Measures of critical exponents in the four dimensional site percolation

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Abstract

Using finite-size scaling methods we measure the thermal and magnetic exponents of the site percolation in four dimensions, obtaining a value for the anomalous dimension very different from the results found in the literature. We also obtain the leading corrections-to-scaling exponent and, with great accuracy, the critical density.

Key words: Lattice. Monte Carlo. Percolation. Critical exponents. Finite size scaling. ϵ-expansion.

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1 Introduction

From the point of view of its definition, the simplest statistical system is perhaps the percolation. In the case of the site percolation, we fill the sites of a given lattice with probability p. Then we construct the clusters as sets of contiguous filled sites.

The critical properties of the system can be described in terms of the clusters. For instance, at the critical percolation the mean cluster size diverges. We define the percolating cluster as the one that contains, in the thermodynamical limit, an infinite number of sites. The strength of this cluster (i.e. the probability of containing an arbitrary point) is the order parameter of the transition: it is zero for $p < p_c$, and finite for $p > p_c$ [1].

Another interesting model is the bond percolation. In this case we fill the lattice bonds with a given probability and construct clusters analogously. It is believed that both models belong to the same Universality Class (share the critical exponents).

It is possible to relate the percolation problem (in the bond version) with the q-states Potts model using the "Fortuin-Kasteleyn" representation of the latter. The bond percolation is obtained in the $q \to 1$ limit [2].

Moreover it is possible to write down a field theoretical description of the percolation. In general, the Potts model is described by means of a ϕ^3 theory, where the coefficient of the cubic term is proportional to $q-2$. For the Ising model ($q=2$) this term vanishes, and the leading term is ϕ^4, recovering the standard field theory representation. For $q \neq 2$ we can write

$$S = \int \mathrm{d}^d x \left[\frac{1}{2}(\nabla\phi_i)(\nabla\phi_i) + \frac{1}{2} m_0^2 \phi_i \phi_i + \frac{1}{3!} g_0 d_{ijk} \phi_i \phi_j \phi_k \right], \tag{1}$$

where the coefficients d_{ijk} depend on the model (Potts, percolation, Lee-Yang singularities, etc.), and $n \equiv q-1$ is the number of components of the field ϕ_i. Thus, the percolation is described by the action (1) in the limit of zero components of the fields.

Using the standard tools it is possible to obtain an ϵ-expansion for this model (and in particular for the percolation). The power counting tells us that the upper critical dimension of the model is six and thereby the expansion parameter is $\epsilon = 6 - d$. Results up to three loops can be found in the literature [3].

For large dimensions ($d = 5$, and, of course, 6) there is a good agreement between the results obtained from the ϵ-expansion (resumed using Padé techniques), the values from numerical simulations, and the results from high temperature expansions.

In lower dimensions, the results disagree for the anomalous dimension, η. The ϵ-expansion predicts a clear negative value, while in the two dimensional case η should be non-negative because the correlation function is decreasing with the distance. In fact, in this case, it has been conjectured [4] that $\eta = 5/24$.

In this paper we will show that the value of the four dimensional η exponent turns out to be large by a 30%, compared to the ϵ-expansion. Thereby it remains as an open problem to understand why the convergence of the ϵ-expansion for this model is so poor even for small values of ϵ [5]. In order to calculate critical exponents we extend some recently developed accurate finite-size scaling techniques [6] to site percolation. As a benchmark we report the two dimensional critical exponents (for which there are almost exact analytical estimates).

A related model with the site percolation is the diluted Ising model [7]. It is defined as a standard Ising model where the spins live only on filled (with probability p) sites. The field theoretical description of this model is a ϕ^4-theory with a random mass term. Using the replica trick it can be related with an $\mathrm{O}(N)$ symmetric ϕ^4 theory with cubic anisotropy, in the limit of zero field components (i.e. $N \to 0$) [8, 9].

The limit of zero temperature (large β) of the diluted Ising model is the site percolation while when $p \to 1$ it is the pure Ising model. A precise determination of the critical exponents of the $d = 4$ percolation is also a very useful first step to understand the phase diagram (β, p) in the diluted Ising model. On the other hand, the site percolation is useful as a benchmark to develop and test different tools to apply to more complicated systems as the $d = 4$ diluted Ising model [10].

Finally, we remark that we are specially interested in these four dimensional models in relation with the triviality issue (is there an interacting continuum limit in four dimensions?). In order to solve the triviality problem is crucial to characterize all the possible fixed points in four dimensions. The site percolation has the unusual feature of having the critical dimension at $d = 6$, thus, it does not present the usual Mean Field exponents at $d = 4$.

2 Numerical Methods

We will work in a hypercubic lattice of linear size L with periodic boundary conditions. The Monte Carlo (MC) procedure for generating configurations in this model is straightforward: we fill each lattice site with probability p. The next step is to build the clusters, what is a deterministic procedure. To save computer memory in the larger lattices, we use a self-recurrent algorithm (in C language). In this way the total memory employed to sketch the clusters is almost negligible (it grows nearly as the lattice size squared).

Due to the absence of MC dynamics, the system is specially vulnerable to eventual pathologies of the random number generator. We have observed significant deviations in some quantities for a commonly used shift register generator [11], specially in the larger lattices. To avoid these effects, we have used as generator the sum (modulus 1) of the output of the generator of ref. [11] and a congruential one, since it is known that their respective drawbacks are very different.[1]

To define the observables that we measure, it is useful to consider a related model that is a diluted Ising model with nearest neighbors infinite coupling, where the spins, $\sigma_i = \pm 1$, live only in filled sites. It is easy to show that the magnetization of the latter model,

$$\mathcal{M} = \frac{1}{V} \sum_i \sigma_i, \tag{2}$$

V being the volume, coincides with the strength of the percolating cluster in the thermodynamical limit and at $T = 0$.

Knowing the size of the clusters, as their spins must take the same sign, we can write

$$\mathcal{M} = \frac{1}{V} \sum_{\mathrm{c}} s_{\mathrm{c}} n_{\mathrm{c}}, \tag{3}$$

where s_{c} is the sign of the cluster c, n_{c} its size, and the sum runs over all clusters. As s_{c} are statistically independent, we can construct an improved estimator for even powers of $\mathcal{M}$ (the only non-vanishing in a finite lattice) averaging over all possible values of $\{s_{\mathrm{c}}\}$, that henceforth we will denote as

[1] We have used $X_{n+1} = 16807 X_n \bmod (2^{31} - 1)$ for the congruential random generator, whereas the shift register formulas read: $X_n = X_{n-24} + X_{n-55}$; using as pseudorandom number X_n XOR X_{n-61}.

$\overline{(\cdots)}$. For the second power we have

$$\overline{\mathcal{M}^2} = \frac{1}{V^2} \sum_{\mathrm{c}} n_{\mathrm{c}}^2. \tag{4}$$

We define the susceptibility as

$$\chi = V \left\langle \overline{\mathcal{M}^2} \right\rangle. \tag{5}$$

To compute the Binder parameter V_M we can construct an improved estimator for the fourth power of the magnetization. Averaging over signs, we obtain after some algebra

$$\overline{\mathcal{M}^4} = 3 \left(\overline{\mathcal{M}^2} \right)^2 - \frac{2}{V^4} \sum_{\mathrm{c}} n_{\mathrm{c}}^4, \tag{6}$$

from which [2]

$$V_M = \frac{3}{2} - \frac{1}{2} \frac{\langle \overline{\mathcal{M}^4} \rangle}{\langle \overline{\mathcal{M}^2} \rangle^2}. \tag{7}$$

For the finite-size scaling (FSS) method that we employ, it is very useful an accurate measure of the correlation length. We have used the second momentum definition [13] in the associated Ising model, that, in a finite lattice, reads

$$\xi = \left(\frac{\chi/F - 1}{4 \sin^2(\pi/L)} \right)^{1/2}, \tag{8}$$

where F is defined in terms of the Fourier transform of the magnetization

$$\widehat{\mathcal{M}}(\boldsymbol{k}) = \frac{1}{V} \sum_{\boldsymbol{r}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \sigma_{\boldsymbol{r}}, \tag{9}$$

as

$$F = \frac{1}{4} \left\langle |\widehat{\mathcal{M}}(2\pi/L, 0, 0, 0)|^2 + \text{permutations} \right\rangle. \tag{10}$$

It is also possible to construct an improved estimator for $|\widehat{\mathcal{M}}|^2$ as

$$\overline{|\widehat{\mathcal{M}}(\boldsymbol{k})|^2} = \sum_{\mathrm{c}} |\widehat{n}_{\mathrm{c}}(\boldsymbol{k})|^2, \qquad \widehat{n}_{\mathrm{c}}(\boldsymbol{k}) \equiv \frac{1}{V} \sum_{\boldsymbol{r}\in\mathrm{c}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{11}$$

[2] For another application of the Binder cumulant in percolation theory see [12].

To measure the critical exponents we use a form of the FSS ansatz that only involves measures on a finite lattice. For an operator O that diverges as $(p - p_\mathrm{c})^{-x_O}$, its mean value in a size L lattice can be written, in the critical region, as

$$O(L,p) = L^{x_O/\nu}\left(F_O(\xi(L,p)/L) + O(L^{-\omega})\right), \tag{12}$$

where F_O is a scaling function and ω is the universal leading corrections-to-scaling exponent. From a Renormalization Group point of view, ω corresponds to the leading irrelevant operator.

We can eliminate the unknown scaling function using the values from two different lattice sizes measuring at a p value where the ξ/L quotients match. Specifically, defining

$$Q_O = O(sL,p)/O(L,p), \tag{13}$$

we can write

$$Q_O|_{Q_\xi=s} = s^{x_O/\nu} + O(L^{-\omega}). \tag{14}$$

Other examples of application of this method can be found in refs. [6].

The form of the scaling corrections allows to parameterize the finite-size effect on the determination of the critical exponents as

$$\left(\frac{x_O}{\nu}\right)_\infty - \left(\frac{x_O}{\nu}\right)_{(L,sL)} \propto L^{-\omega}. \tag{15}$$

To compute the ω exponent, we can use equation (12) for an operator with $x_O = 0$ (as, for instance, V_M or ξ/L) obtaining for the shift of the crossing point of lattice sizes L and sL [14]

$$\Delta p^{L,sL} \equiv [p_c(L,sL) - p_c(\infty)] \propto \frac{1 - s^{-\omega}}{s^{\frac{1}{\nu}} - 1} L^{-\omega-\frac{1}{\nu}}. \tag{16}$$

To efficiently use the FSS formulas, it is necessary to use a reweighting method to move in the critical region. For this model there is not a Boltzmann weight, but the role of the energy is carried out by the density of the configuration, and the probability distribution is binomial.

The probability of finding a density q when filling sites with a probability p is

$$\rho_p(q) = \frac{V!}{(qV)!((1-q)V)!} p^{qV}(1-p)^{(1-q)V}. \tag{17}$$

From a set of N measures of an observable O and the actual density of the configuration $\{(O_i, q_i)\}$ we can compute the mean value of the observable for a neighbor density p' as

$$O(p') = \frac{1}{N}\sum_i \frac{\rho_{p'}(q_i)}{\rho_p(q_i)} O_i = \frac{1}{N}\sum_i \left(\frac{p'}{p}\right)^{q_i V} \left(\frac{1-p'}{1-p}\right)^{(1-q_i)V} O_i. \qquad (18)$$

Using equation (18) p-derivatives of observables can also be computed.

Obviously we cannot extrapolate much further than $\sqrt{p(1-p)/V}$, which is the dispersion of the distribution (17). Therefore the visible region decreases as $L^{-d/2}$. Fortunately, it is enough for our purposes since to use eq. (14) we need to move in a neighborhood of the critical point whose size decreases as $L^{-\omega-1/\nu}(\approx L^{-2.5})$.

3 Numerical Results

We have produced a million of independent samples for each L^4 lattices, with $L = 8, 12, 16, 24, 32$ and 48.

To measure the thermal critical exponent we have used as operators: $\mathrm{d}\log\chi/\mathrm{d}p$ ($x_{\mathrm{d}\log\chi/\mathrm{d}p} = 1$) and $\mathrm{d}\xi/\mathrm{d}p$ ($x_{\mathrm{d}\xi/\mathrm{d}p} = 1+\nu$). For the magnetic exponents we have used the susceptibility χ ($x_\chi = \gamma$). We remark that, although χ is a fast varying function of p at the critical region (see refs. [6]), the use of eq. (14) allows a very precise measure. Moreover as what we directly measure is the quotient $\gamma/\nu = 2 - \eta$, we can obtain a very accurate determination of the anomalous dimension η.

We have checked the method in the $d = 2$ case, where there is a very solid conjecture [4] for the values of the critical exponents, which is confirmed by conformal group analysis. We present the measured critical exponents for the two dimensional site percolation in table 1, obtained from a million of samples for each lattice size. The conjectured values by Nienhuis [4] are $\eta = 5/24 = 0.20833\ldots$, $\nu = 4/3$ and $\omega = 2$. The agreement is very good.

In the four dimensional case (see table 2), we observe a very stable value for the ν exponent when using the operator $\mathrm{d}\xi/\mathrm{d}p$. However, the results for the exponents η or ν computed from measures of other operators do need an infinite volume extrapolation, what will be considered next.

To measure the critical density and the corrections-to-scaling exponent ω, we have studied the crossing points of V_M and ξ/L for different pairs of lattice sizes, fitting the displacements to the functional form (16). As the behavior of V_M and ξ/L is very different regarding the corrections-to-scaling, we obtain a great improvement performing a joint fit.

	ν		η
L	$\mathrm{d}\xi/\mathrm{d}p$	$\mathrm{d}\log(\chi)/\mathrm{d}p$	χ
24	1.324(9)	1.326(14)	0.2155(5)
32	1.330(8)	1.30(2)	0.2121(4)
48	1.344(10)	1.36(2)	0.2085(4)
64	1.330(9)	1.36(2)	0.2082(4)

Table 1: Estimates for the critical exponents of two dimensional site percolation obtained from the finite-size scaling analysis using data from lattice sizes L and $2L$. In the second row we show the operator used for each column. Practically we can read from the last row the conjectured values.

We show in figure 1 the crossing points of V_M and ξ/L as a function of $L^{-(\omega+1/\nu)}$, where we have used $\nu = 0.689$ and $\omega = 1.13$.

We fix the lattices ratio to $s = 2$ and perform the fit twice, for $L \geq 8$ and for $L \geq 12$. In both cases we obtain compatible values for the ω exponent and for the critical density. We get acceptable fits, for example $\chi^2/\mathrm{d.o.f.} = 4.7/4$ for the former. We give the central values from the former fit but with the error bars coming from the latter fit:

$$\omega = 1.13(10), \quad p_c(\infty) = 0.196901(5). \tag{19}$$

The error bars have been slightly (20%) increased to take into account the error in the value of ν.

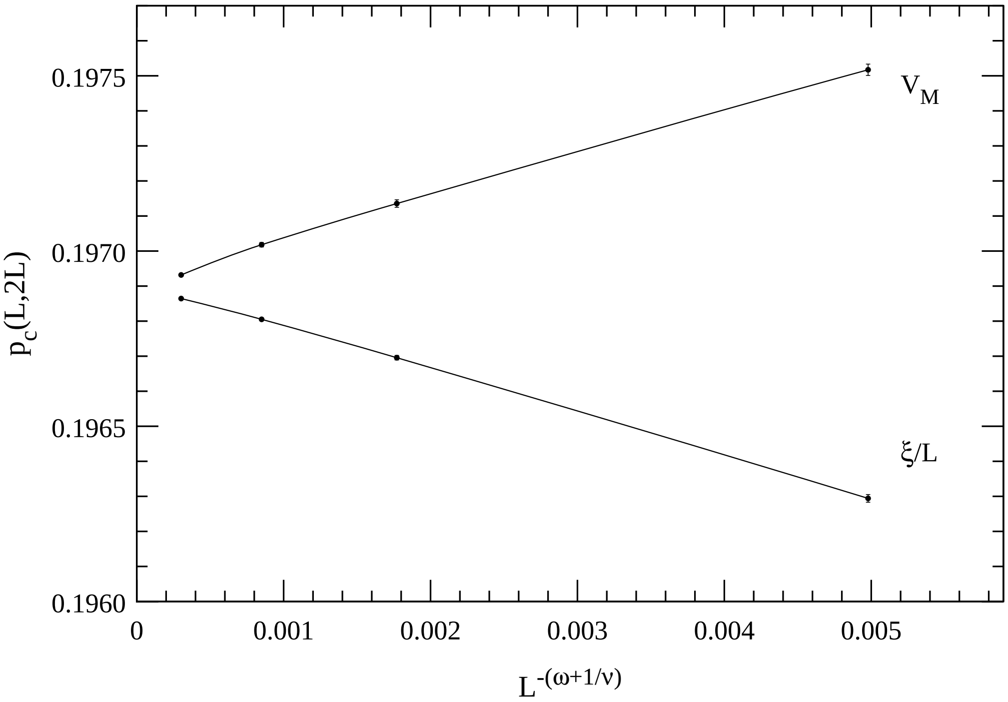

Figure 1: $p_c(L,2L)$ as a function of $L^{-(\omega+1/\nu)}$ for the observables V_M and ξ/L.

L	ν		η
	$\mathrm{d}\xi/\mathrm{d}p$	$\mathrm{d}\log(\chi)/\mathrm{d}p$	χ
8	0.689(3)	0.668(3)	-0.0687(7)
12	0.687(3)	0.666(4)	-0.0775(7)
16	0.688(4)	0.681(5)	-0.0823(6)
24	0.691(5)	0.683(6)	-0.0868(8)
∞	0.689(10)	0.683(12)	-0.0944(17+11)

Table 2: Critical exponents obtained using data from lattice sizes L and $2L$ for the four dimensional site percolation. In the second row we show the operator used for each column. The last row corresponds to the infinite volume extrapolation using (15).

d	ω	ν	γ	η
6	0	0.5	1	0
5	0.79	0.57	1.18	-0.07
4	1.52	0.68	1.44	-0.12
3	2.23	0.83	1.81	-0.18
2	2.95	1.07	2.41	-0.25

Table 3: Our results using [2,1]-Padé resummation for the ω exponent. We also report (columns three to five) the results for the critical exponents (ν, γ and $\eta \equiv 2-\gamma/\nu$) obtained in ref. [3] using the [2,1]–Padé-Borel resummation.

Using these values, we can obtain an infinite volume extrapolation for the critical exponents by means of (15). To control that higher order scaling-corrections can be neglected, we use an objective criterium. We perform the fit considering data from lattices of sizes $L \geq L_{\min}$ and then repeat it discarding the smallest lattices data. If both fits parameters (extrapolated value and slope) are compatible, we keep the central values from the former fit and error bars from the latter. We have found that $L_{\min} = 8$ is enough for our data.

The results are displayed in the last row of table 2. For η the first term in the error have been obtained considering ω fixed, and the second one corresponds to the variation when ω moves within its error bars. In figure 2 we show the behavior of $\eta(L, 2L)$ as a function of $L^{-\omega}$, with $\omega = 1.13$, together with the extrapolated value.

At this point we can use the results of ref. [3] obtained with the ϵ-expansion. We are specially interested in the corrections-to-scaling exponent, that is, the derivate of the β-function at the non trivial fixed point (i.e. $g^* \neq 0$). Using the β-function and the non trivial fixed point from ref. [3] we have obtained

$$\omega \equiv \left.\frac{\mathrm{d}\beta(g)}{\mathrm{d}g}\right|_{g^*} = \epsilon - 0.760767\epsilon^2 + 2.00886\epsilon^3 + O(\epsilon^4). \tag{20}$$

As $\epsilon(= 6 - d)$ is large, we have analyzed this series using the Padé technique. Only the [2,1]–Padé gives consistent results (i.e. $\omega = O(\epsilon) > 0$). This agrees with the results of ref. [3] where in the final Padé analysis of their series for the critical exponents only the [2,1]–Padé is reported (the results of the other Padés turned out to be incompatible with the numerical simulation results).

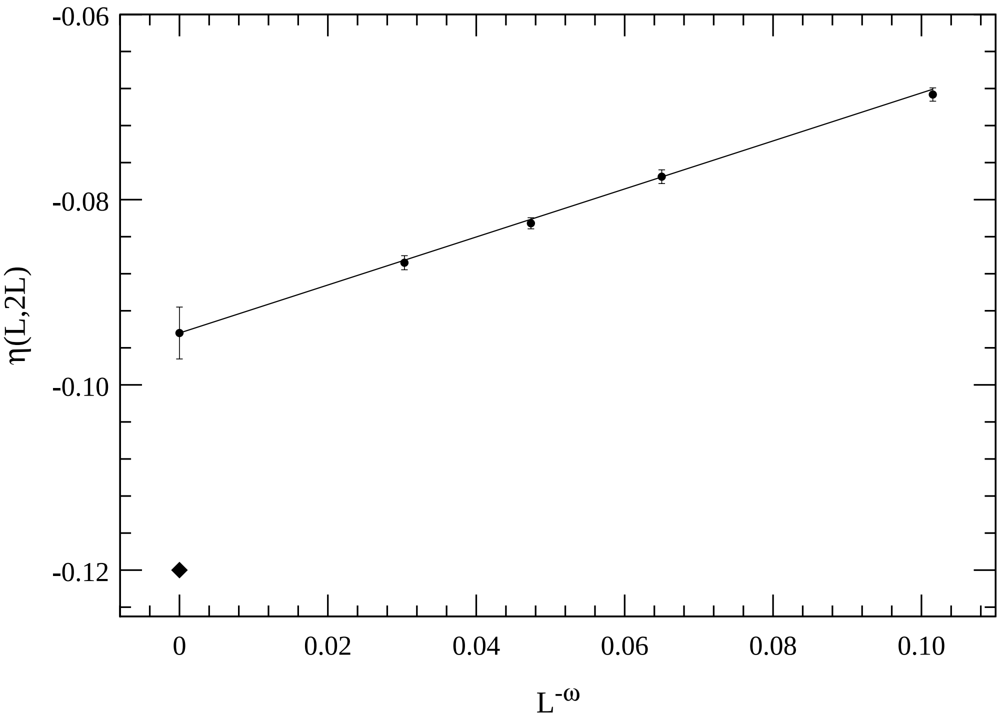

Figure 2: $\eta(L, 2L)$ as a function of $L^{-\omega}$, where ω is the corrections-to-scaling exponent, that we have fixed in this plot to 1.13 (see text for more details). The numerical value found in the literature, $\eta = -0.12$, is also displayed.

We show our results for ω in table 3. We also display in this table the results for the exponents ν, γ and η calculated in ref. [3] using the [2,1]–Padé-Borel resummation.

In other cases with $\epsilon = 2$ a good agreement has been found between resumed series and numerical results. For the two dimensional Ising model the differences in η and ω are 1% and $5 - 35\%$ respectively (taking into account the error bars of the ϵ-expansion estimate of ω)[15]. However, we have obtained a discrepancy of 30% in the anomalous dimension and of 50% in the ω-exponent. Linking this discrepancy with the behavior of η with the dimension, reported in the introduction, we find the ϵ-expansion not trustworthy in this case.

4 Conclusions

Using FSS techniques we have obtained accurate values for the critical exponents of the four dimensional site percolation. We have been able to parameterize the leading corrections-to-scaling what allows to largely reduce the systematic errors coming from finite-size effects.

We have obtained an anomalous dimension that is 30% far away from previous numerical and analytical (ϵ-expansion) approaches.

We project to extend these methods to the case of the diluted Ising model in four dimensions, in order to study the possible variation of the critical exponents on the critical line [10].

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