LECTURE 1

1.0 Introduction

The typical density of terrestrial macroscopic objects is $\rho \gtrsim 1$ g/cm^3. Their structure is mainly dictated by electromagnetic interactions.

At much higher density, in the region $\rho \geq 10^4$ g/cm^3, the picture changes dramatically: the role of electromagnetic interactions becomes negligible and the structure of matter is driven by quantum effects and nuclear interactions.

In these lectures, we will discuss the density region $10^4 \leq \rho \leq 10^{16}$ g/cm^3, relevant to the understanding of the structure of stars ranging from white dwarfs to neutron stars.

The formation of a star is believed to be triggered by the contraction of a self-gravitating hydrogen cloud. As the density increases, the temperature also increases, and eventually becomes high enough to ignite the nuclear reactions turning hydrogen into helium:

$$
\begin{aligned}
p + p &\rightarrow {}^2\mathrm{H} + e^+ + \nu + 0.4 \text{ MeV} \ , \\
e^+ + e^- &\rightarrow \gamma + 1.0 \text{ MeV} \ , \\
{}^2\mathrm{H} + p &\rightarrow {}^3\mathrm{He} + \gamma + 5.5 \text{ MeV} \ , \\
{}^3\mathrm{He} + {}^3\mathrm{He} &\rightarrow {}^4\mathrm{He} + 2\nu + 26.7 \text{ MeV} \ .
\end{aligned}
$$

Note that the above reactions are all exothermic, and energy is released in form of kinetic energy of the produced particles (1 MeV $= 1.6021917 \times 10^{-6}$ erg). Equilibrium is reached as soon as gravitational attraction is balanced by matter pressure.

When the nuclear fuel is exhausted the core stops producing heat, the internal pressure cannot be sustained and the contraction produced by gravitational attraction resumes. If the mass of the helium core is large enough its contraction, associated with a further increase of the temperature, can then lead to the ignition of new fusion reactions, resulting in the appearance of heavier nuclei (carbon and oxygen). Depending on the mass of the stars, this process can take place several times, the final result being the formation of a core made of the most stable nuclear species, nickel and iron, at density $\sim 10^{14}$ g/cm^3 (note that the central density of atomic nuclei is $\rho_0 \approx 2 \times 10^{14}$ g/cm^3). Larger densities are believed to occur in the interior of neutron stars, astrophisical objects resulting from the contraction of the iron core in very massive stars (M > 4 M$_\odot$).

If the star is sufficiently small, so that the gravitational contraction of the core does not produce a temperature high enough to ignite the burning of heavy nuclei, it will eventually turn into a white dwarf, i.e. a star consisting mainly of helium, carbon and oxygen, with mass M ~ 1 M$_\odot$, radius R ~ 5000 Km and average density of the order of 10^6 g/cm^3.

In white dwarfs, as well as in neutron stars, there is no nuclear fuel burning to generate the thermal pressure needed to balance gravitational attraction. The mechanism producing the pressure responsible for the stability of white dwarfs and neutron stars against gravitational collapse is of purely quantum mechanical origin. As an example, in the next section we will discuss the case of white dwarfs, whose stability is due to the pressure of a degenerate electron gas.

1.1 Energy-density and pressure of a degenerate Fermi gas

Let us consider a system of noninteracting electrons uniformly distributed in a cubic box of volume $\Omega = \mathrm{L}^3$. If the temperature is sufficiently low, so that thermal energies can be neglected, the lowest quantum levels are occupied by two electrons, one for each spin state. Both electrons have the same energy, i.e. they are *degenerate*. This configuration corresponds to the ground state of the system. A gas of noninteracting electrons in its ground state is said to be *fully degenerate*. At higher temperature, the thermal energy can excite electrons to higher energy states, leaving some of the lower lying levels not fully degenerate.

As the electrons are uniformly distributed, their wave functions must exhibit translational invariance. Hence, they must be eigenfunctions of the generator of spacial translation, i.e. the momentum operator, of the form

$$
\phi_{\mathbf{p}}(\mathbf{r}) = \sqrt{\frac{1}{\Omega}} \, \mathrm{e}^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} \ , \tag{1}
$$

satisfying periodic boundary conditions (x, y and z denote the components of the vector $\mathbf{r}$, specifying the electron position)

$$\phi_{\mathbf{p}}(x, y, z) = \phi_{\mathbf{p}}(x + n_x L, y + n_y L, z + n_z L) \ , \tag{2}$$

with $n_x, n_y, n_z = 0, \pm 1, \pm 2, \ldots$. The above equation obviously implies the relations ($\mathbf{p} \equiv (p_x, p_y, p_z)$)

$$\frac{p_x}{\hbar} = \frac{2\pi n_x}{L}, \quad \frac{p_y}{\hbar} = \frac{2\pi n_y}{L}, \quad \frac{p_z}{\hbar} = \frac{2\pi n_z}{L} \ , \tag{3}$$

which in turn determine the momentum eigenvalues.

Each quantum state is associated with an eigenvalue of the momentum $\mathbf{p}$, i.e. with a specific triplet of integers (n_x, n_y, n_z). The corresponding energy eigenvalue is ($p^2 = |\mathbf{p}|^2 = p_x^2 + p_y^2 + p_z^2$)

$$\epsilon_p = \frac{p^2}{2m_e} = \left(\frac{2\pi}{L}\right)^2 \frac{\hbar^2}{2m_e} \ (n_x^2 + n_y^2 + n_z^2) \ , \tag{4}$$

m_e being the electron mass ($m_e = 9.11 \times 10^{-28}$ g). The highest energy reached, called the Fermi Energy of the system, is denoted by ϵ_F, and the associated momentum, the Fermi momentum, is $p_F = \sqrt{2m_e\epsilon_F}$.

The number of quantum states with energy less or equal to ϵ_F can be easily calculated. Since each triplet (n_x, n_y, n_z) corresponds to a point in a cubic lattice with unit lattice spacing, the number of momentum eigenstates is equal to the number of lattice points within a sphere of radius $R = p_F L/(2\pi\hbar)$. The number of electrons in the system can then be obtained from (note: the factor 2 takes into account spin degeneracy, i.e. the fact that there are two electrons with opposite spin projections sitting in each momentum eigenstate)

$$N = 2 \ \frac{4\pi}{3} \ R^3 = \Omega \ \frac{1}{3\pi^2} \ \left(\frac{p_F}{\hbar}\right)^3 \ , \tag{5}$$

and the electron number density, i.e. the number of electrons per unit volume, is given by

$$n_e = \frac{N}{\Omega} = \frac{1}{3\pi^2} \left(\frac{p_F}{\hbar}\right)^3 \ . \tag{6}$$

The total ground state energy can be easily evaluated from

$$E = 2 \sum_{p < p_F} \frac{p^2}{2m_e} \tag{7}$$

replacing (use (3) again and take the limit of large L, corresponding to vanishingly small level spacing)

$$\sum_{p < p_F} \to \frac{\Omega}{(2\pi^3)} \ \frac{1}{\hbar^3} \ \int_{p < p_F} d^3p \tag{8}$$

to obtain

$$E = 2 \ \frac{\Omega}{(2\pi^3)} \ \frac{1}{\hbar^3} \ 4\pi \ \int_0^{p_F} p^2 dp \ \frac{p^2}{2m_e} \ . \tag{9}$$

The resulting energy density is

$$\epsilon = \frac{E}{\Omega} = \frac{1}{(2\pi)^3} \ \frac{1}{\hbar^3} \ 4\pi \ \frac{p_F^5}{5m_e} \ . \tag{10}$$

From eq.(6) it follows that the Fermi energy can be written in terms of the number density according to

$$\epsilon_F = \frac{p_F^2}{2m_e} = \frac{\hbar^2}{2m_e} \ \left(3\pi^2 n_e\right)^{2/3} \ . \tag{11}$$

The above equation can be used to define a density n_0 such that for $n_e \gg n_0$ the electron gas at given temperature T is fully degenerate. Full degeneracy is realized when the thermal energy $K_B T$ (K_B is the Boltzman constant: K_B = 1.38×10^{-16} erg/°K) is much smaller than the Fermi energy ϵ_F, i.e. when

$$n_e \gg n_0 = \frac{1}{3\pi^2} \left(\frac{2m_e}{\hbar^2} \; K_B T \right)^{3/2} \quad . \tag{12}$$

For an ordinary star at the stage of hydrogen burning, like the Sun, the interior temperature is $\sim 10^7$ °K, and the corresponding value of n_0 is $\sim 10^{26}$ cm^{-3}. If we assume that the electrons come from a fully ionized hydrogen gas, the *matter* density of the proton-electron plasma is

$$\rho = (m_p + m_e) \; n_0 \sim 200 \; g/cm^3 \; , \tag{13}$$

m_p being the proton mass ($m_p = 1.67 \times 10^{-24}$ g). This density is high for most stars in the early stage of hydrogen burning, while for ageing stars that have developed a substantial helium core the density (m_n denotes the neutron mass: $m_n \approx m_p$).

$$\rho = (m_p + m_n + m_e) \; n_0 \sim 400 \; g/cm^3 \tag{14}$$

can be largely exceeded within the core. For example, white dwarfs have core densities of the order of 10^7 g/cm^3. As a consequence, in the study of their structure thermal energies can be safely neglected, the primary role being played by the degeneracy energy $p^2/2m_e$.

The pressure P of the electron gas, i.e. the force per unit area on the walls of the box, is defined in kinetic theory as the rate of momentum transferred by the electrons colliding on a surface of unit area. The pressure generated on the wall of the box lying on the yz plane by electrons moving with momentum p_x and velocity v_x parallel to the x axis is

$$P(p_x) = \frac{1}{L^2} \; \frac{dp_x}{dt} = \frac{1}{L^2} \; (2p_x) \; \left(\frac{1}{2} n_e v_x L^2 \right) = \frac{N}{\Omega} p_x v_x \; . \tag{15}$$

In the above equation, the first terms in round brackets is the momentum transfer associated with the reflection of one electron off the box wall, while the second term is the electron flux, i.e. the number of electrons hitting the wall over the time dt (the factor $1/2$ accounts for the fact that half of the electrons go the wrong way). The total pressure can therefore be obtained from

$$P = 2 \frac{1}{N} \sum_{p_x < p_F} P(p_x) = 2 \; \frac{1}{\hbar^3} \int_{p \leq p_F} \frac{d^3p}{(2\pi)^3} \; p_x \; v_x \; . \tag{16}$$

Since the system is isotropic

$$p_x v_x = \frac{1}{3} \; (p_x v_x + p_y v_y + p_z v_z) = \frac{1}{3} \; (\mathbf{p} \cdot \mathbf{v}) = \frac{1}{3} \; (pv) \; , \tag{17}$$

and eq.(16) can be rewritten (use $v = (\partial \epsilon_p / \partial p) = p/m_e$)

$$P = \frac{2}{3} \; \frac{1}{(2\pi)^3} \; \frac{1}{\hbar^3} \; 4\pi \int_0^{p_F} p^2 dp \; (pv) \; = \frac{1}{(2\pi)^3} \; \frac{1}{\hbar^3} \; 4\pi \; \frac{2p_F^5}{15 m_e} \; . \tag{18}$$

Note that the above result can also be obtained from the standard thermodynamical definition of pressure

$$P = - \left(\frac{\partial E}{\partial \Omega} \right)_N \; , \tag{19}$$

using E given by eq.(9) and $(\partial p_F / \partial \Omega)_N = -p_F/(3\Omega)$.

Eq.(18) shows that the pressure of a degenerate Fermi gas decreases linearly as the mass of the constituent particle increases. For example, the pressure of an electron gas at number density n_e is ~ 2000 times larger than the pressure of a gas of protons at the same number density.

So far, we have been assuming that the electrons in the degenerate gas be nonrelativistic. However, the properties of the system depend primarily on the distribution of quantum states, which is dictated by translation invariance only, and is not affected by this assumption. Releasing the nonrelativivstic approximation simply amounts to replace the nonrelativistic energy with its relativistic counterpart:

$$\frac{p^2}{2m_e} \to \sqrt{(pc)^2 + (m_e c^2)^2} - m_e c^2 \; . \tag{20}$$

The transition from the nonrelativistic regime to the relativistic regime occurs when the electron energy becomes comparable to the energy assciated with the electron rest mass, $m_e c^2$. It is therefore possible to define a density n_c such that at $n_e \ll n_c$ the system is nonrelativistic, while $n_e \gg n_c$ corresponds to the relativistic regime. The value of n_c can be found requiring that the Fermi energy at $n_e = n_c$ be equal to $m_e c^2$. The resulting expression is

$$n_c = \frac{2^{3/2}}{3\pi^2} \left(\frac{m_e c^2}{\hbar c} \right)^3 \sim 1.6 \times 10^{30} cm^{-3} \ . \tag{21}$$

The energy density of a fully degenerate gas of relativistic electrons can be obtained from (compare to eqs.(9) and (10))

$$\epsilon = 2 \ \frac{1}{(2\pi^3)} \ \frac{1}{\hbar^3} \ 4\pi \ \int_0^{p_F} p^2 dp \ \left[\sqrt{(pc)^2 + (m_e c^2)^2} - m_e c^2 \right] \ , \tag{22}$$

while the equation for the pressure reads (compare to eq.(18) and use again $v = \partial \epsilon_p / \partial p$ with relativistic ϵ_p)

$$P = \frac{2}{3} \ \frac{1}{(2\pi)^3} \ \frac{1}{\hbar^3} \ 4\pi \ \int_0^{p_F} p^2 dp \ \left(p \frac{\partial \epsilon_p}{\partial p} \right) \ . \tag{23}$$

Carrying out the integrations involved in eqs.(22) and (23) we find:

$$\epsilon = \frac{\pi m_e c^2}{\lambda_e^3} \left[t \left(2t^2 + 1 \right) \sqrt{t^2 + 1} - ln \left(t + \sqrt{t^2 + 1} \right) \right] \tag{24}$$

and

$$P = \frac{\pi m_e c^2}{\lambda_e^3} \left[\frac{1}{3} \ t \left(2t^2 + 1 \right) \sqrt{t^2 + 1} + ln \left(t + \sqrt{t^2 + 1} \right) \right] , \tag{25}$$

where $\lambda_e = 2\pi\hbar / m_e c$ is the electron Compton wavelength and (see eq.(11))

$$t = \frac{p_F}{m_e c} = \frac{\hbar}{m_e c} \left(3\pi^2 n_e \right)^{1/3} \tag{26}$$

Eqs.(24) and (25) give the energy density and pressure of a fully degenerate electron gas as a function of the variable t, which can in turn be written in terms of the number density n_e according to eq.(26).

1.2 Equation of state of a degenerate Fermi gas

The *equation of state* describes the relationship between the pressure of the system (P) and its matter density (ρ), which is in turn related to the electron number density n_e by the equation

$$\rho = \frac{m_p}{Y_e} \ n_e \ , \tag{27}$$

where Y_e is the number of electrons per nucleon in the system. For example, for a fully ionized helium plasma $Y_e = 0.5$, whereas for a plasma of iron nuclei $Y_e = Z/A = 26/56 = 0.464$ (Z and A denote the nuclear charge and mass number, respectively).

The equation of state of a fully degenerate electron gas (see eq.(25)) takes a particularly simple form in the nonrelativistic limit (corresponding to $t \ll 1$), as well as in the extreme relativistic limit (coresponding to $t \gg 1$). Using eq.(25) we find

$$P = \frac{8}{15} \ \frac{\pi m_e c^2}{\lambda_e^3} \left(\frac{3\pi^2 Y_e}{m_p} \right)^{5/3} \rho^{5/3} \tag{28}$$

for $e_F \ll m_e c^2$ and

$$P = \frac{2}{3} \ \frac{\pi m_e c^2}{\lambda_e^3} \left(\frac{3\pi^2 Y_e}{m_p} \right)^{4/3} \rho^{4/3} \tag{29}$$

for $e_F \gg m_e c^2$.

An equation of state of the form

$$P \propto \rho^\Gamma \ , \tag{30}$$

is said to be *polytropic*. The exponent Γ is called *adiabatic index*, whereas the quantity n, defined through

$$\Gamma = 1 + \frac{1}{n} \ , \tag{31}$$

goes under the name of *polytropic index*.

The adiabatic index, whose definition for a generic equation of state reads

$$\Gamma = \frac{d \ (lnP)}{d \ (ln\rho)} \ . \tag{32}$$

is related to the compressibility χ, characterizing the change of pressure with volume according to

$$\frac{1}{\chi} = -\Omega \left(\frac{\partial P}{\partial \Omega} \right)_N = \rho \left(\frac{\partial P}{\partial \rho} \right)_N \tag{33}$$

through

$$\Gamma = \frac{1}{\chi P}. \tag{34}$$

The compressibility is also simply related to speed of sound in matter, c_s, defined as

$$c_s = \left(\frac{\partial P}{\partial \rho} \right)^{1/2} = \frac{1}{\chi \rho} \ . \tag{35}$$

The magnitude of the adiabatic index reflects the so called *stiffnes* of the equation of state. Larger stiffness corresponds to more incompressible matter. As we will see in the following lectures, stiffness turns out to be critical in determining a number of stellar properties.

1.3 Hydrostatic equilibrium and structure of white dwarfs

Let us assume that white dwarfs consist of a plasma of fully ionized helium at zero temperature. The pressure of the system, P, is provided by the electrons, the contribution of the helium nuclei being negligible due to their large mass. For any given value of the matter density ρ, P can be computed from eqs.(25) and (26) (in this case $Y_e = 0.5$, implying $n_e = \rho/2m_p$). The results of this calculation are shown by the diamonds in fig. 1. For comparison, the nonrelativistic (eq.(28)) and extreme relativistic (eq.(29)) limits are also shown by the solid and dashed line, respectively.

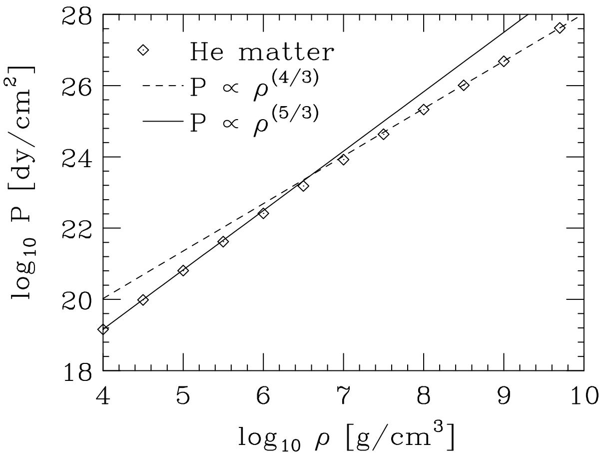

FIG. 1. Equation of state of a fully ionized helium plasma at zero temperature (diamonds). The solid and dashed line correspond to the nonrelativistic and extreme relativistic limits, respectively. Note that the value of matter density corresponding to n_c defined in eq.(21) is $\rho \sim 5.3$ g/cm^3.

In order to show the sensitivity of the equation of state to the value of Y_e, in fig. 2 the equation of state of the fully ionized helium plasma ($Y_e = 0.5$) is compared to that of a hydrogen plasma ($Y_e = 1$).

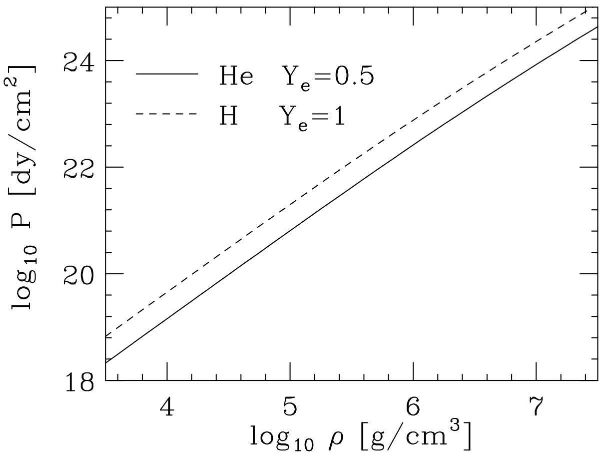

FIG. 2. Comparison between the equations of state of a fully ionized plasma of helium (solid line) and hydrogen (dashed line).

The surface gravity of white dwarfs, GM/c^2R (G is the gravitational constant), is small, of order $\sim 10^{-4}$. Hence, their structure can be studied assuming that they consist of a spherically symmetric fluid in hydrostatic equilibrium.

Equilibrium requires that the gravitational force acting on an volume element at distance r from the center of the star, $F_G(r)$, be balanced by the force produced by the spacial variation of the pressure. From

$$F_G(r) = -\ \rho(r)\ \frac{GM(r)}{r^2}\ , \tag{36}$$

with

$$M(r) = 4\pi \int_0^r dr'\ {r'}^2\ \rho(r')\ , \tag{37}$$

it then follows the equilibrium equation

$$\frac{dP}{dr} = -\ \rho(r)\ \frac{GM(r)}{r^2}\ . \tag{38}$$

Given an equation of state, eq.(38) can be integrated numerically for any value of the central density ρ_c to obtain the radius of the star, i.e. the value of r corresponding to vanishingly small density. The mass can then be obtained from eq.(37).

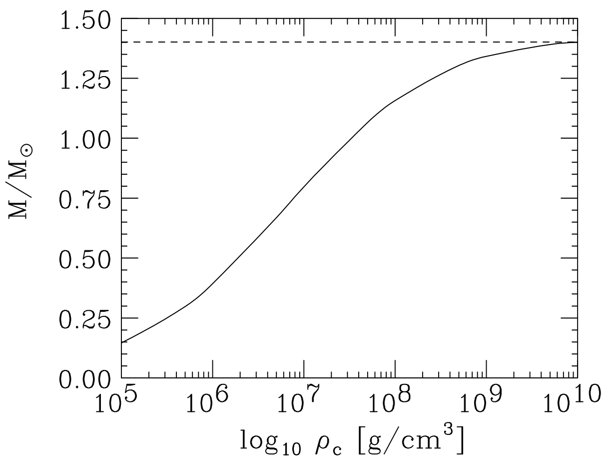

FIG. 3. Dependence of the mass of a white dwarf upon its central density, obtained from the integration of eq.(38) using the equation of state of a fully ionized helium plasma.

The dependence of the mass of the star upon its central density, obtained from integration of eq.(38) using the equation of state of a fully ionized helium plasma, is illustrated in fig. 3. The figure shows that the mass increases as the central density increases, until a limiting value $M \sim 1.44\ M_\odot$ is reached at $\rho_0 \sim 10^{10}\ \mathrm{g/cm^3}$. The existence of this limiting mass was first pointed out by Chandrasekhar. However, as we will see in the following lectures, at $\rho \sim 10^8\ \mathrm{g/cm^3}$ the *neutronization* process sets in, and the validity of the description in terms of a helium plasma breaks down. At $\rho \geq 10^8\ \mathrm{g/cm^3}$, matter does not support pressure as effectively as predicted by the equation of state of the helium plasma. As a consequence, a more realistic estimate of the limiting mass, generally referred to as the *Chandrasekhar mass*, is given by the mass corresponding to a central density of $10^8\ \mathrm{g/cm^3}$, i.e. $\sim 1.2\ M_\odot$.