Gaussian Integral with Complex Exponent

Consider the integral

$$I = \int_{-\infty}^{+\infty} e^{-zp^2}\, dp \qquad (1)$$

with $z = x + iy$ a complex number, and $x > 0$. The calculation of I can be carried out using the standard procedure

$$I^2 = \int_{-\infty}^{+\infty} dp\, e^{-zp^2} \int_{-\infty}^{+\infty} dq\, e^{-zq^2}$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} \rho\, d\rho\, e^{-z\rho^2} \qquad (2)$$

where $\rho^2 = p^2 + q^2$ and $\theta = \operatorname{atan} q/p$. Note that the Jacobian determinant of the transformation is $|J| = \rho$.

From (2) it follows that

$$I^2 = 2\pi \frac{1}{-2z} e^{-z\rho^2} \Big|_0^{\infty} \qquad (3)$$

$$= 2\pi \frac{1}{-2z} e^{-(x+iy)\rho^2} \Big|_0^{\infty}$$

The contribution of the exponential at $\rho \to \infty$ only vanishes for $x > 0$. In this case

$$I^2 = \frac{\pi}{2} \rightarrow I = \left(\frac{\pi}{2}\right)^{1/2} \qquad (4)$$

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The integral occurring in the calculation of the quantum mechanical amplitude is

$$I = \int_{-\infty}^{+\infty} e^{-i\alpha p^2} \, dp \qquad \text{with} \quad \alpha = \frac{T}{2m}$$

Replacing $T \to (1 - i\eta)T$, with $\eta > 0$ we obtain

$$I \to \int_{-\infty}^{+\infty} e^{-i(1-i\eta)\frac{T}{2m} p^2} dp$$

and the integration can be done using Eqs. (1)-(4), with

$$z = \frac{T}{2m}(\eta + i) \quad , \quad \eta > 0 .$$