_{\mu\nu} = F_{\alpha\mu}F^{\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}.$$

If the field is null $F_{\alpha\beta}F^{\alpha\beta} = 0$ and $k_\alpha p^\alpha = k_\mu k^\mu = 0$. In addition we have seen that $F_{\alpha\nu} = p_\alpha k_\nu - p_\nu k_\alpha$ and

$$F_{\alpha\nu}F^{\alpha}_{\mu} = (p_\alpha k_\nu - p_\nu k_\alpha)(p^\alpha k_\mu - p_\mu k^\alpha) = p_\alpha p^\alpha k_\nu k_\mu - p_\nu k_\alpha p^\alpha k_\mu - p_\alpha k_\nu p_\mu k^\alpha + p_\nu k_\alpha p_\mu k^\alpha =$$

$$p_\alpha p^\alpha k_\nu k_\mu$$

Setting $p_\mu p^\mu = \Phi$,

$$T_{\mu\nu} = \Phi k_\mu k_\nu. \tag{16.33}$$

In a locally inertial frame if, for example, the wave travels along the x-axis,

$k^\mu = (1,1,0,0)$,

$T_{00} = \Phi(ct - x)$ is the energy-density carried by the wave,

$cT_{0x} = c\Phi(ct - x)$ is the energy which flows across a surface perpendicular to the x-axis per unit time.

The Poynting vector is $S^\nu = cT^{0\nu}$, $W = T^{00} = \Phi$, then

$$S = cW$$

A null field is pure radiation which propagates at the speed of light.

16.3 Gravitational waves generated by null fields

Let us now assume that the gravitational wave is generated by some kind of source. As a first example we shall compute the gravitational field of a plane electromagnetic wave, with stress-energy tensor given by eq. (16.33). Choosing the coordinates as in section 16.2, so that the wave vector is $k_\mu = (1,0,0,0)$, the Einstein equations plus the "parallel-ray" condition give

$$R_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad \rightarrow \quad R_{uu} = \frac{8\pi G}{c^4} T_{uu}, \qquad k_{\mu;\nu} = 0. \tag{16.34}$$

Since the only non-vanishing component of the stress-energy tensor is T_{uu}, the metric can be put in the same form as in (16.24), and eq. (16.34) becomes

$$-\frac{1}{2}\Delta H(x,y,u) = \frac{8\pi G}{c^4}\Phi(u) \qquad \rightarrow \quad h(u)\Delta f(x,y) = -\frac{16\pi G}{c^4}\Phi(u), \tag{16.35}$$

the solution of which is

$$\begin{aligned} f = x^2 + y^2 \qquad h(u) &= -\frac{r84\pi G}{c^4}\Phi(u), \\ H(x,y,u) &= -\frac{8\pi G}{c^4}(x^2+y^2)\Phi(u). \end{aligned} \tag{16.36}$$

Hence the metric describing a plane electromagnetic wave is

$$ds^2 = 2dudv + dx^2 + dy^2 + h(u)(x^2+y^2)du^2. \tag{16.37}$$

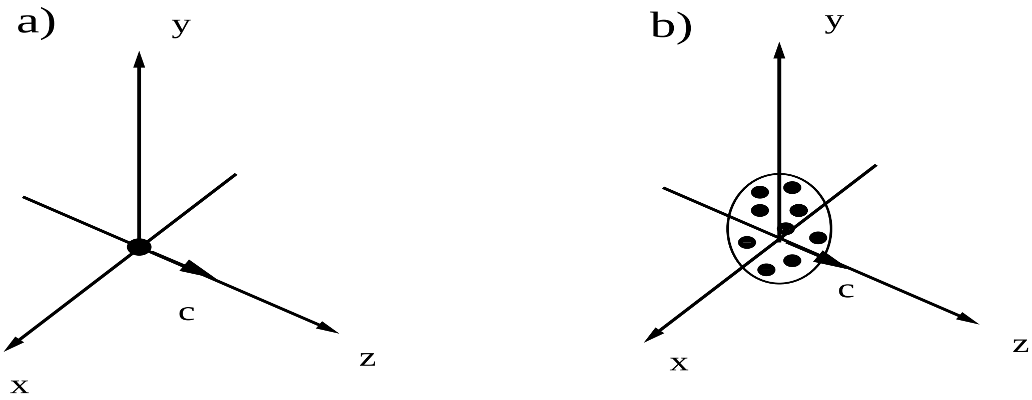
Incidentally, it should be noted that in this case $h(u)$ is negative, a condition which we will use in the following.

Let us now assume that the wave is generated by

- a) a massless particle

- b) a beam of massless particles

traveling along the z-direction.



a) The stress-energy tensor of a massless particle has the form of eq. (16.33), and the only non vanishing component is

$$T_{uu} = p\delta(u)\delta(r), \tag{16.38}$$

where p is the particle momentum If we use polar coordinates on the plane (x, y) and write $H(x, y, u) = h(u)f(r)$ Einstein's equations give

$$\frac{1}{2}h(u)\Delta f(r) = -\frac{8\pi G}{c^4}p\delta(u)\delta(r), \tag{16.39}$$

and consequently

$$a) \quad h(u) = \delta(u), \qquad b) \quad \Delta f(r) = -\frac{16\pi G}{c^4} p\delta(r). \tag{16.40}$$

The solution of eq. (16.40b) can easily be found, since we know the Green function of the problem. In fact

$$\Delta \log(r^2) = 2\pi\delta(r), \tag{16.41}$$

therefore

$$f = -\frac{8G}{c^4} \log(r^2), \qquad r^2 = x^2 + y^2, \tag{16.42}$$

and

$$H = -\frac{8G}{c^4} p\delta(u) \log(r^2). \tag{16.43}$$

This solution has been derived in an alternative way by Bonnor and Aichelburg & Sexl [1] , by boosting the Schwarzschild solution in the limiting case when the velocity of the boost tends to the speed of light and the mass of the particle tends to zero, keeping its energy finite. In this case the gravitational field gets squashed on the surface orthogonal to the line of motion, which becomes the wavefront, and we are left with the impulsive plane wave (16.43) generated by a massless particle.

[1]W.B. Bonnor *Commun. Math. Phys.* **13** (1981) 29

P.C. Aichelburg and R.U. Sexl *Gen. Relativ. Grav.* **2** (1971) 303

b) If the source of the wave is a beam of particles of constant density ρ for $r < r_0$

$$T_{uu} = \rho\delta(u), \qquad r < r_0 \tag{16.44}$$

and the solution of the Einstein equations is

$$H = -\frac{8G}{c^4}\rho r^2\delta(u), \qquad r < r_0. \tag{16.45}$$

In all these cases the only nonvanishing Weyl scalar is $\Psi_0 \neq 0$ and the solution is Petrov Type N.

It should be noted that the two solutions (16.43) and (16.45) are impulsive gravitational waves.

16.4 Impulsive waves and spacetime shifts

Impulsive, plane-fronted, pure gravitational waves are the most elementary wave-like solutions of Einstein's equations. The idea of interpreting these waves as shifts in space and time is originally due to Penrose, and it is known as the 'scissor-and-paste' procedure. It operates in the following way. Consider a flat spacetime

$$ds^2 = 2dudv + dx^2 + dy^2, \tag{16.46}$$

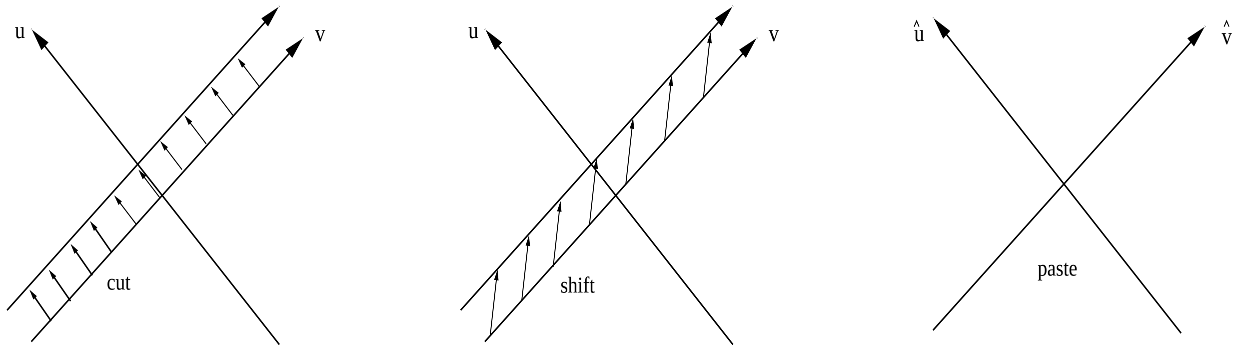
where u and v are null coordinates

$$u = \frac{z-t}{\sqrt{2}}, \qquad v = \frac{z+t}{\sqrt{2}}. \tag{16.47}$$

Cut the spacetime along the surface $u = 0$, and assume that for $u > 0$ the coordinate v is replaced by a shifted coordinate

$$v \to v + \Theta(u) f(x, y), \tag{16.48}$$

where Θ is the Heaviside step-function, and $f = f(x, y)$ is an unspecified function.



For $u > 0$ the metric becomes

$$ds^2 = 2du[dv + \Theta(u) f_{,i} dx^i] + dx^2 + dy^2, \qquad i = 1, 2. \tag{16.49}$$

If we now introduce the following set of new coordinates

$$\begin{aligned} \hat{u} &= u \\ \hat{v} &= v + \Theta(\hat{u}) f \\ \hat{x}^i &= x^i, \end{aligned} \tag{16.50}$$

the metric (16.49) takes the form

$$ds^2 = 2d\hat{u}[d\hat{v} - \delta(\hat{u})f(x,y)d\hat{u}] + d\hat{x}^2 + d\hat{y}^2, \tag{16.51}$$

which is flat everywhere except on the null surface $\hat{u} = u = 0$. Therefore the transformation (16.50) "glues" together the two parts of flat spacetime, $u < 0$ and $u > 0$, that have been shifted with respect to each other, by introducing an impulsive "disturbance" traveling along $u = 0$ at the speed of light.

The solution (16.51) has the same form as eq. (16.24), and represents a wave traveling in flat spacetime along the positive z-direction. The only non-vanishing component of the Ricci tensor is

$$R_{\hat{u}\hat{u}} = \frac{1}{2}\delta(\hat{u})\Delta f, \tag{16.52}$$

where Δ is the laplacian operator in the transverse coordinates $\hat{x}$ and $\hat{y}$. Consequently, the metric (16.51) represents a pure gravitational impulsive wave if

$$\Delta f = 0, \qquad \rightarrow \qquad f = \hat{x}^2 - \hat{y}^2, \tag{16.53}$$

or an impulsive wave associated to a distribution of massless particles if

$$\Delta f \neq 0. \tag{16.54}$$

16.5 The focussing properties of a gravitational plane wave

We shall now study the focussing properties of a gravitational plane wave, following a paper published by Roger Penrose in 1965 [2].

We write the metric in a general form originally due to Brinkman, which includes both the metric of a pure gravitational wave, (16.24), and that generated by a plane electromagnetic wave (16.37)

$$\begin{aligned} ds^2 &= 2dudv + h_{ij}(u)x^i x^j du^2 + \delta_{ij}dx^i dx^i \\ &= 2dudv + (h_{11}x^2 + h_{22}y^2 + 2h_{12}xy)du^2 + dx^2 + dy^2, \qquad i,j = 1,2. \end{aligned} \tag{16.55}$$

All coordinates vary in the range $(-\infty, +\infty)$, thus covering the entire manifold. Without loss of generality we can assume that h_{ij} is symmetric. If h_{ij} is traceless (vanishing Ricci tensor)

$$h_{11} + h_{22} = 0 \quad \to \quad h_{11}x^2 + h_{22}y^2 = h(u)(x^2 - y^2), \tag{16.56}$$

the metric represents *a pure gravitational wave*, linearly polarized if $h_{12} = 0$. If

$$h_{ij}(u) = h(u)\delta_{ij}, \qquad \text{with} \qquad h(u) \le 0. \tag{16.57}$$

then

$$h_{11}x^2 + h_{22}y^2 = h(u)(x^2 + y^2). \tag{16.58}$$

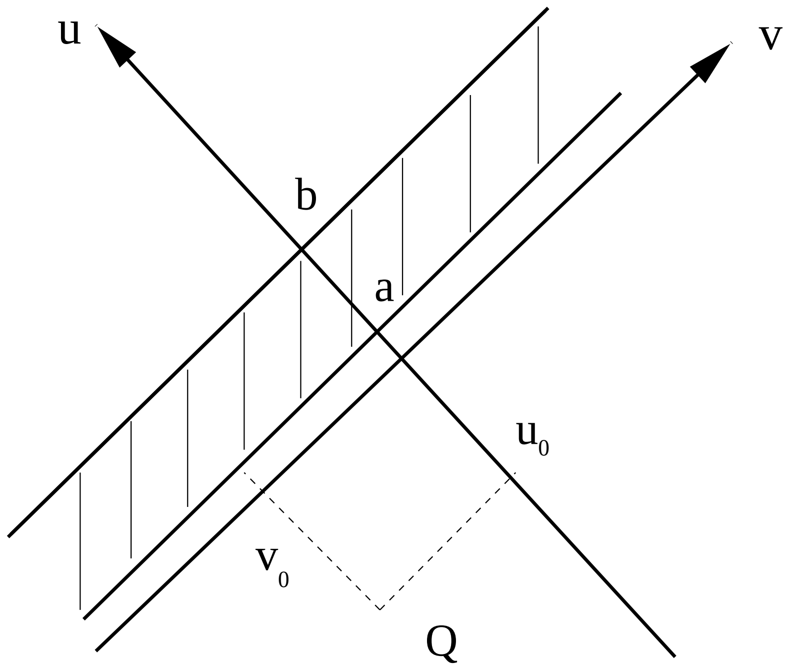
[2] R.Penrose, *Rev. of Mod. Phys.* **37** (1965) 215-220;

In this case the Weyl tensor vanishes, and the metric represents *a pure electromagnetic wave.*

In the following we shall use u as a parameter continuously increasing with time on any timelike or null curve that is not parallel to the world lines

$$u = const \qquad x_i = const, \qquad i = 1, 2. \tag{16.59}$$

These lines can be parametrized with the function v.



A "sandwich" wave is a wave in which h_{ij} vanishes outside a certain range (a, b) of values of u. So the spacetime is flat for $u < a$ and for $u > b$, but curved in between. We shall now see, following Penrose's approach, how a null cone can be forced to focus again to a second vertex by the passage of a plane wave.

a) Before the passage of the wave

Let Q be a point in the manifold and let us consider *the complete null cone* through Q, i.e. the set of points lying on all null geodesics through Q. Assume Q has coordinates

$$\mathrm{Q} : (x_i = 0, \qquad v = v_0, \qquad u = u_0 < a), \tag{16.60}$$

so that Q lies in the flat region preceeding the passage of the wave. Near Q the equation of the null cone will be

$$ds^2 = 2dudv + \delta_{ij}dx^i dx^j = 0 \quad \rightarrow \quad 2(u-u_0)(v-v_0) + \delta_{ij}x^i x^j = 0 \tag{16.61}$$

or

$$v = -\frac{1}{2}f_{ij}x^i x^j + v_0, \tag{16.62}$$

where

$$f_{ij} = \frac{\delta_{ij}}{(u-u_0)}. \tag{16.63}$$

Be n_α the vector normal to the hypersurface

$$\Sigma(x^\alpha) = v + \frac{1}{2}f_{ij}x^i x^j - v_0 = 0 \tag{16.64}$$

$$n_u = \frac{1}{2}f'_{ij}x^i x^j \tag{16.65}$$

$$n_v = 1$$

$$n_i = f_{ij}x^j, \qquad i,j = 1,2$$

[3] where we have assumed that f_{ij} is symmetric. Since Σ is a null surface, n_α is a null vector.

b) Inside the 'sandwich' wave

We shall assume that in the curved region the equation of the light-cone has the same structure of eq. (16.64), with f_{ij} function of u, and determine the equation for f_{ij} by requiring that the vector n_α, remains null, i.e. that Σ remains a null surface. This condition gives:

$$\begin{aligned} g^{\alpha\beta} n_\alpha n_\beta &= 2g^{uv} n_u n_v + g^{vv} n_v^2 + g^{xx} n_x^2 + g^{yy} n_y^2 \\ &= f'_{ij} x^i x^j - h_{ij} x^i x^j + \delta^{ij} (f_{ik} x^k)(f_{jl} x^l) \\ &= \left[f'_{ij} - h_{ij} + f^k{}_i f_{kj} \right] x^i x^j. \end{aligned} \tag{16.66}$$

Hence

$$f'_{ij} - h_{ij} + f^k{}_i f_{kj} = 0, \qquad u \geq a. \tag{16.67}$$

This equation says how the light-cones are deformed by the passage of the wave. The trace of the previous equation is

$$f'^{\,i}_i - h^i_i + f^k{}_i f_k{}^i = 0, \tag{16.68}$$

and, since $h^i_i \leq 0$ ($=0$ for a pure gravitational wave (eq. 16.56), and <0 for an electromagnetic wave (eq. 16.37)), it must be

$$f'^{\,i}_i + f^k{}_i f_k{}^i \leq 0. \tag{16.69}$$

[3] remember that $\frac{\partial}{\partial x^k}(x^i x^j) = \delta^i_k x^j + \delta^j_k x^i$ consequently $n_i = f_{ij}(x^i x^j)_{,i} = f_{ij} x^j, \quad i,j = 1,2.$

By the Schwarz inequality

$$f^k{}_i f_k{}^i \geq \frac{1}{2}(f_i^i)^2, \tag{16.70}$$

hence

$$f_i^{'i} + \frac{1}{2}(f_i^i)^2 \leq 0. \tag{16.71}$$

By introducing a new function

$$z(u) = exp\{\frac{1}{2}\int f_i^i(u)du\} \qquad z(u) \geq 0, \tag{16.72}$$

and using eq. (16.71) we find:

$$2\frac{z^{''}}{z} = \frac{1}{2}(f_i^i)^2 + f_i^{'i} \leq 0, \qquad \rightarrow \qquad z^{''} \leq 0. \tag{16.73}$$

Consequently

$$(exp\{\frac{1}{2}\int f_i^i(u)du\})^{''} \leq 0, \qquad u \geq a \tag{16.74}$$

where the strict inequality holds at least for some value of u.

The consequences of eq. (16.74) are the following. Let us consider a sandwich wave confined between a and b along the u-axis, and the point Q given by eq. (16.60) such that $u_0 = -\infty$. Then $f_{ij} = 0$ (see eq. 16.63) for $u < a$. That means that in the region near Q the null cone coincides with the null hyperplane $v = v_0$, and that

$$(exp\{\frac{1}{2}\int f_i^i(u)du\})^{'} = 0 \quad \rightarrow \quad (exp\{\frac{1}{2}\int f_i^i(u)du\}) = const. \tag{16.75}$$

In the region $u > a$ the function $(exp\{\frac{1}{2}\int f^i_i(u)du\})$ has negative second derivative (eq. 16.74), therefore we espect that it becomes zero, and consequently f_{ij} becomes infinite, for some value of $u = u_1$, with $u_1 > a$.

We shall assume that the wave is "weak" in the sense that h_{ij} becomes zero before u_1, so that $u_1 > b$. In this case, if u_0 remains very large and negative, the form of the null cone in the vicinity of $u = a$ will not change very much and u_1 will remain finite and greater than b. That is to say that f_{ij} will become infinite on the other side of the sandwich wave.

We shall now compute f_{ij} for an electromagnetic and a gravitational wave.

1) *Electromagnetic wave.* Let us consider, for example, a sandwich wave of constant amplitude

$$ds^2 = 2dudv - a^2[\Theta(u) - \Theta(u-b)](x^2+y^2)du^2 - dx^2 - dy^2 \qquad (16.76)$$

where a is a constant and Θ is the Heaviside step-function. According to (16.76) $h_{12} = 0$ and $h_{11} = h_{22} = -a^2$, and eq. (16.67) in the curved region becomes

$$\begin{aligned} &f'_{11} + f^2_{12} + f^2_{11} + a^2 = 0 \qquad (16.77)\\ &f'_{12} + f_{12}f_{11} + f_{21}f_{22} = 0\\ &f'_{22} + f^2_{12} + f^2_{22} + a^2 = 0 \end{aligned}$$

Since for $u \leq 0$ $f_{ij} = \frac{\delta_{ij}}{(u-u_0)}$, if $u_0 \to \infty$ then $f_{12} = f'_{12} = 0$, and consequently $f_{12} = 0$ also inside the sandwich. Thus eqs. (16.77) become

$$f'_{11} + f^2_{11} + a^2 = 0 \tag{16.78}$$

$$f'_{22} + f^2_{22} + a^2 = 0$$

These equations are easily integrated

$$-\int_0^u du = \int \frac{df_{11}}{f^2_{11} + a^2} \quad \to \quad u = -\frac{1}{a}\tan^{-1}\frac{f_{11}}{a}, \tag{16.79}$$

hence

$$f_{11} = f_{22} = -a\tan(au). \tag{16.80}$$

Finally

$$f_{ij} = \begin{pmatrix} -a\tan(au) & 0 \\ 0 & -a\tan(au) \end{pmatrix}, \qquad a \leq u \leq b. \tag{16.81}$$

2) *Pure gravitational wave.* Let us again consider a sandwich wave of constant amplitude and constant polarization

$$ds^2 = 2dudv + a^2[\Theta(u) - \Theta(u-b)](x^2 - y^2)du^2 + dx^2 + dy^2 \tag{16.82}$$

According to (16.82), $h_{12} = 0$, $h_{11} = -h_{22} = a^2$ and eq. (16.67) gives

$$f'_{11} + f^2_{11} + a^2 = 0 \tag{16.83}$$

$$f'_{22} + f^2_{22} - a^2 = 0$$

whose solution is

$$f_{11} = -a\tan(au) \qquad \text{and} \qquad f_{22} = a\tanh(au). \tag{16.84}$$

$$f_{ij} = \begin{pmatrix} -a\tan(au) & 0 \\ 0 & +a\tanh(au) \end{pmatrix}, \qquad a \leq u \leq b. \tag{16.85}$$

c) After the passage of the wave

The spacetime is flat and eq. (16.67) for f_{ij} becomes

$$f'_{ij} + f^k{}_i f_{kj} = 0, \qquad u > b. \tag{16.86}$$

Let us introduce the matrix p_{ij} inverse to f_{ij}.

$$p^i_k f_{ij} = \delta_{kj} \quad \rightarrow \quad p^{i'}{}_k f_{ij} + p^i{}_k f'{}_{ij} = 0 \quad \rightarrow \quad p^{i'}{}_k f_{ij} = -p^i{}_k f'{}_{ij}, \tag{16.87}$$

where the prime indicates differentiation with respect to u. By multiplying eq. (16.86) by $p^i{}_l$ we get

$$p^i{}_l f'_{ij} + p^i{}_l f^k{}_i f_{kj} = 0, \quad \rightarrow \quad -p^{i'}{}_l f_{ij} + f_{lj} = 0 \tag{16.88}$$

and multiplying again by p^{ij} to the right

$$-p^{i'}{}_l f_{ij} p^{ij} + f_{lj} p^{ij} = 0, \quad \rightarrow \quad p^{i'}{}_l = \delta_l{}^i \quad \rightarrow \quad p'{}_{il} = \delta_{il}. \tag{16.89}$$

The solution of eq. (16.89) is

$$p_{ij}(u) = u\delta_{ij} - q_{ij}, \tag{16.90}$$

where q_{ij} is constant, symmetric (since f_{ij} is symmetric), and it is determined by the values that p_{ij} assumes at $u = b$.

Let us assume that $au^* = \frac{\pi}{2}$ when $u^* > b$, i.e. the components of f_{ij} do not diverge inside the sandwich (in other words, the wave is weak). Then for $u \geq b$ we have (compare with eqs. (16.81) and (16.85))

1) *For a pure electromagnetic wave.*

$$p_{ij} = \begin{pmatrix} \frac{1}{-a\tan(au)} & 0 \\ 0 & \frac{1}{-a\tan(au)} \end{pmatrix}, \qquad a \leq u \leq b. \tag{16.91}$$

In this case eq. (16.90) gives

$$q_{12} = 0, \qquad q_{11} = b + \frac{1}{a\tan(au)}, \qquad q_{11} = b + \frac{1}{a\tan(au)}. \tag{16.92}$$

The eigenvalues are coincident.

2) *For a pure gravitational wave.*

$$p_{ij} = \begin{pmatrix} \frac{1}{-a\tan(au)} & 0 \\ 0 & \frac{1}{+a\tanh(au)} \end{pmatrix}, \qquad a \leq u \leq b. \tag{16.93}$$

In this case eq. (16.90) gives

$$q_{12} = 0, \qquad q_{11} = b + \frac{1}{a\tan(au)}, \qquad q_{11} = b - \frac{1}{a\tanh(au)}. \tag{16.94}$$

The eigenvalues are distinct.

We shall now compute the null geodesics which generate the null cone from Q in the two cases. The equation of the light-cone is

$$ds^2 = 2dudv + h_{ij}x^ix^jdu^2 + dx^2 + dy^2 = 0. \tag{16.95}$$

If we use u as a parameter it becomes

$$2v' + h_{ij}x^i x^j + x_i' x^{i'} = 0. \tag{16.96}$$

v' can be computed by differentiating eq. (16.62)

$$v = -\frac{1}{2} f_{ij} x^i x^j + v_0 \quad \rightarrow \quad v' = -\frac{1}{2}\left[f'_{ij} x^i x^j + f_{ij} (x^i x^j)' \right], \tag{16.97}$$

and by making use of eq. (16.67)

$$f'_{ij} = h_{ij} - f^k{}_i f_{kj} \quad \rightarrow \quad v' = -\frac{1}{2} h_{ij} x^i x^j + \frac{1}{2} f^k{}_i f_{kj} x^i x^j - \frac{1}{2} f_{ij} (x^i x^j)'. \tag{16.98}$$

By substituting this expression of v' in eq. (16.96) we find

$$f^k{}_i f_{kj} x^i x^j - f_{ij} (x^i x^j)' + x_i' x^{i'} = 0, \tag{16.99}$$

whose solution is

$$x_i' = f_{ij} x^j. \tag{16.100}$$

In the flat region behind the wave $u > b$, eq. (16.100) can easily be integrated by using the following properties of the matrices $\mathbf{f}$ and $\mathbf{p}$:

i) $\mathbf{p}' = \mathbf{I}$, where $I_{ij} = \delta_{ij}$,

ii) $\mathbf{pf} = \mathbf{I} \quad \rightarrow \quad \mathbf{f} = -\mathbf{p}\mathbf{f}'$.

Therefore eq. (16.100) can be written as

$$\mathbf{x}' = -\mathbf{p}\mathbf{f}'\mathbf{x}, \tag{16.101}$$

and multiplying by $\mathbf{f}$ on both sides

$$\mathbf{f}\mathbf{x}' = -\mathbf{f}\ \mathbf{p}\ \mathbf{f}'\mathbf{x} = -\mathbf{f}'\mathbf{x} \quad \rightarrow \quad (\mathbf{f}\mathbf{x})' = 0. \tag{16.102}$$

The solution is clearly

$$x_i = (u\delta_{ij} - q_{ij})m^j, \tag{16.103}$$

where the constants m^i are different for each geodesic.

If the eigenvalues of q_{ij} are coincident, say $q_i = q$, when $u = q$ all geodesics will pass through the point

$$\mathrm{R} : (u = q, \qquad x_i = 0, \qquad v = v_0).$$

This is the case of *anastigmatic* focusing induced by a pure electromagnetic wave: all null geodesics through Q will be focused in R after passing through the wave. There is one exception, the geodesic which is parallel to the propagation direction

$$u = u_0, \qquad x_i = 0$$

which cannot be parametrized by u and is not included in eq. (16.103).

If the q_i are distinct, say (q_1, q_2) with $q_1 < q_2$ and $q_1 > b$, choosing the axis x_1, x_2 in such a way that q_{ij} is diagonal as in example 2, we see that

$$x_1 = 0 \qquad \text{when} \qquad u = q_1. \tag{16.104}$$

Thus all null geodesics passing through Q (with the exception of that parallel to the propagation direction) will pass through some point of the *spacelike* line

$$x_1 = 0, \qquad x_2 \neq 0 \qquad u = q_1, \qquad v = v_0. \tag{16.105}$$

This is the case of *astigmatic* focusing induced by a pure gravitational wave.

The focusing properties of plane waves lead to the non existence of a global Cauchy surface for the evolution of initial data. The reason is the following. A Cauchy hypersurface is acceptable to specify initial data if every null geodesic intersect the surface only once. Let us consider a spacelike surface through Q: it must lie entirely to the past of the future light-cone from Q. However, after the passage of the wave the cone will fold down to focuse to the point R (or to a line passing through R). Since the spacelike surface cannot cross the null cone and remain spacelike everywhere, it will be forced to remain trapped beneath it. As we have seen before, the only geodesic through Q which does not converge to R is the one parallel to the propagation world-line. Let us call it α. But the role of Q and R can be interchanged and in this case the light-cone through R will be focused to Q with the exception of the null geodesic parallel to the propagation worldline through R, say β. α and β are the two limiting geodesics of the congruence, in the sense that any null geodesic passing through Q, as close as we want to α, will converge to R, except α itself, and conversely, any null geodesic passing through R, as close as we want to β, will converge to Q, except β itself. Each null geodesic of this sequence will intersect the Cauchy surface through Q just once, except the limiting geodesic β. As a consequence the evolution of Cauchy data given on the trapped suface will provide no information about a parallel wave which lies beyond the critical geodesic β.