Finite-size scaling study of the $d = 4$ site-diluted Ising model*.

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We study the four dimensional site-diluted Ising model using finite-size scaling techniques. We explore the whole parameter space (density-coupling) in order to determine the Universality Class of the transition line. Our data are compatible with Mean Field behavior plus logarithmic corrections.

1. Introduction

If a pure system has a specific heat exponent $\alpha > 0$ the Universality Class changes when the dilution is introduced in the model (Harris criterion [1]), while it remains unchanged if $\alpha < 0$ (i.e. the Universality Class is that of the pure model). In the Ising model in 4 (or 2) dimensions $\alpha = 0$ and the Harris criterion does not apply.

Perturbative renormalization group (PRG) computations for $d = 4$ predict Mean Field with Logarithmic Corrections. On the other hand, previous Monte Carlo (MC) results pointed to non Mean Field behavior [2]. In $d = 2$ there are also MC studies that conclude a change of the Universality Class.

We describe here the results of a higher statistics MC study [3]. A Finite-Size Scaling (FSS) approach has been used in order to study large lattices in the critical region. Results on the $d = 2$ case are also briefly described [4].

2. The model

We work in a hypercubic four dimensional lattice. The action is:

$$S = -\beta \sum_{<i,j>} \epsilon_i \epsilon_j \sigma_i \sigma_j , \tag{1}$$

where ϵ_i are quenched uncorrelated random variables whose value is 1 with probability p and 0 otherwise.

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For each $\{\epsilon_i\}$ configuration (sample) we perform an Ising model simulation.

There are two types of averaging. The first corresponds to averaging in Ising configurations, and will be denoted with brackets, the second is associated to the ϵ_i variables (sample average) and will be denoted by overlines. We first perform the Ising average, then the sample one.

2.1. Observables

For each spin configuration we measure the magnetization and first neighbor energy, defined respectively as

$$\mathcal{M} = \frac{1}{V} \sum_i \epsilon_i \sigma_i , \quad \mathcal{E} = \sum_{\langle i,j \rangle} \epsilon_i \sigma_i \epsilon_j \sigma_j . \tag{2}$$

We have focused our study in the following mean values (specific heat, susceptibility, Binder parameter, and correlation length, respectively)

$$C = V^{-1} \left(\overline{\langle \mathcal{E}^2 \rangle} - \overline{\langle \mathcal{E} \rangle}^2 \right) , \tag{3}$$

$$\chi = V \overline{\langle \mathcal{M}^2 \rangle} , \tag{4}$$

$$g_4 = \frac{3}{2} - \frac{1}{2} \frac{\overline{\langle \mathcal{M}^4 \rangle}}{\overline{\langle \mathcal{M}^2 \rangle}^2} , \tag{5}$$

$$\xi = \left(\frac{\chi / F - 1}{4 \sin^2(\pi/L)} \right)^{\frac{1}{2}} , \tag{6}$$

where F is the Fourier transform of the magnetization at $k = \frac{2\pi}{L}$.

3. Finite-size scaling techniques

In a finite lattice at the the critical region, the FSS ansatz states that

$$\langle O(L,\beta)\rangle = L^{\frac{x_O}{\nu}}\left[F_O\left(\frac{\xi(L,\beta)}{L}\right)+O(L^{-\omega})\right], \quad (7)$$

where ω is the corrections-to-scaling exponent, F_O is a (smooth) scaling function and x_O is the critical exponent. For instance, $x_\chi=\gamma$, $x_\xi=\nu$, and $x_{\partial_\beta\xi}=\nu+1$.

We study the quotient of $O(L_1)$ and $O(L_2)$

$$Q_O \equiv \frac{\langle O(L_2,\beta)\rangle}{\langle O(L_1,\beta)\rangle} = s^{\frac{x_O}{\nu}}\frac{F_O(\frac{\xi(L_2,\beta)}{L_2})}{F_O(\frac{\xi(L_1,\beta)}{L_1})}+O(L^{-\omega}) \quad (8)$$

where $s=\frac{L_2}{L_1}$. The unknown F_O can be eliminated in this way: we look for $\beta(L_1,L_2)$ such that $\frac{\xi_{L_2}}{L_2}=\frac{\xi_{L_1}}{L_1}$ so

$$Q_O|_{Q_\xi=s} = s^{\frac{x_O}{\nu}}+O(L^{-\omega}). \quad (9)$$

Now, this picture is slightly modified due to the presence of logarithmic corrections. Using the PRG analysis we obtain for the diluted model [3]

$$\begin{array}{lll} \xi & \propto & L(\log L)^{\frac{1}{8}} \\ \partial_\beta\xi & \propto & L^3\,(\log L)^{\frac{1}{4}}\left(\frac{\xi}{L}\right)^3 \mathrm{e}^{-2\sqrt{\frac{3\log L}{53}}} \\ \chi & \propto & L^2(\log L)^{\frac{1}{4}+\frac{1}{106}} \\ C & \propto & (\log L)^{\frac{1}{2}}\,\mathrm{e}^{-\sqrt{\frac{48}{53}|\log L|}} \end{array} \quad (10)$$

The scaling behavior for the pure model is:

$$\begin{array}{lll} \xi & \propto & L(\log L)^{\frac{1}{4}} \\ \chi & \propto & L^2(\log L)^{\frac{1}{2}} \\ C & \propto & (\log L)^{\frac{1}{3}} \end{array} \quad (11)$$

It can be shown that $\xi(L,\beta,p)/L$ remains as the scaling variable.

The expected leading logarithmic correction for the ν exponent goes as $1/\log L$ for the pure model, but changes to $1/\sqrt{\log L}$ for the diluted case. For the η exponent, the correction is always order $1/\log L$.

4. Numerical methods

We store individual measures of the energy and magnetization for extrapolating in a neighborhood of the simulation parameters.

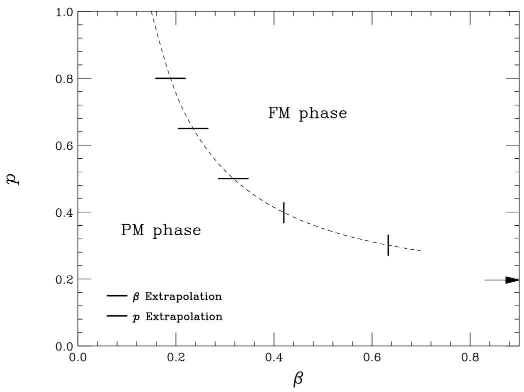

Figure 1. Phase diagram of the $d=4$ site-diluted Ising model. The tics are plotted at the simulated values along the extrapolation direction. The black arrow indicates the percolation limit.

The β-derivatives are obtained from

$$\partial_\beta\overline{\langle\mathcal{O}\rangle} = \overline{\partial_\beta\langle\mathcal{O}\rangle} = \overline{\langle\mathcal{O}\mathcal{E}\rangle} - \overline{\langle\mathcal{O}\rangle\langle\mathcal{E}\rangle}, \quad (12)$$

which is a biased estimator. The statistical error behaves as $1/\sqrt{N_S}$, N_S being the number of samples and the bias as $1/N_I$ (N_I being the number of Ising independent measures in a sample). In our calculations $\sqrt{N_S}\sim N_I$ so a $N_I\to\infty$ extrapolation is performed.

As we gained statistics in a large number of samples with slightly different number of filled sites, in addition to a β-extrapolation a p one is also possible.

The probability of finding q site density for dilution p is binomial.

The observable value from a set of N_S samples at p' near p is

$$\overline{\langle\mathcal{O}\rangle}(p',\beta) = \frac{1}{N_S}\sum_i^{N_S}\left(\frac{p'}{p}\right)^{q_iV}\left(\frac{1-p'}{1-p}\right)^{(1-q_i)V}\langle\mathcal{O}\rangle_i(\beta) \quad (13)$$

It is also possible to compute p-derivatives but the statistical error is much larger than for the β-derivatives (eight times typically).

5. Results

We use cluster methods to update the spin variables. The diluted model is simulated in lattice

Table 1
The ν exponent for $(L, 2L)$ pairs at the different dilutions. The last column corresponds to an inverse linear extrapolation.

p	$L=8$	$L=12$	$L=16$	$L=\infty$
0.8	.5175(11)	.5154(11)	.5142(13)	.5110(25)
0.65	.5308(13)	.5270(13)	.5251(12)	.5194(26)
0.5	.5482(16)	.5428(19)	.5412(19)	.534(4)
≈ 0.4	.5604(15)	.5532(19)	.5478(18)	.536(4)
≈ 0.3	.5700(26)	.5647(22)	.5583(26)	.549(5)

sizes $L \leq 32$. We also studied the pure one ($L \leq 64$) as a check.

We generate 10,000 samples at p=0.8, 0.65, 0.5, 0.4, 0.3. For each sample we take 100 nearly independent measures after equilibration.

Let first assume hyperscaling (there are not logarithmic corrections). We use $\partial_\beta \xi$ and χ to obtain the critical exponents. In the ν case we perform a $L \to \infty$ extrapolation (see table 1), using $\omega = 1$ (near the percolation value [5]). For comparison, in the pure model we obtain $\nu = 0.5019(14)$.

For $\eta = 2 - \gamma/\nu$ we find a weaker evolution with L and the dilution (see ref.[3]).

We clearly discard the percolation values ($\nu = 0.686(2)$ and $\eta = -0.094(3)$ [5]) and a new fixed point for all the critical line is unlikely.

Weak universality (ν varying on the line) cannot be ruled out, but a more economic picture is a MF behavior plus logarithmic corrections (see fig. 2).

We can check the expected logarithmic corrections directly measuring ξ and C at the critical point. To determine the critical coupling (or dilution), we study the Binder g_4 parameter.

The hyperscaling violations for ξ at the critical point are

$$\xi(L, \beta_c) \propto L \, (\log L)^{\delta_\xi} \, . \tag{14}$$

In table 2 we show the fitted δ_ξ values, which are reasonably close to the predicted values.

6. The $d=2$ model

We perform 10^4 samples for $L \leq 384$ at different dilutions. PRG predicts logarithmic corrections to the Ising behavior: ν, equal to one, is corrected by $1/\log L$ and $C \propto \log(\log L)$. We find

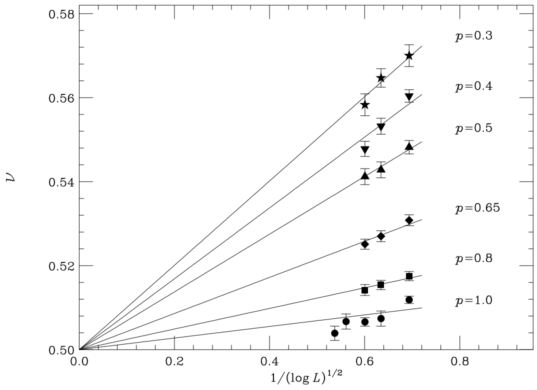

Figure 2. Exponent ν estimation for $(L, 2L)$ pairs. The solid lines are linear fits constrained to be $\nu = 0.5$ in the $L \to \infty$ limit.

Table 2
δ_ξ for the different dilutions. The second error bar is due to the uncertainty in the critical point and the first is the statistical one.

p	δ_ξ	p	δ_ξ
1.0	.198(3+5)	0.8	.203(5+10)
0.65	.181(6+14)	0.5	.189(7+12)
0.4	.241(8+12)	0.3	.281(10+23)

also good agreement to the predicted behavior in both cases.

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