Notes on the the infrared divergence

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1. Electron scattering by a static field

Let us consider electron scattering by a static external field, whose potential can be written

$$A^\mu(x) = A^\mu(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3}\ A^\mu(\mathbf{q})\ \mathrm{e}^{i\mathbf{qx}}\ . \tag{1}$$

The corresponding S-matrix expansion reads:

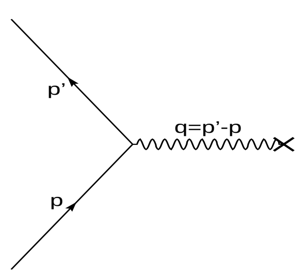
$$S = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int d^4x_1 ... d^4x_n\ T\{N[\bar\psi(x_1) A\!\!\!/(x_1)\psi(x_1)] ... T\{N[\bar\psi(x_n) A\!\!\!/(x_n)\psi(x_n)]\}\ , \tag{2}$$

where ψ is the electron field and, as usual, T and N denote time-ordered and normal product, respectively.

At lowest order, i.e. order 1 in the fine structure constant $\alpha = e^2/4\pi$, eq.(2) reduces to

$$S = ie \int d^4x\ \bar\psi^-(x) A\!\!\!/(x)\psi^+(x)\ . \tag{3}$$

The process described by the above S-matrix is depicted by the Feynman diagram G0, representing the transition of an electron from the state $|i\rangle = |pr\rangle$ to the state $|f\rangle = |p'r'\rangle$, $p \equiv (E, \mathbf{p})$ and $p' \equiv (E', \mathbf{p}')$ being the initial and final four momenta.



(G0)

The matrix element of the lowest order S, given by eq.(3), between the states $|i\rangle$ and $|f\rangle$ reads

$$
\begin{aligned}
S_{if} = \langle f|S|i\rangle &= (2\pi)\ \delta(E-E')\ N_p N_{p'}\ \mathcal{M}_{if} \\
&= (2\pi)\ \delta(E-E')\ \left(\frac{m}{VE}\right)^{1/2} \left(\frac{m}{VE'}\right)^{1/2}\ \mathcal{M}_{if}\ ,
\end{aligned} \tag{4}
$$

where V is the volume of the normalization box. Note that the above equation includes only the energy conserving δ-function, while momentum is not conserved. This is a consequence of the fact in our treatment the momentum of the source of the static field is ignored. In fact, introducing a source breaks translation invariance, thus immediately leading to breaking momentum conservation. Energy conservation, as stated by the δ-function in eq.(4), requires

$$
|\mathbf{p}| = |\mathbf{p}'|\ , \tag{5}
$$

implying that the recoil energy of the source is consistently neglected.

As usual, the transition probability per unit time w_{if} is written in terms of the invariant amplitude $\mathcal{M}_{if}$ as

$$
w_{if} = \frac{|S_{if}|^2}{T} = (2\pi)\ \delta(E-E')\left(\frac{m}{VE}\right)^2 |\mathcal{M}_{if}|^2\ , \tag{6}
$$

where T denotes the interaction time and

$$
\mathcal{M}_{if} = ie\ \bar{u}^{(r')}(\mathbf{p}')\not{A}(\mathbf{p}'-\mathbf{p})u^{(r)}(\mathbf{p})\ . \tag{7}
$$

The differential cross section

$$
d\sigma = \frac{1}{\text{flux}}\ w_{if}\ \frac{V}{(2\pi)^3}\ d^3p' \tag{8}
$$

can be readily obtained from the above equations using

$$
|\mathbf{p}'|^2 d|\mathbf{p}'| = |\mathbf{p}'|E'dE' \tag{9}
$$

and

$$
\text{flux} = \frac{v}{V} = \frac{|\mathbf{p}|}{VE}\ , \tag{10}
$$

with the result

$$d\sigma = \frac{VE}{|\mathbf{p}|} \, (2\pi) \, \delta(E-E') \, \left(\frac{m}{VE}\right)^2 |\mathcal{M}_{if}|^2 \frac{V}{(2\pi)^3} \, |\mathbf{p}'| E' dE' d\Omega_{p'} \; . \tag{11}$$

Using the δ-function to carry out the E' integration we can finally write the differential cross section, yielding the probability that the electron be scattered into the element of solid angle $d\Omega_{p'}$:

$$\frac{d\sigma}{d\Omega_{p'}} = \left(\frac{m}{2\pi}\right)^2 |\mathcal{M}_{if}|^2 = \left(\frac{me}{2\pi}\right)^2 |\bar{u}^{(r')}(\mathbf{p}') \not{A}(\mathbf{q}) u^{(r)}(\mathbf{p})|^2 \; , \tag{12}$$

wth $\mathbf{q} = \mathbf{p}' - \mathbf{p}$.

To be more specific, we will now identify the static field with the Coulomb filed of a heavy nucleus,

$$A^\mu(\mathbf{x}) \equiv \left(\frac{Z}{4\pi}\frac{e}{|\mathbf{x}|}, 0, 0, 0\right) \; , \tag{13}$$

whose Fourier transform to momentum space is

$$A^\mu(\mathbf{q}) \equiv \left(Z\frac{e}{|\mathbf{q}|^2}, 0, 0, 0\right) \; . \tag{14}$$

Substituting eq.(14) into eq.(12), summing over the final electron spin and averaging over the initial electron spin we get

$$\begin{aligned} \frac{d\sigma}{d\Omega_{p'}} &= \left(\frac{me}{2\pi}\right)^2 Z^2 \, \frac{e^2}{|\mathbf{q}|^4} \frac{1}{2} \sum_{rr'} |\bar{u}^{(r')}(\mathbf{p}') \gamma^0 u^{(r)}(\mathbf{p})|^2 \\ &= \frac{(2m\alpha Z)^2}{|\mathbf{q}|^4} \, \frac{1}{(2m)^2} \, \frac{1}{2} Tr \left[(\not{p}' + m) \gamma^0 (\not{p} + m) \gamma^0 \right] \; , \end{aligned} \tag{15}$$

where the r and r' sums have been carried out using the completeness relation fulfilled by Dirac's spinors. The trace involved in the above equation can be easily obtained from

$$\begin{aligned} Tr \left[(\not{p}' + m) \gamma^0 (\not{p} + m) \gamma^0 \right] &= Tr(\not{p}' \gamma^0 \not{p} \gamma^0) + m^2 Tr(\gamma^0 \gamma^0) \\ &= 4 p'_\mu p_\nu (2 g^{\mu 0} g^{\nu 0} - g^{\mu\nu} g^{00}) + 4m^2 \\ &= 4[2EE' - (pp') + m^2] = 4[EE' + (\mathbf{p} \cdot \mathbf{p}') + m^2] \; , \end{aligned} \tag{16}$$

and the differential cross section can be rewritten

$$\begin{aligned}\frac{d\sigma}{d\Omega_{p'}} &= 2\ \frac{(\alpha Z)^2}{|\mathbf{q}|^4}\ [E^2 + (\mathbf{p}\cdot\mathbf{p}') + m^2] \\ &= 2\ \frac{(\alpha Z)^2}{|\mathbf{q}|^4}\ 2E^2 \left(1 - \frac{|\mathbf{p}|^2}{E^2}\sin\frac{\theta}{2}\right) \\ &= 2\ \frac{(\alpha Z^2)}{|\mathbf{q}|^4}\ 2E^2 \left(1 - v^2\sin\frac{\theta}{2}\right)\ ,\end{aligned} \tag{17}$$

where v and θ denote the electron velocity and scattering angle (i.e. the angle between $\mathbf{p}$ and $\mathbf{p}'$), respectively. Using the relation

$$|\mathbf{q}|^2 = |\mathbf{p}' - \mathbf{p}|^2 = 2|\mathbf{p}|^2(1-\cos\theta) = 4E^2v^2\sin^2\frac{\theta}{2} \tag{18}$$

we can finally recast eq.(18) in the form originally obtained by Mott.

$$\frac{d\sigma}{d\Omega_{p'}} = \frac{\alpha Z^2}{4E^2v^4\sin^4(\theta/2)}\ \left(1 - v^2\sin\frac{\theta}{2}\right)\ , \tag{19}$$

The Mott cross section describes elastic electron-nucleus scattering to lowest order in α. Obviously, it is a rather poor approximation for heavy nuclei, having large $Z\alpha$.

In the nonrelativistic limit, i.e. setting $v << 1$ and $E \sim m$, from eq.(19) we recover the celabrated Rutherford cross section

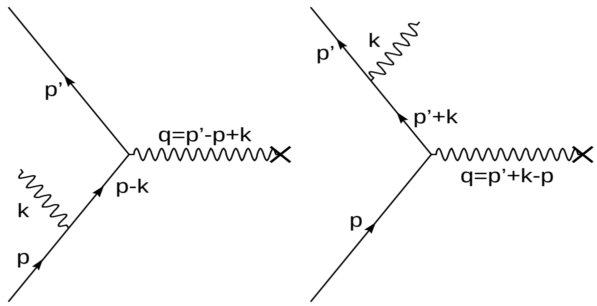
$$\frac{d\sigma}{d\Omega_{p'}} = \frac{aZ^2}{4m^2v^4\sin^4(\theta/2)}\ , \tag{20}$$

whose experimental verification led to the development of the planetary model of the atom.

Note that the fact that Rutherford's cross section has been obtained from a calculation carried out at lowest order in α is somewhat surprising, since the same expression can be obtained in classical nonrelativistic mechanics as an exact result. The highly nontrivial solution of this puzzle was found by Dalitz, who was able to show that in the nonrelativistic limit higher order corrections contribute a phase factor to the scattering amplitude, thus leaving the cross section unaffected.

2. Bremsstrahlung

In the previous section, we have seen that the interaction with a static field deflects the incoming electron. As the electron is accelerated, it may also radiate, i.e. emit real photons. The occurrence of this process, called "bremsstrahlung" ("braking radiation"), has important experimental implications and needs to be carefully taken into account.



(G1) (G2)

In this section we will consider the processes represented by Feynman diagrams G1 and G2, in which the electron radiates a single photon of momentum $|\mathbf{k}|$ (and energy $\omega = |\mathbf{k}|$) either before or after interacting with the external field. The correspondig S-matrix elements reads

$$S_{if} = \langle f|S|i\rangle = (2\pi)\ \delta(E' + \omega - E)\ N_p N_{p'} N_k\ (2m)\ \mathcal{M}_{if}\ , \tag{21}$$

where (use $S_F(p) = 1/(\not{p} - m) = (\not{p} + m)/(p^2 - m^2)$)

$$\begin{aligned}\mathcal{M}_{if} = -ie^2\ &\Big[\bar{u}^{(r')}(\mathbf{p}')\not{\epsilon}(\mathbf{k})S_F(p'+k)\not{A}(\mathbf{q})u^{(r)}(\mathbf{p}) \\ &+ \bar{u}^{(r')}(\mathbf{p}')\not{A}(\mathbf{q})S_F(p-k)\not{\epsilon}(\mathbf{k})u^{(r)}(\mathbf{p})\ \Big] \\ = -ie^2\ \bar{u}^{(r')}(\mathbf{p}')&\left[\not{\epsilon}(\mathbf{k})\frac{\not{p}' + \not{k} + m}{2(p'k)}\not{A}(\mathbf{q}) + \not{A}(\mathbf{q})\frac{\not{p} - k + m}{-2(pk)}\not{\epsilon}(\mathbf{k})\right] u^{(r)}(\mathbf{p})\ .\end{aligned} \tag{22}$$

The differential cross section can be written

$$\begin{aligned}d\sigma &= \frac{1}{\text{flux}}\ \frac{|S_{if}|^2}{T}\ \frac{V^2}{(2\pi)^6} d^3k d^3p' \\ &= \frac{VE}{|\mathbf{p}|}\ (2\pi)\ \delta(E' + \omega - E)\ \left(\frac{m}{VE}\right)\left(\frac{m}{VE'}\right)\left(\frac{1}{2V\omega}\right)\ |\mathcal{M}_{if}|^2\ \frac{V^2}{(2\pi)^6}\ d^3k d\Omega_{p'} |\mathbf{p}'| E' dE'\ .\end{aligned} \tag{23}$$

Using the δ-function to integrate over the energy of the outgoing electron, E', we finally obtain

$$\frac{d\sigma}{d\Omega_{p'}} = \frac{m^2}{(2\pi)^5} \frac{1}{2\omega} \frac{|\mathbf{p}'|}{|\mathbf{p}|} |\mathcal{M}_{if}|^2 \, d^3k \ . \tag{24}$$

The calculation of the above differential cross section with the invariant amplitude of eq.(22) involves lenghty algebraic manipulations. Here we will only discuss the limit of small energy of the radiated photon, i.e. $\omega \approx 0$. In this limit

$$|\mathbf{p}'| \approx |\mathbf{p}|, \qquad \mathbf{q} \approx \mathbf{p}' - \mathbf{p} \ , \tag{25}$$

and $\not{k}$ can be neglected in the numerator of the fermion propagators. As a consequence, the invariant amplitude takes the simplified form

$$\mathcal{M}_{if} = -ie^2 \ \bar{u}^{(r')}(\mathbf{p}')\not{A}(\mathbf{q})u^{(r)}(\mathbf{p}) \left[\frac{p'\epsilon}{p'k} - \frac{p\epsilon}{pk} \right] \ . \tag{26}$$

The above equation can be easily obtained starting from ($\omega \approx 0$)

$$\bar{u}^{(r')}(\mathbf{p}') \ \not{\epsilon}(\mathbf{k}) \frac{\not{p}' + \not{k} + m}{2(p'k)} \not{A}(\mathbf{q})u^{(r)}(\mathbf{p}) \approx \bar{u}^{(r')}(\mathbf{p}') \ \not{\epsilon}(\mathbf{k}) \frac{\not{p}' + m}{2(p'k)} \not{A}(\mathbf{q})u^{(r)}(\mathbf{p}) \tag{27}$$

and using

$$\not{\epsilon}\not{p}' = \gamma^\mu \gamma^\nu \epsilon_\mu p'_\nu = (2g^{\mu\nu} - \gamma^\nu \gamma^\mu)\epsilon_\mu p'_\nu = 2p'\epsilon - \not{p}'\not{\epsilon} \ . \tag{28}$$

Substitution of eq.(28) into eq.(27) yields

$$\bar{u}^{(r')}(\mathbf{p}') \ \frac{2p'\epsilon - (\not{p}' - m)}{2(p'k)} \not{\epsilon}\not{A}(\mathbf{q})u^{(r)}(\mathbf{p}) \ , \tag{29}$$

leading to (use Dirac's equation $\bar{u}^{(r')}(\mathbf{p}')(\not{p}' - m) = 0$)

$$\bar{u}^{(r')}(\mathbf{p}')\not{A}(\mathbf{q})u^{(r)}(\mathbf{p}) \ \frac{2p'\epsilon}{2(p'k)} \ . \tag{30}$$

Applying the same procedure to the second term in the last line of eq.(22) we finally obtain eq.(26).

Comparison between the amplitude of eq.(26) and that in absence of radiation (see eq.(7)),

$$\mathcal{M}^0_{if} = ie \ \bar{u}^{(r')}(\mathbf{p}')\not{A}(\mathbf{p}' - \mathbf{p})u^{(r)}(\mathbf{p}) \ , \tag{31}$$

shows that

$$\mathcal{M}_{if} = \mathcal{M}^0_{if} \; (-e) \; \left[\frac{p'\epsilon}{p'k} - \frac{p\epsilon}{pk}\right] \tag{32}$$

and (use eq.(24) and compare to eq.(12))

$$\begin{aligned} \frac{d\sigma}{d\Omega_{p'}} &= \frac{m^2}{(2\pi)^5} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \; |\mathcal{M}_{if}|^2 \; \frac{d^3k}{2\omega} \\ &= \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 \frac{\alpha}{(2\pi)^2} \left[\frac{p'\epsilon}{p'k} - \frac{p\epsilon}{pk}\right]^2 \frac{d^3k}{\omega} \; , \end{aligned} \tag{33}$$

where $(d\sigma/d\Omega_{p'})_0$ denotes the differential cross section in absence of radiation of eq.(12).

Besides displaying a simple factorized form, the right hand side of the above equation exhibits a singularity at $\omega = 0$. For this reason it is said to be *infrared divergent.* Before discussing this feature in detail we will complete the calculation of the differential cross section of eq.(33) carrying out the sum over the polarization states of the emitted photon.

The procedure to perform the polarization sum exploits the requirement that the final result of our calculation be gauge invariant and the fact that the amplitude of any process involving an external photon of polarization r can be cast in the form (to simplify the notation the dependence of ϵ upon $\mathbf{k}$ is omitted)

$$\mathcal{M}_{if} = \epsilon_{r,\mu} \mathcal{M}^{\mu}_{if} \; . \tag{34}$$

The polarization vector ϵ_r is gauge dependent, as can be easily seen considering a change of gauge

$$A_\mu(x) \rightarrow {A_\mu}'(x) = A_\mu(x) + \partial_\mu \Lambda(x), \tag{35}$$

where, for a free photon described in the Lorentz gauge

$$A_\mu(x) = \epsilon_{r,\mu} \mathrm{e}^{ikx} \; . \tag{36}$$

Choosing $\Lambda(x) = a\mathrm{e}^{ikx}$, the above equations leads to

$${A_\mu}'(x) = (\epsilon_{r,\mu} + iak_\mu)\mathrm{e}^{ikx} = {\epsilon_{r,\mu}}'\mathrm{e}^{ikx} \; , \tag{37}$$

showing that the gauge transformation changes the polarization vector ϵ_r into ${\epsilon_r}'$.

Requiring that the amplitude (34) be unaffected by the replacement $\epsilon_r \to {\epsilon_r}'$ amounts to require that

$$\epsilon_{r,\mu}\mathcal{M}^{\mu}_{if} = {\epsilon_{r,\mu}}'\mathcal{M}^{\mu}_{if} = (\epsilon_{r,\mu} + iak_\mu)\mathcal{M}^{\mu}_{if} \ , \tag{38}$$

i.e. that

$$k_\mu \mathcal{M}^{\mu}_{if} = 0 \ . \tag{39}$$

The above result can be immediately employed to perform the sum

$$\sum_r |\mathcal{M}_{if}|^2 = (\mathcal{M}^{\mu}_{if})^* \mathcal{M}^{\nu}_{if} \sum_{r=1}^{2} \epsilon_{r,\mu}\epsilon_{r,\nu} \ . \tag{40}$$

The photon polarization vectors fulfill the completeness relation (see notes on the covariant quantization of the electronagnetic field)

$$\sum_{r=0}^{3} \ \zeta_r \epsilon_{r,\mu}\epsilon_{r,\nu} = -g_{\mu\nu} \ , \tag{41}$$

with $\zeta \equiv (-1, 1, 1, 1)$, implying

$$\sum_{r=1}^{2} \zeta_r \epsilon_{r,\mu}\epsilon_{r,\nu} = -g_{\mu\nu} + \epsilon_{0,\mu}\epsilon_{0,\nu} - \epsilon_{3,\mu}\epsilon_{3,\nu} \ . \tag{42}$$

Choosing the polarization vectors in the usual form:

$$\epsilon^{\mu}_{0} \equiv n^\mu \equiv (1, 0, 0, 0) \ ,$$
$$\epsilon^{\mu}_{r} \equiv (0, \boldsymbol{\epsilon}_r) \ , \quad r = 1, 2, 3$$

with

$$\boldsymbol{\epsilon}_r \cdot \boldsymbol{\epsilon}_s = \delta_{rs} \ , \quad r, s = 1, 2 \ ,$$
$$\mathbf{k} \cdot \boldsymbol{\epsilon}_r \ = 0 \ , \quad r = 1, 2 \ ,$$
$$\boldsymbol{\epsilon}_3 = \frac{\mathbf{k}}{|\mathbf{k}|} \ ,$$

and

$$\epsilon_3^\mu = \frac{k^\mu - (kn)n^\mu}{[(kn)^2 - k^2]^{1/2}} \tag{43}$$

we find (use the real photon condition $k^2 = 0$)

$$\sum_{r=1}^{2} \epsilon_{r,\mu}\epsilon_{r,\nu} = -g_{\mu\nu} - \frac{k_\mu k_\nu - (kn)(k_\mu n_\nu + k_\nu n_\mu)}{(kn)^2} \ , \tag{44}$$

implying in turn (use eq.(39))

$$\sum_{r=1}^{2} |\mathcal{M}_{if}|^2 = (\mathcal{M}_{if}^\mu)^* \mathcal{M}_{if}^\nu \sum_{r=1}^{2} \epsilon_{r,\mu}\epsilon_{r,\nu} = -(\mathcal{M}_{if}^\mu)^* \mathcal{M}_{if}^\nu \ g_{\mu\nu} = -|\mathcal{M}_{if}|^2 \ . \tag{45}$$

In conclusion, after summing over photon polarization the cross section reads (compare to eq.(33))

$$\frac{d\sigma}{d\Omega_{p'}} = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 \frac{(-\alpha)}{(2\pi)^2} \left[\frac{p'}{p'k} - \frac{p}{pk}\right]^2 \frac{d^3k}{\omega} \ . \tag{46}$$

The divergence of the bremsstrahlung cross section at ω=0 is a consequence of the fact that real photons have vanishing rest mass. Therefore, it can be formally removed giving the photon a finite mass λ, i.e. setting $k^2 = \lambda^2$ and replacing

$$\omega = |\mathbf{k}| \to \omega_\lambda = \sqrt{|\mathbf{k}|^2 + \lambda^2} \ , \tag{47}$$

leading to (compare again to eq.(33))

$$\frac{d\sigma}{d\Omega_{p'}} = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 \frac{\alpha}{(2\pi)^2} \left[\frac{2p'\epsilon}{2p'k + \lambda^2} + \frac{2p\epsilon}{-2pk + \lambda^2}\right]^2 \frac{d^3k}{\omega_\lambda} \ . \tag{48}$$

To carry out the sum over polarization states in the case of massive photons, we have to include both transverse ($r = 1, 2$) and longitudinal ($r = 3$) polarization. The sum can be easily obtained writing

$$\sum_{r=1}^{3} \epsilon_r^\mu \epsilon_r^\nu = A g^{\mu\nu} + B k^\mu k^\nu \tag{49}$$

and requiring ($r = 1, 2, 3$)

$$k_\mu \epsilon_r^\mu = 0 \ , \qquad g_{\mu\nu}\epsilon_r^\mu \epsilon_r^\nu = -1 \ , \tag{50}$$

yielding $B = -A/\lambda^2$ and $A = -1$, i.e.

$$\sum_{r=1}^{3} \epsilon_r^\mu \epsilon_r^\nu = -g^{\mu\nu} + \frac{k^\mu k^\nu}{\lambda^2} \ . \tag{51}$$

The resulting cross section reads

$$\frac{d\sigma}{d\Omega_{p'}} = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 \frac{(-\alpha)}{(2\pi)^2} \left[\frac{2p'}{2p'k + \lambda^2} + \frac{2p}{-2pk + \lambda^2}\right]^2 \frac{d^3k}{\omega_\lambda} \ . \tag{52}$$

Obviously, as $\lambda \to 0$ the above cross section reduces to the QED result and the infrared divergence reappears.

3. Radiative corrections

We have seen that the cross-section describing electron bremsstrahlung in a static external field behaves as $d|\mathbf{k}|/|\mathbf{k}|$, implying that the probability of emitting a photon carrying zero energy is infinite. This unpleasant feature goes under the name of *infrared catastrophe.*

To better understand the origin of the infrared divergence and the mechanism driving its dipappearance from the measured cross section, we have to carefully analyze the experimental conditions in which bremsstrahlung is observed.

The main point to be kept in mind is that the energy resolution of any experimental device is finite. Hence, the experimental apparatus always accepts electrons scattered *both* elastically *and* inelastically, provided the energy of the radiated photons is in the range $0 \leq \omega \leq \Delta\mathrm{E}$, $\Delta\mathrm{E}$ being the experimental energy resolution. The experimental cross section can be written in the form

$$\left(\frac{d\sigma}{d\Omega_{p'}}\right)_{expt} = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_{el} + \left(\frac{d\sigma}{d\Omega_{p'}}\right)_B \ , \tag{53}$$

where $(d\sigma/d\Omega_{p'})_{el}$ is the elastic cross section and $(d\sigma/d\Omega_{p'})_B$ denotes the inelastic bremsstrahlung cross section integrated over the experimental energy resolution. In order to carry out a meaningful comparison between theory and experiment, the two contributions appearing in the right hand side of the above equation must be calculated consistently, i.e. *at the same order in the fine structure constant* α.

Using the results obtained in the previous sections and denoting again by $(d\sigma/d\Omega_{p'})_0$ the lowest order elastic cross section (eq.(12)) we can write

$$\left(\frac{d\sigma}{d\Omega_{p'}}\right)_B = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 \alpha B \ , \tag{54}$$

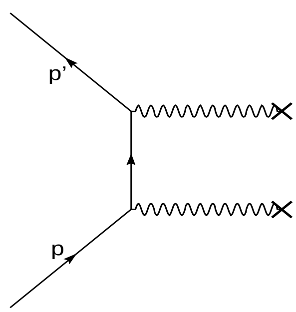
with (to avoid the $\omega = 0$ singularity we give again the photon a finite mass λ)

$$B = -\frac{1}{(2\pi)^2} \int_{\lambda \le \omega \le \Delta \mathrm{E}} \left[\frac{2p'}{2(p'k) + \lambda^2} + \frac{2p}{-2(pk) + \lambda^2}\right]^2 \frac{d^3k}{\omega_\lambda} \ . \tag{55}$$

Hence, consistency with the bremsstrahlung cross section requires that the elastic contribution be calculated including corrections to $(d\sigma/d\Omega_{p'})_0$ of order α, i.e. including R defined by

$$\left(\frac{d\sigma}{d\Omega_{p'}}\right)_{el} = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 (1 + \alpha R) \ . \tag{56}$$

There are several types of corrections contributing to R.



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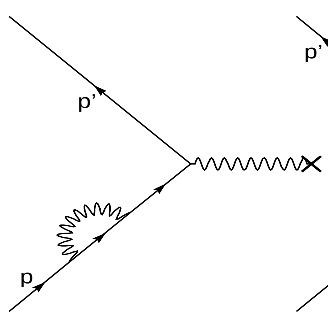
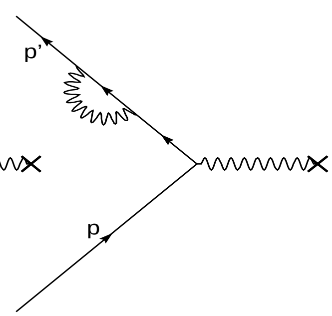


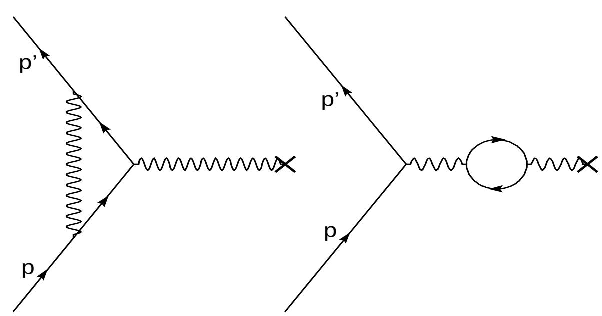
(G4)



(G5)

In principle, one should include the process in which the electron interacts twice with the external field (represented by the second Born approximation diagram G3), processes in which the electron interacts with itself emitting and readsorbing a photon (e.g. diagrams G4-G6) and processes in in which the virtual photon fluctuates to an electron-positron pair (diagram G7). However, as we are only interested to the combined effect of higher order corrections and bremsstrahlung at $\omega \approx 0$, we will consider radiative corrections only.

The Feynman diagrams associated with the lowest order radiative corrections are shown in Figs. G4-G7. The processes represented by diagrams G4 and G5 contribute only to the mass and wave function renormalization, since the radiative correction occurs in the external lines. Diagram G6, besides contributing to the spurious charge renormalization, provides the largest observable effect. Finally, diagram G7 contributes a true charge renormalization and an observable vacuum polarization effect.



(G6) (G7)

After renormalization, the amplitude receives contributions only from diagrams G0, G6 and G7, the resulting expression being

$$ie\ \bar{u}^{(r')}(\mathbf{p}')\gamma^\mu u^{(r)}(\mathbf{p})A_\mu(\mathbf{q}) + ie\ \bar{u}^{(r')}(\mathbf{p}')\left[e^2\ \Lambda_c^\mu(p',p)\right]u^{(r)}(\mathbf{p})A_\mu(\mathbf{q})$$
$$+\ ie\ \bar{u}^{(r')}(\mathbf{p}')\gamma^\mu u^{(r)}(\mathbf{p})\left[-e^2\ \Pi_c(q^2)\right]A_\mu(\mathbf{q})\ . \tag{57}$$

While $\Pi_c(q^2)$ remains finite as $\omega \to 0$, the term containing $\Lambda_c^\mu(p',p)$ (diagram G6) is infrared divergent. Interference between this term and that associated with the lowest order process (diagram G0) leads to the appearance of a correction of order α to the lowest order cross section. We will show that this correction contains an infrared divergent contribution that exactly cancels the divergence of the bremsstrahlung cross section at $\omega = 0$.

Consider the vertex function appearing in the invariant amplitude associated with diagram G6. As we will be focusing on the infrared behavior only, we will omit the cutoff factor taking care of the ultraviolet divergence, and use once more the standard trick of giving the photon a finite mass λ. QED will be recovered taking the limit $\lambda \to 0$ in the final result. The vertex function can then be written

$$
\begin{aligned}
e^2\,\Lambda^\mu(p',p) &= (ie)^2 \int \frac{d^4k}{(2\pi)^4}\, iD_F^{\alpha\beta}(k)\gamma_\alpha\, iS_F(p'-k)\gamma^\mu\, iS_F(p-k)\gamma_\beta \\
&= ie^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{-g^{\alpha\beta}}{k^2-\lambda^2+i\epsilon}\gamma_\alpha\, \frac{1}{(\not{p}'-\not{k})-m+i\epsilon}\gamma^\mu\, \frac{1}{(\not{p}-\not{k})-m+i\epsilon}\,\gamma_\beta \\
&= -ie^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2-\lambda^2+i\epsilon}\gamma_\alpha\, \frac{1}{(\not{p}'-\not{k})-m+i\epsilon}\gamma^\mu\, \frac{1}{(\not{p}-\not{k})-m+i\epsilon}\,\gamma^\alpha\ .
\end{aligned}
\tag{58}
$$

To isolate the observable part $\Lambda_c^\mu(p',p)$ we have to identify the ultraviolet divergent part. In the case of free initial and final electrons, i.e. for $p^2 = p'^2 = m^2$, and in the limit $p' \to p$, Lorentz invariance requires (to simplify the notation, the spin indices of the Dirac's spinors will be omitted hereafter)

$$
\bar{u}(\mathbf{p})\Lambda^\mu(p,p)u(\mathbf{p}) = A\ \bar{u}(\mathbf{p})\gamma^\mu u(\mathbf{p}) + B\ p^\mu \bar{u}(\mathbf{p})u(\mathbf{p})\ . \tag{59}
$$

The above equation can be rewritten using Gordon's identity (see Appendix)

$$
2m\ \bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p}) = \bar{u}(\mathbf{p}')\left[(p+p')^\mu + i\sigma^{\mu\nu}(p-p')_\nu\right]u(\mathbf{p})\ , \tag{60}
$$

implying

$$
p^\mu \bar{u}(\mathbf{p})u(\mathbf{p}) = m\bar{u}(\mathbf{p})\gamma^\mu u(\mathbf{p})\ . \tag{61}
$$

The resulting expression is

$$
\bar{u}(\mathbf{p})\Lambda^\mu(p,p)u(\mathbf{p}) = (A+mB)\bar{u}(\mathbf{p})\gamma^\mu u(\mathbf{p}) = L\bar{u}(\mathbf{p})\gamma^\mu u(\mathbf{p})\ , \tag{62}
$$

L being a scalar constant. At $p' \neq p$ we can therefore define $\Lambda_c^\mu(p',p)$ through

$$\Lambda^\mu(p',p) = L\ \gamma^\mu + \Lambda^\mu_c(p',p)\ , \tag{63}$$

with the obvious condition

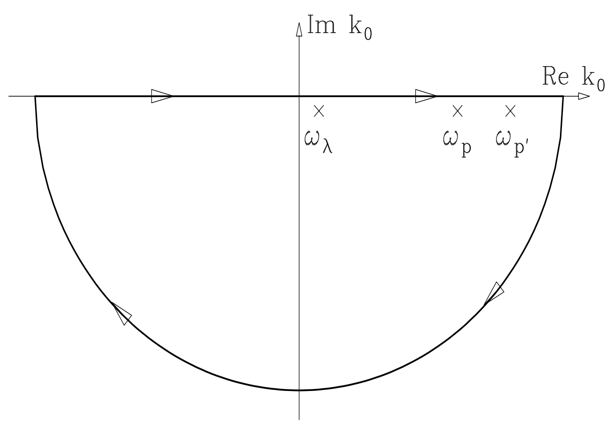
$$\bar{u}(\mathbf{p})\Lambda^\mu_c(p,p)u(\mathbf{p}) = 0\ . \tag{64}$$

As already stated, after renormalization only $\Lambda^\mu_c(p',p)$ appears in the amplitude we are interested in. We will come back to this point later.

As we are only interested in the limit $\omega \approx 0$ we can rewrite eq.(59) neglecting terms linear in k in the numerator of the fermion propagators, in exactly the same way as we did in the calculation of the bremsstrahlung cross section of the previous section. As a result, we find

$$\begin{aligned} e^2\ \bar{u}(\mathbf{p}')\Lambda^\mu(p',p)u(\mathbf{p}) \approx -ie^2 &\int \frac{d^4k}{(2\pi)^4}\ \frac{1}{k^2-\lambda^2+i\epsilon} \\ &\times \bar{u}(\mathbf{p}')\gamma_\alpha\ \frac{\not{p}'+m}{(p'-k)^2-m^2+i\epsilon}\gamma^\mu\ \frac{\not{p}+m}{(p-k)^2-m^2+i\epsilon}\ \gamma^\alpha u(\mathbf{p}) \\ &= -ie^2\ \bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p})\ \int \frac{d^4k}{(2\pi)^4}\ \frac{1}{k^2-\lambda^2+i\epsilon} \\ &\times \frac{4(pp')}{[(p'-k)^2-m^2+i\epsilon]\,[(p-k)^2-m^2+i\epsilon]}\ . \end{aligned}$$

The k_0 integration involved in the above equation can be carried out using Cauchy's theorem.



In principle, to evaluate the integral along the contour shown in the figure, one should collect the contributions of three simple poles located at $\omega_\lambda - i\eta$, $\omega_p - i\eta$ and $\omega_{p'} - i\eta$, with

$$\omega_\lambda = \sqrt{|\mathbf{k}|^2 + \lambda^2} \ ,$$

and

$$\omega_p = E_p + \sqrt{|\mathbf{p} - \mathbf{k}|^2 + m^2} \ \ , \ \ \omega_{p'} = E_{p'} + \sqrt{|\mathbf{p'} - \mathbf{k}|^2 + m^2} \ ,$$

where $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$ and $E_{p'} = \sqrt{|\mathbf{p'}|^2 + m^2}$. However, in the limit $\omega \approx 0$, implying in turn $|\mathbf{k}| \approx 0$, only the residue of the pole at ω_λ contributes to the final result, since in this limit $\omega_p \approx 2E_p$ and $\omega_{p'} \approx 2E_{p'}$. In conclusion, we can write

$$\begin{aligned} e^2 \ \bar{u}(\mathbf{p'})\Lambda^\mu(p',p)u(\mathbf{p}) &= e^2 \ \bar{u}(\mathbf{p'})\gamma^\mu u(\mathbf{p}) \left(-\frac{1}{2}\right) \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega_\lambda} \frac{4pp'}{[-2(p'k) + \lambda^2][-2(pk) + \lambda^2]} \\ &= e^2 \ \bar{u}(\mathbf{p'})\gamma^\mu u(\mathbf{p}) \ A(p',p) \ , \end{aligned} \tag{65}$$

the three-dimensional integration being restricted to the region corresponding to $|\mathbf{k}| \approx 0$. We are now in the condition of subtracting the $L\gamma^\mu$ piece from $\Lambda^\mu(p',p)$ to get the observable piece $\Lambda^\mu_c(p',p)$. From

$$e^2 \ \bar{u}(\mathbf{p'})\Lambda^\mu_c(p',p)u(\mathbf{p}) = e^2 \ \bar{u}(\mathbf{p'}) \left[\Lambda^\mu(p',p) - L\gamma^\mu\right] u(\mathbf{p}) \tag{66}$$

and

$$e^2 \ \bar{u}(\mathbf{p})\Lambda^\mu(p,p)u(\mathbf{p}) = e^2 \ L \ \bar{u}(\mathbf{p})\gamma^\mu u(\mathbf{p}) = e^2 \ \bar{u}(\mathbf{p})\gamma^\mu u(\mathbf{p}) \ A(p,p) \tag{67}$$

it follows that

$$L = A(p,p) = A(p',p') = \frac{1}{2}\left[A(p,p) + A(p',p')\right] \ , \tag{68}$$

implying in turn

$$\begin{aligned} e^2 \ \bar{u}(\mathbf{p'})\Lambda^\mu_c(p',p)u(\mathbf{p}) &= e^2 \ \bar{u}(\mathbf{p'})\gamma^\mu u(\mathbf{p}) \left[A(p',p) - \frac{1}{2}A(p,p) - \frac{1}{2}A(p',p')\right] \\ &= e^2 \ \bar{u}(\mathbf{p'})\gamma^\mu u(\mathbf{p}) \left(\frac{1}{4}\right) \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega_\lambda} \left[\frac{2p'}{-2(p'k) + \lambda^2} - \frac{2p}{-2(pk) + \lambda^2}\right]^2 . \end{aligned} \tag{69}$$

In conclusion, substituting the above result into eq.(57) and using the lowest order amplitude given by eq.(31) we can write the amplitude associated with diagram G6 in the factorized form

$$ie\;\bar{u}(\mathbf{p}')\not{A}(\mathbf{q})u(\mathbf{p}) \times \frac{e^2}{4(2\pi)^3}\int \frac{d^3k}{\omega_\lambda}\left[\frac{2p'}{-2(p'k)+\lambda^2} - \frac{2p}{-2(pk)+\lambda^2}\right]^2$$

$$= \mathcal{M}^0_{if} \times \frac{1}{2}\frac{\alpha}{(2\pi)^2}\int \frac{d^3k}{\omega_\lambda}\left[\frac{2p'}{-2(p'k)+\lambda^2} - \frac{2p}{-2(pk)+\lambda^2}\right]^2 . \qquad (70)$$

Interference bwtween the above amplitude and $\mathcal{M}^0_{if}$ produces a correction of order α to the lowest order elastic the cross section, to be included in the quantity R appearing in eq.(56). After integration over the experimental energy resolution we can finally write

$$R = \frac{1}{(2\pi)^2}\int_{\lambda\leq\omega_\lambda\leq\Delta E} \frac{d^3k}{\omega_\lambda}\left[\frac{2p'}{-2(p'k)+\lambda^2} - \frac{2p}{-2(pk)+\lambda^2}\right]^2 + \cdots , \qquad (71)$$

where the dots denote terms that remain finite in the $\lambda \to 0$, $\omega \to 0$ limit.

Finally, comparison between eq.(71) and eq.(55) shows that in the $\lambda \to 0$, $\omega \to 0$ limit, R and B cancel one another and the observed cross section

$$\left(\frac{d\sigma}{d\Omega_{p'}}\right)_{expt} = \left(\frac{d\sigma}{d\Omega_{p'}}\right)_0 \left[1 + \alpha\,(B+R)\right] \qquad (72)$$

is finite.

APPENDIX

Proof of Gordon's identity

Dirac's equation obviously implies

$$\bar{u}(\mathbf{p}')\left[\not{a}(\not{p}-m) + (\not{p}'-m)\not{a}\right]u(\mathbf{p}) = 0 .$$

It follows that

$$\bar{u}(\mathbf{p}')\,(\not{a}\not{p} + \not{p}'\not{a})\,u(\mathbf{p}) = 2m\not{a}\bar{u}(\mathbf{p}')u(\mathbf{p}) = 2ma_\mu\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p}) .$$

The quantity enclosed in round brackets in the left hand side of the above equation can be rewritten using

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} - i\sigma^{\mu\nu}$$

and

$$\gamma^\nu \gamma^\mu = g^{\mu\nu} + i\sigma^{\mu\nu} \ ,$$

with the result

$$\not{a}\not{p} + \not{p}'\not{a} = a_\mu \left(p_\nu \gamma^\mu \gamma^\nu + p'_\nu \gamma^\nu \gamma^\mu\right) = a_\mu \left[(p + p')_\nu g^{\mu\nu} + i(p - p')_\nu \sigma^{\mu\nu}\right] \ .$$

leading to

$$\bar{u}(\mathbf{p}') \left[(p + p')_\nu g^{\mu\nu} + i(p - p')_\nu \sigma^{\mu\nu}\right] u(\mathbf{p}) = 2m\bar{u}(\mathbf{p}')\gamma^\mu u(\mathbf{p}) \ .$$