Reduction formulae

CONTENTS

D.1 REDUCTION FORMULAE FOR THE COMPTON SCATTERING AMPLITUDE

As we saw in Chapter 10, the generating functional that is obtained to first order from the QED perturbative expansion can be written in the form[1]

$$
\begin{aligned}
Z_1[J^\rho, \bar{J}, J] &= ie \int d^4u \, \frac{-i\delta}{\delta J(u)} \gamma^\mu \frac{i\delta}{\delta \bar{J}(u)} \, \frac{i\delta}{\delta J^\mu(u)} Z_0[J^\rho, \bar{J}, J] \\
&= ie \int d^4u \; C_{1\mu}[J^\rho] C_2^\mu[\bar{J}, J] Z_0[J^\rho, \bar{J}, J] \ , \qquad \text{(D.1)}
\end{aligned}
$$

where $Z_0[J^\rho, \bar{J}, J] = Z_0[J^\rho] Z_0[\bar{J}, J]$ is the generating functional of the free theory, equation (10.9),

$$
C_{1\mu}[J^\rho] = -\int d^4x J^\rho(x) \Delta_{\rho\mu}(x-u) \ , \qquad \text{(D.2)}
$$

$$
C_2^\mu[\bar{J}, J] = -i\gamma^\mu S_F(u-u) - \int d^4y \; d^4z \bar{J}(y) S_F(y-u) \gamma^\mu S_F(u-z) J(z) \ , \quad \text{(D.3)}
$$

and we have introduced the notation $\Delta_{\lambda\mu}(x-u) = g_{\lambda\mu} \Delta_F(x-u; 0)$.

To obtain the generating functional to second order, we must calculate the expression

$$
\frac{1}{2} ie \int d^4v \, \frac{-i\delta}{\delta J(v)} \gamma^\nu \frac{i\delta}{\delta \bar{J}(v)} \, \frac{i\delta}{\delta J^\nu(v)} Z_1[J^\rho, \bar{J}, J] \ . \qquad \text{(D.4)}
$$

The result of this operation is the sum of a large number of terms, which correspond to the different physical processes discussed in Section 10.4. Here, we only consider the contribution corresponding to the generating functional from which the Compton scattering amplitude is obtained, denoted $Z_2^C[J^\rho, \bar{J}, J]$.

[1] The presence of mass counterterms in the interaction Lagrangian is not relevant for the result that we propose to derive, and will therefore be neglected.

In this context the relevant functional derivative with respect to $J^\nu(v)$ can operate only on $Z_0[J^\rho]$, with the result

$$\frac{i\delta}{\delta J^\nu(v)} Z_0[J^\rho] = -\int d^4x'\ \Delta_{\sigma\nu}(v-x') J^\sigma(x') Z_0[J^\rho]\ , \tag{D.5}$$

while the functional derivatives with respect to $\bar{J}(v)$ and $J(v)$ can operate, respectively, on $C_2[\bar{J}, J]$ and $Z_0[\bar{J}, J]$, or vice versa. In the first case we find (for clarity, we make the Dirac indices explicit)

$$\gamma^\nu_{\alpha\beta} \frac{i\delta}{\delta \bar{J}_\beta(v)} C_2[\bar{J}, J] = -i \int d^4z\ \gamma^\nu_{\alpha\beta} S_{F\,\beta\delta}(v-u) \gamma^\mu_{\delta\rho} S_{F\,\rho\sigma}(u-z) J_\sigma(z)\ , \tag{D.6}$$

and

$$\frac{-i\delta}{\delta J_\alpha(v)} Z_0[\bar{J}, J] = -\int d^4y\ \bar{J}_\xi(y) S_{F\,\xi\alpha}(y-v) Z_0[\bar{J}, J]\ . \tag{D.7}$$

Combining equations (D.6)–(D.7) with the analogous expressions obtained by differentiating $Z_0[\bar{J}, J]$ with respect to $\bar{J}(v)$ and $C_2[\bar{J}, J]$ with respect to $J(v)$, we obtain

$$\begin{aligned} Z_2^C = \frac{1}{2}(ie)^2 i \int & d^4u\, d^4v\, d^4x\, d^4x' d^4y\, d^4z\, J^\lambda(x) \Delta_{\lambda\mu}(x-u) \Delta_{\sigma\nu}(v-x') J^\sigma(x') \\ & \times \left[\bar{J}(y) S_F(y-v) \gamma^\nu S_F(v-u) \gamma^\mu S_F(u-z) J(z) \right. \\ & \qquad \left. + (y \leftrightharpoons z,\ u \leftrightharpoons v,\ \mu \leftrightharpoons \nu)\right]. \end{aligned} \tag{D.8}$$

It is immediately seen that the two terms of (D.8) give identical contributions, which together eliminate the factor $\frac{1}{2}$. The final result can be written in the form

$$Z_2^C = (ie)^2 \int d^4u\, d^4v D_{1\mu\nu}[J^\rho] D_2^{\mu\nu}[\bar{J}, J] Z_0[J^\rho, \bar{J}, J]\ , \tag{D.9}$$

with

$$D_{1\mu\nu}[J^\rho] = \int d^4x\, d^4x'\ J^\lambda(x) \Delta_{\lambda\mu}(x-u) \Delta_{\sigma\nu}(v-x') J^\sigma(x')\ , \tag{D.10}$$

and

$$D_2^{\mu\nu}[\bar{J}, J] = i \int d^4y\, d^4z\ \bar{J}(y) S_F(y-v) \gamma^\nu S_F(v-u) \gamma^\mu S_F(u-z) J(z)\ . \tag{D.11}$$

The Compton scattering amplitude is obtained, by means of the LSZ reduction formulae, from the four-point Green's function

$$G_{\alpha\beta}(x_1, x_2, x_1', x_2') = \langle 0 | T\{A_\alpha(x_1) A_\beta(x_1') \bar{\psi}(x_2) \psi(x_2')\} | 0 \rangle\ , \tag{D.12}$$

which in the path integral formalism can be rewritten in the form

$$G_{\alpha\beta}(x_1,x_2,x_1',x_2') = \frac{1}{Z[0,0,0]}\frac{-i\delta}{\delta J(x_2)}\frac{i\delta}{\delta \bar{J}(x_2')}\frac{i\delta}{\delta J^\alpha(x_1)}\frac{i\delta}{\delta J^\beta(x_1')}Z[J^\rho,\bar{J},J]\Big|_{J^\rho=\bar{J}=J=0} \ . \quad \text{(D.13)}$$

Using the expression for the generating functional to second order, given by equations (D.9)–(D.11), it is immediately seen that the two functional derivatives with respect to $J^\alpha(x_1)$ and $J^\beta(x_1')$ must operate on $D_1[J^\rho]$, with the result

$$\frac{i\delta}{\delta J^\alpha(x_1)}\frac{i\delta}{\delta J^\beta(x_1')}D_1[J^\rho] = -\left[\Delta_{\alpha\mu}(x_1-u)\Delta_{\beta\nu}(v-x_1') + \Delta_{\beta\mu}(x_1'-u)\Delta_{\alpha\nu}(v-x_1)\right] \ . \quad \text{(D.14)}$$

We now take the functional derivatives with respect to $\bar{J}(x_2)$ and $J(x_2')$ which can operate only on $D_2[\bar{J},J]$. The result obtained is

$$\begin{aligned}\frac{-i\delta}{\delta J(x_2)}\frac{i\delta}{\delta \bar{J}(x_2')}D_2[\bar{J},J] &= \frac{i\delta}{\delta J(x_2)}\int d^4z\ S_F(x_2'-v)\gamma^\nu S_F(v-u)\gamma^\mu S_F(u-z)J(z)\\ &= iS_F(x_2'-v)\gamma^\nu S_F(v-u)\gamma^\mu S_F(u-x_2) \ . \quad \text{(D.15)}\end{aligned}$$

From equations (D.13)–(D.15) it follows that

$$\begin{aligned}G_{\alpha\beta}(x_1,x_2,x_1',x_2') &= -i(ie)^2\int d^4u\, d^4v\, [\Delta_{\alpha\mu}(x_1-u)\Delta_{\beta\nu}(v-x_1') + \Delta_{\beta\mu}(x_1'-u)\Delta_{\alpha\nu}(v-x_1)]\\ &\quad \times S_F(x_2'-v)\gamma^\nu S_F(v-u)\gamma^\mu S_F(u-x_2) \ . \quad \text{(D.16)}\end{aligned}$$

Now we want to use the expression for the Green's function, (D.16), to obtain the amplitude of the process

$$\gamma(k,r) + e(p,s) \to \gamma(k',r') + e(p',s') \ , \quad \text{(D.17)}$$

where (k,r), (k',r'), (p,s) and (p',s') are the 4-momenta and the electron and photon polarisation in the initial and final states. Using the reduction formulae discussed in Chapter 10, we can write the S-matrix element in the form

$$\begin{aligned}S_{if} &= N\int dx_1 dx_1' dx_2 dx_2' e^{-i(kx_1+px_2)}e^{i(k'x_1'+p'x_2')}\\ &\quad \times \epsilon^\beta(k',r')\overrightarrow{\Box}_{x_1'}\bar{u}(p',s')\overrightarrow{(i\partial\!\!\!/ - m)}_{x_2'}G_{\alpha\beta}(x_1,x_2,x_1',x_2')\\ &\quad \times \overleftarrow{(-i\partial\!\!\!/ - m)}_{x_2}u(p,s)\overleftarrow{\Box}_{x_1}\epsilon^\alpha(k,r) \ , \quad \text{(D.18)}\end{aligned}$$

where N is a normalisation factor that we will discuss later. The integrals over x_1, x'_1, x_2 and x'_2 are carried out by using the relations

$$\Box_x \Delta_F(x-y;0) = -\delta^{(4)}(x-y) \ , \quad (i\partial\!\!\!/ - m)_x S_F(x-y) = \delta^{(4)}(x-y) \ , \tag{D.19}$$

with the result

$$\begin{aligned} S_{if} &= -i(ie)^2 N \int d^4u\, d^4v \\ &\quad \times \Big\{ e^{-i[(p+k)u-(p'+k')v]} \epsilon_\nu(k',r')\bar{u}(p',s')\gamma^\nu S_F(v-u)\gamma^\mu u(p,s)\epsilon_\mu(k,r) \\ &\qquad + e^{-i[(p-k')u-(p'-k)v]} \epsilon_\mu(k',r')\bar{u}(p',s')\gamma^\nu S_F(v-u)\gamma^\mu u(p,s)\epsilon_\nu(k,r) \Big\} \ . \end{aligned} \tag{D.20}$$

For the last two integrations, the new variables $W = (v+u)/2$ and $w = v - u$ are used. Integrating over W, the δ-function expressing 4-momentum conservation is obtained and, using the concise notation

$$\epsilon_\mu(k,r) = \epsilon_\mu \ , \quad \epsilon_\nu(k',r') = \epsilon'_\nu \ , \quad u(p,s) = u \ , \quad \bar{u}(p',s') = \bar{u}' \ , \tag{D.21}$$

we can write the S-matrix element in the form

$$\begin{aligned} S_{if} &= N\ (2\pi)^4 \delta(k+p-k'-p')\ ie^2 \int d^4w \\ &\qquad \times \left[e^{i(p+k)w} \bar{u}' \epsilon\!\!/' S_F(w) \epsilon\!\!/ u + e^{i(p-k')w} \bar{u}' \epsilon\!\!/ S_F(w) \epsilon\!\!/' u \right] \\ &= N\ (2\pi)^4 \delta(k+p-k'-p')\ ie^2 \bar{u}' \left[\epsilon\!\!/' S_F(p+k) \epsilon\!\!/ + \epsilon\!\!/ S_F(p-k') \epsilon\!\!/' \right] u \ , \end{aligned} \tag{D.22}$$

with

$$S_F(p) = \frac{p\!\!/ + m}{p^2 - m^2 + i\epsilon} \ . \tag{D.23}$$

The normalisation factor in equation (D.20) has the form

$$N = \frac{1}{\sqrt{(2\pi)^3 2\omega_k Z_3}} \frac{1}{\sqrt{(2\pi)^3 2\omega_p Z_2}} \frac{1}{\sqrt{(2\pi)^3 2\omega_{k'} Z_3}} \frac{1}{\sqrt{(2\pi)^3 2\omega_{p'} Z_2}} \ , \tag{D.24}$$

which contains the normalisations of the states describing the particles in the initial and final states, and we may set $Z_2 = Z_3 = 1$ to the present perturbative order.

Equation D.22 reproduces the Compton scattering amplitude to lowest order, well known in the literature (see e.g. [1], equations (14.82) and (14.83)).