The integrals in (5.12) can be computed in the low temperature limit, leading to [13]:

$$\eta = \frac{\dfrac{1}{\beta^3 J^3(\tilde{a}-q)^3}\sqrt{\dfrac{\tilde{a}-q}{\tilde{t}\pi}}\dfrac{\mathrm{e}^{-\tilde{z}^2}\mathrm{c_N}}{\beta J}}{\dfrac{4q(\tilde{a}-q)}{4\beta^3 J^3(\tilde{a}-q)^3}} = \frac{\mathrm{e}^{-\tilde{z}^2}\mathrm{c_N}}{\beta J q\sqrt{\tilde{t}\pi(\tilde{a}-q)}}, \tag{5.13}$$

where $\tilde{z}$ is defined in (3.7), and the numerical constant $\mathrm{c_N}$ can be easily evaluated, and is equal to 0.0167066... Equation (5.13) shows that the scaling behaviuor of η is precisely $\eta = \mathcal{O}(\beta^{-1})$ when $\beta \to +\infty$.

6. Conclusions

We have shown that in the replicant model replica symmetry is broken. The predictions based on one step replica symmetry breaking are in better agrement with the numerical data than those coming from exact symmetry. Contrary to what happens in most of the cases also the one step replica symmetry breaking is not able to capture the behaviour of the system in the limit of very small a. It would be rather interesting to obtain the results from full replica symmetry breaking in this region. This task should not be impossible using the techniques of [13].

References

[1] P Schuster & K Sigmund; *J. Theor. Biol.* **100** (1983), 533.

[2] J Hofbauer & K Sigmund; in *Evolutionstheorie und Dynamische Systeme,* ed. Parey, (1984).

[3] M Peschel & W Mende; in *The Predator-Prey Model,* ed. Springer-Verlag, (1986).

[4] H Mühlenbein, M Gorges-Schleuter & O Krämer; *Parallel Computing* **7** (1988), 65.

[5] S Diederich & M Opper; *Phys. Rev. A* **39** (1989), 4333.

[6] S F Edwards & P W Anderson; *J. Phys. F* **5** (1975), 965.

[7] D Sherrington & S Kirkpatrick; *Phys. Rev. Lett.* **35** (1975), 1792.

[8] S Kirkpatrick & D Sherrington; *Phys. Rev. B* **17** (1978), 4384.

[9] J R L de Almeida & D J Thouless; *J. Phys. A* **11** (1978), 983.

[10] G Parisi; *Phys. Lett.* **73** (1979), 203.

[11] G Parisi; *J. Phys. A* **13** (1980), L115.

[12] G Parisi; *J. Phys. A* **13** (1980), 1101.

[13] P Biscari; in *The replicant model and the physics of disordered systems,* Tesi di Perfezionamento at the Scuola Normale Superiore, Pisa (1993).

order terms in n, the stationarity equation

$$\frac{\delta}{\delta Q(x)} f[q] = 0, \tag{5.8}$$

that must be verified $\forall x \in [0,1]$, can be given the form [13]:

$$\begin{aligned}
2\dot{Q}(x)\Big\{ & 1 - \beta^2 \big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big)^2 + \\
& + 2\ell\beta^2 \big(\overline{\langle x^3\rangle_z} - 3\overline{\langle x^2\rangle_z\langle x\rangle_z} + 2\overline{\langle x\rangle_z^3}\big)\big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big) + \\
& - 2\rho\beta^3 \big(\overline{\langle x^4\rangle_z} - 2\overline{\langle x^3\rangle_z\langle x\rangle_z} - \overline{\langle x^2\rangle_z^2} + 2\overline{\langle x\rangle_z^2\langle x^2\rangle_z}\big)\big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big) + \\
& + 2\beta^3 Q(x) \bigg(2\big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big)^3 x - \big(\overline{\langle x^3\rangle_z} - \overline{\langle x^2\rangle_z\langle x\rangle_z}\big)\big(\overline{\langle x^3\rangle_z} - 5\overline{\langle x^2\rangle_z\langle x\rangle_z} + \\
& \qquad + 4\overline{\langle x\rangle_z^3}\big) + 4\overline{\langle x\rangle_z^2}\big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big)^2 \bigg) + \\
& + 4\beta^3 \int_0^1 \mathrm{d}x'\, Q(x') \big(\overline{\langle x^3\rangle_z} - 3\overline{\langle x^2\rangle_z\langle x\rangle_z} + 2\overline{\langle x\rangle_z^3}\big)\big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big) + \\
& + 4\beta^3 \int_x^1 \mathrm{d}x'\, Q(x') \big(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\big)^3 \Big\} = 0,
\end{aligned} \tag{5.9}$$

where

$$\dot{Q}(x) := \frac{\mathrm{d}Q}{\mathrm{d}x}. \tag{5.10}$$

This expression can be derived again with respect to x to obtain a necessary condition for the equilibrium:

$$\begin{cases} \dot{Q}(x) = 0 \qquad \text{or} \\ -4\beta^3 \left[2x\left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\right)^3 - \left(\overline{\langle x^3\rangle_z} - 3\overline{\langle x^2\rangle_z\langle x\rangle_z} + 2\overline{\langle x\rangle_z^3}\right)^2 \right] = 0. \end{cases} \tag{5.11}$$

This is exactly what we were looking for: the order parameter function $Q(x)$ must be a constant in all $x \in [0,1]$, except for

$$x = \eta := \frac{\left(\overline{\langle x^3\rangle_z} - 3\overline{\langle x^2\rangle_z\langle x\rangle_z} + 2\overline{\langle x\rangle_z^3}\right)^2}{2\left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z^2}\right)^3}, \tag{5.12}$$

where a jump can happen. Note that in Ising spin glasses with two spins interaction no solution of this type can be found, while a similar phenomenon happens in the Potts model with p components, when $p > 4$.

Figure 4. Numerical solutions obtained for the product of the breaking parameters η and r at zero temperature.

Figure 5. Improvement led by the 1-step replica symmetry broken solution in the prediction for the free energy at zero temperature.

The second order term $f^{(2)}$ was studied in the preceding section; considering the third order term as a functional of the order parameter function $Q(x)$, and neglecting higher

The stationarity equations related to $f_{\rm H}$ become:

$$\begin{aligned}
1 &= \overline{\Big[\langle x\rangle_{(z,z_r)}\Big]_z}\,, \\
t &= \frac{\beta J}{2}\overline{\Big[\langle x\rangle_{(z,z_r)}\Big]_z^2}\,, \\
r &= \frac{\beta J}{2}\overline{\Big(\Big[\langle x\rangle^2_{(z,z_r)}\Big]_z - \Big[\langle x\rangle_{(z,z_r)}\Big]_z^2\Big)}\,, \\
q &= \frac{\beta J}{2}\overline{\Big(\Big[\langle x^2\rangle_{(z,z_r)}\Big]_z - \Big[\langle x\rangle^2_{(z,z_r)}\Big]_z\Big)}\,, \\
\eta^2 r\,(r+2t) &= \overline{\left(\eta\left[\ln\int_0^{+\infty}{\rm d}x\;\mathcal{L}_{\rm H}\right]_z - \ln\int_{-\infty}^{+\infty}{\rm d}z_r\frac{e^{-z_r^2}}{\sqrt{\pi}}{\rm P}(z,z_r)\right)}\,,
\end{aligned} \tag{5.5}$$

where

$$\begin{aligned}
{\rm P}(z,z_r) &:= \left(\int_0^{+\infty}{\rm d}x\;\mathcal{L}_{\rm H}(q,t,r,\lambda,x,z,z_r)\right)^{\eta}, \\
\langle\,\cdot\,\rangle_{(z,z_r)} &:= \frac{\displaystyle\int_0^{+\infty}{\rm d}x\;\cdot\;\mathcal{L}_{\rm H}(q,t,r,\lambda,x,z,z_r)}{\displaystyle\int_0^{+\infty}{\rm d}x\;\mathcal{L}_{\rm H}(q,t,r,\lambda,X,z,z_r)}\,, \\
[\,\cdot\,]_z &:= \frac{\displaystyle\int_{-\infty}^{+\infty}{\rm d}z_r\;\frac{e^{-z_r^2}}{\sqrt{\pi}}\;\cdot\;{\rm P}(z,z_r)}{\displaystyle\int_{-\infty}^{+\infty}{\rm d}z_r\;\frac{e^{-z_r^2}}{\sqrt{\pi}}\;{\rm P}(z,z_r)}.
\end{aligned} \tag{5.6}$$

Solving numerically equations (5.5) in the low temperature limit, we find that the product ηr of the two breaking parameters remains finite in the $\beta\to+\infty$ limit, and that it becomes different from zero as soon as $\tilde a<\tilde a_{\rm c}$, as it is shown in figure 4.

In figure 5 we show one of the results that we found in the numerical simulations that we have performed on this model, and that we will describe in more detail elsewhere: the triangles represent the free energy density obtained from the simulations at zero temperature, the continuous line corresponds to the replica symmetric prediction, and the dashed line illustrates the broken symmetry results. Figure 5 clearly shows how the first step of the replica symmetry breaking improves the symmetric predictions, even if it fails when $\tilde a$ goes to zero.

To conclude the study of the replica symmetry broken solution we will now show that, in the low temperature limit, η goes to zero as $\mathcal{O}\big(\beta^{-1}\big)$, so that the breaking parameter r scales as t and λ do, *i.e.* that $r=\beta\tilde r$ with $\tilde r$ finite as T goes to zero. We will prove this result near the critical value of a where we have a better analytic control. To this end, we push our expansion of f to the third order in $\boldsymbol{\delta\lambda}$, $\boldsymbol{\delta q}$, $\boldsymbol{\delta t}$, obtaining

$$-\beta f = -\beta f_{\rm RS} + \lim_{n\to 0^+}\frac{f^{(2)}+f^{(3)}}{n}. \tag{5.7}$$

Figure 3. Critical eigenvalue at zero temperature. The symmetric solution becomes unstable when it is positive.

This ansatz can be improved by iterating the breaking scheme in all the blocks introduced in the first step, but we will see that even a single breaking improves drastically the symmetric predictions. We recall that, in the limit $n \to 0^+$, the hierarchical parametrization can be written in terms of an order parameter function $Q(x)$, defined in the interval $x \in [0,1]$, which, at this point of symmetry breaking, is equal to

$$Q(x) = \begin{cases} t & \text{if } x \in [0,\eta), \\ t+r & \text{if } x \in (\eta,1). \end{cases} \tag{5.2}$$

In (5.2) we have omitted the diagonal term containing q (corresponding to $Q(1)$), because it involves the term in the Hamiltonian that contains a and it can always be treated separately. The introduction of the breaking parameters η and r changes the free energy density as follows:

$$\begin{aligned} -\beta f_{\mathrm{H}} = \max_{q,t,r,\lambda,\eta} \Big[& -(q+t+r)^2 - (\eta-1)(t+r)^2 + \eta t^2 + \lambda + \\ & + \frac{1}{\eta}\int_{-\infty}^{+\infty} \mathrm{d}z \frac{\mathrm{e}^{-z^2}}{\sqrt{\pi}} \ln \int_{-\infty}^{+\infty} \mathrm{d}z_r \frac{\mathrm{e}^{-z_r^2}}{\sqrt{\pi}} \left(\int_0^{+\infty} \mathrm{d}x \exp \mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,x,z,z_r) \right)^{\eta} \Big], \end{aligned} \tag{5.3}$$

with

$$\mathcal{L}_{\mathrm{H}}(q,t,r,\lambda,x,z,z_r) := -\beta J(\tilde{a}-q)x^2 + \big(2z\sqrt{\beta J t} + 2z_r\sqrt{\beta J r} - \lambda\big)x. \tag{5.4}$$

(ii) 1-asymmetry eigenvectors, with:

$$\begin{aligned}
\delta\lambda_\alpha &= \begin{cases} \ell_1 & \text{if } \alpha = \tilde{\alpha} \\ \ell_0 & \text{otherwise,} \end{cases} \\
\delta q_\alpha &= \begin{cases} \rho_1 & \text{if } \alpha = \tilde{\alpha} \\ \rho_0 & \text{otherwise,} \end{cases} \\
\delta t_{\alpha\gamma} &= \begin{cases} \tau_1 & \text{if } \alpha = \tilde{\alpha} \text{ or } \gamma = \tilde{\alpha} \\ \tau_0 & \text{otherwise;} \end{cases}
\end{aligned} \tag{4.4}$$

(iii) 2-asymmetries eigenvectors, of the type:

$$\begin{aligned}
\delta\lambda_\alpha &= \begin{cases} \ell_1 & \text{if } \alpha = \tilde{\alpha} \text{ or } \alpha = \tilde{\gamma} \\ \ell_0 & \text{otherwise,} \end{cases} \\
\delta q_\alpha &= \begin{cases} \rho_1 & \text{if } \alpha = \tilde{\alpha} \text{ or } \alpha = \tilde{\gamma} \\ \rho_0 & \text{otherwise,} \end{cases} \\
\delta t_{\alpha\gamma} &= \begin{cases} \tau_2 & \text{if } \alpha\gamma = \tilde{\alpha}\tilde{\gamma} \text{ or } \alpha\gamma = \tilde{\gamma}\tilde{\alpha} \\ \tau_0 & \text{if } \alpha \neq \tilde{\alpha},\ \alpha \neq \tilde{\gamma},\ \gamma \neq \tilde{\alpha} \text{ and } \gamma \neq \tilde{\gamma} \\ \tau_1 & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.5}$$

The eigenvalues of $\mathcal{M}$ must be negative in order to ensure the stability of the symmetric ansatz. The biggest among the eigenvalues associated with the families above comes from the 2-asimmetries eigenvectors, and is given by [13]

$$\mu_{\rm cr} = \frac{1}{2}\big(K + K' - 2L + M\big) = -1 + \beta^2\overline{\big(\langle x^2\rangle_z - \langle x\rangle_z^2\big)^2}. \tag{4.6}$$

In the low temperature limit $\mu_{\rm cr}$ can be easily computed, and is equal to

$$\mu_{\rm cr} = \frac{2q - \tilde{a}}{\tilde{a} - q}. \tag{4.7}$$

In particular, as figure 3 shows, $\mu_{\rm cr}$ becomes positive when $\tilde{a} < \tilde{a}_{\rm c}$:

$$\mu_{\rm cr} = \pi(\tilde{a} - \tilde{a}_{\rm c}) - \sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\rm c})^2 + o\big((\tilde{a} - \tilde{a}_{\rm c})^2\big). \tag{4.8}$$

5. Replica symmetry breaking

Having proved the instability of the symmetric solution, we must now search a more general ansatz to describe the system when $\tilde{a} < \tilde{a}_{\rm c}$. To obtain it, we will follow the guidelines of the hierarchical ansatz of spin glasses [10, 11, 12]. In this paper we study only the first step of the replica symmetry breaking, testing order parameter matrices of the type

$$Q_{\alpha\gamma} = \begin{cases} t & \text{if } \mathrm{Int}\left(\dfrac{\alpha}{\eta}\right) \neq \mathrm{Int}\left(\dfrac{\gamma}{\eta}\right), \\[2ex] t + r & \text{if } \mathrm{Int}\left(\dfrac{\alpha}{\eta}\right) = \mathrm{Int}\left(\dfrac{\gamma}{\eta}\right) \text{ but } \alpha \neq \gamma, \\[2ex] q + t + r & \text{if } \alpha = \gamma. \end{cases} \tag{5.1}$$

4. Instability of the symmetric solution

In the preceding section we have shown that equations (2.4) admit a symmetric solution, but we must also check the Hessian of the free energy to determine whether our solution is a minimum of f or just a saddle-point. To find the eigenvalues of the Hessian we generalize the calculus made by De Almeida and Thouless [9] for the SK model of spin glass: let

$$\begin{aligned} Q_{\alpha\gamma} &= (q+\delta q_\alpha)\,\delta_{\alpha\gamma} + t + \delta t_{\alpha\gamma}, \qquad \text{with } \begin{cases} \delta t_{\alpha\alpha} = 0, \\ \delta t_{\alpha\gamma} = \delta t_{\gamma\alpha}; \end{cases} \\ \lambda_\alpha &= \lambda + \delta\lambda_\alpha. \end{aligned} \tag{4.1}$$

If we denote by $\boldsymbol{\delta\xi}$ the vector $(\boldsymbol{\delta\lambda};\ \boldsymbol{\delta q};\ \boldsymbol{\delta t})$ and we substitute (4.1) in (2.2) we obtain, after some tedious calculations [13], that the second order term in the expansion of f in terms of $\boldsymbol{\delta\xi}$ is given by $-\beta\delta_2 f = \frac{1}{2}\,\boldsymbol{\delta\xi}^T \cdot \mathcal{M}\ \boldsymbol{\delta\xi}$, where $\mathcal{M}$ is a real symmetric matrix with the following fourteen different types of elements:

$$\begin{aligned}
A := \mathcal{M}_{\delta\lambda_\alpha\delta\lambda_\alpha} &= \left(\overline{\langle x^2\rangle_z} - \overline{\langle x\rangle_z}^2\right), \\
B := \mathcal{M}_{\delta\lambda_\alpha\delta\lambda_\gamma} &= \left(\overline{\langle x\rangle_z^2} - \overline{\langle x\rangle_z}^2\right), \\
C := \mathcal{M}_{\delta\lambda_\alpha\delta q_\alpha} &= -\beta J\left(\overline{\langle x^3\rangle_z} - \overline{\langle x^2\rangle_z}\ \overline{\langle x\rangle_z}\right), \\
D := \mathcal{M}_{\delta\lambda_\alpha\delta q_\gamma} &= -\beta J\left(\overline{\langle x^2\rangle_z\langle x\rangle_z} - \overline{\langle x^2\rangle_z}\ \overline{\langle x\rangle_z}\right), \\
E := \mathcal{M}_{\delta q_\alpha\delta q_\alpha} &= -2 + \beta^2 J^2\left(\overline{\langle x^4\rangle_z} - \overline{\langle x^2\rangle_z}^2\right), \\
F := \mathcal{M}_{\delta q_\alpha\delta q_\gamma} &= -\beta^2 J^2\left(\overline{\langle x^2\rangle_z^2} - \overline{\langle x^2\rangle_z}^2\right), \\
G := \mathcal{M}_{\delta\lambda_\alpha\delta t_{\alpha\gamma}} &= -\beta J\left(\overline{\langle x^2\rangle_z\langle x\rangle_z} - \overline{\langle x\rangle_z^2}\ \overline{\langle x\rangle_z}\right), \\
H := \mathcal{M}_{\delta\lambda_\alpha\delta t_{\gamma\delta}} &= -\beta J\left(\overline{\langle x\rangle_z^3} - \overline{\langle x\rangle_z^2}\ \overline{\langle x\rangle_z}\right), \\
I := \mathcal{M}_{\delta q_\alpha\delta t_{\alpha\gamma}} &= \beta^2 J^2\left(\overline{\langle x^3\rangle_z\langle x\rangle_z} - \overline{\langle x^2\rangle_z}\ \overline{\langle x\rangle_z^2}\right), \\
J := \mathcal{M}_{\delta q_\alpha\delta t_{\gamma\delta}} &= \beta^2 J^2\left(\overline{\langle x^2\rangle_z\langle x\rangle_z^2} - \overline{\langle x^2\rangle_z}\ \overline{\langle x\rangle_z^2}\right), \\
K := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\alpha\gamma}} &= -2 + \beta^2 J^2\left(\overline{\langle x^2\rangle_z^2} - \overline{\langle x\rangle_z^2}^2\right), \\
K' := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\gamma\alpha}} &= \beta^2 J^2\left(\overline{\langle x^2\rangle_z^2} - \overline{\langle x\rangle_z^2}^2\right) = K + 2\,, \\
L := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\gamma\delta}} &= \beta^2 J^2\left(\overline{\langle x^2\rangle_z\langle x\rangle_z^2} - \overline{\langle x\rangle_z^2}^2\right), \\
M := \mathcal{M}_{\delta t_{\alpha\gamma}\delta t_{\delta\eta}} &= \beta^2 J^2\left(\overline{\langle x\rangle_z^4} - \overline{\langle x\rangle_z^2}^2\right).
\end{aligned} \tag{4.2}$$

Furthermore, $\mathcal{M}$ has three different types of eigenvectors:

(i) symmetric eigenvectors of the type

$$\boldsymbol{\delta\xi} = (\ell,\ldots,\ell;\rho,\ldots,\rho;\tau,\ldots,\tau); \tag{4.3}$$

In the former region the equilibrium configurations become trivial, with $x_i^{\mathrm{eq}} \simeq 1$, $\forall i$. The replica symmetric solution, which we will prove to be stable in this region, predicts:

$$\begin{aligned}
q &= \frac{1}{2}\left(\tilde{a} - \sqrt{\tilde{a}^2 - 1}\right) + \mathcal{O}\big(\mathrm{e}^{-\tilde{a}^2}\big), \\
\tilde{t} &= \frac{1}{4}\left(1 + \frac{\tilde{a}}{\sqrt{\tilde{a}^2 - 1}}\right) + \mathcal{O}\big(\mathrm{e}^{-\tilde{a}^2}\big), \\
\tilde{\lambda} &= -\tilde{a} - \sqrt{\tilde{a}^2 - 1} + \mathcal{O}\big(\mathrm{e}^{-\tilde{a}^2}\big),
\end{aligned} \tag{3.9}$$

and the free energy density becomes

$$f^o_{\mathrm{RS}} = \frac{1}{2}\left(\tilde{a} + \sqrt{\tilde{a}^2 - 1}\right) + \mathcal{O}\big(\mathrm{e}^{-\tilde{a}^2}\big). \tag{3.10}$$

Instead, the latter is the transition point to the glassy phase, as we will show below; in its neighbourhood we have:

$$\begin{aligned}
q &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2}(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + o\big((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\big), \\
\tilde{t} &= \frac{\pi}{2} - \sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}}) + \pi^2(3\pi - 8)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + o\big((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\big), \\
\tilde{\lambda} &= -2\pi(\tilde{a} - \tilde{a}_{\mathrm{c}}) + 2\sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + o\big((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\big), \\
f^o_{\mathrm{RS}} &= \pi(\tilde{a} - \tilde{a}_{\mathrm{c}}) - \sqrt{2}\pi(\pi - 2)(\tilde{a} - \tilde{a}_{\mathrm{c}})^2 + o\big((\tilde{a} - \tilde{a}_{\mathrm{c}})^2\big).
\end{aligned} \tag{3.11}$$

Finally, figure 2 shows the numerical results that we obtained for the order parameters q and t by solving equations (3.5) for different finite values of β.

Figure 2. Numerical solutions of the replica symmetric equations at finite temperature

the stationarity equations (3.5) become:

$$\begin{cases} 4\,q\,(\tilde{a}-q) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{\tilde{z}}^{+\infty} \mathrm{d}z\ \mathrm{e}^{-z^2}\ ; \\ 4\,\tilde{t}\,(2q-\tilde{a}) = \tilde{\lambda}\ ; \\ \dfrac{\sqrt{\tilde{t}}\ \mathrm{e}^{-\tilde{z}^2}}{\sqrt{\pi}} - 2(\tilde{a}-q) = \dfrac{\tilde{\lambda}}{\sqrt{\pi}} \displaystyle\int_{\tilde{z}}^{+\infty} \mathrm{d}z\ \mathrm{e}^{-z^2}, \end{cases} \tag{3.8}$$

leading to $f_{\mathrm{RS}} = 2J\tilde{t}(\tilde{a}-2q)$. Figure 1 shows how q, $\tilde{t}$, $\tilde{\lambda}$, and f_{RS} behave as functions of $\tilde{a}$ in this limit. We give also the approximate expressions of these parameters in two particularly interesting cases: the "classical" regime ($\tilde{a} \gg 1$), and the neighbourhood of the critical point $\tilde{a}_{\mathrm{c}} = 1/\sqrt{2}$.

Figure 1. Numerical solutions of the replica symmetric equations in the low temperature limit

3. Replica symmetric solution

Both the free energy density and the stationarity equations above are invariant under the action of the group $\mathcal{S}_n$ of permutations between the n replicas. This implies that at least one of the solutions of (2.4) is invariant under $\mathcal{S}_n$, so that the first ansatz that is to be tried is certainly the symmetric one, which is given by:

$$Q_{\alpha\gamma} = q\,\delta_{\alpha\gamma} + t \qquad \text{and} \qquad \lambda_\alpha = \lambda. \tag{3.1}$$

Introducing (3.1) into (2.2), and denoting by $f_{\rm RS}$ the resulting the free energy density we have, after the manipulations described in [13],

$$-\beta f_{\rm RS} = \max_{q,\tilde{t},\tilde{\lambda}} \left[q^2 + 2\beta J\, q\tilde{t} - \beta J\,\tilde{\lambda} - \overline{\ln\left(\int_0^{+\infty} {\rm d}x \exp \mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z)\right)}\,\right], \tag{3.2}$$

where

$$\mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z) := -\beta J \left[(\tilde{a}-q)x^2 - \left(2z\sqrt{\tilde{t}} - \tilde{\lambda}\right)x \right], \tag{3.3}$$

$\tilde{t} := t/(\beta J)$, $\tilde{\lambda} := \lambda/(\beta J)$, $\tilde{a} := a/J$, and we have introduced the notation

$$\overline{G(z)} := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} {\rm d}z\, {\rm e}^{-z^2}\, G(z). \tag{3.4}$$

In a similar way, the stationarity equations become:

$$\begin{cases} \overline{\langle x\rangle_z} = 1, \\[2ex] \overline{\langle x\rangle_z^2} = 2\tilde{t}, \\[2ex] \overline{\langle x^2\rangle_z - \langle x\rangle_z^2} = \dfrac{2q}{\beta J}, \end{cases} \tag{3.5}$$

where

$$\langle\, G(x)\,\rangle_z := \frac{\displaystyle\int_0^{+\infty} {\rm d}x\; G(x)\; \exp \mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z)}{\displaystyle\int_0^{+\infty} {\rm d}x\; \exp \mathcal{L}_{\rm RS}(q,\tilde{t},\tilde{\lambda},x,z)}. \tag{3.6}$$

The low temperature limit of the symmetric solution, first studied by Diederich and Opper [5], is particularly interesting because it allows us to prove analytically the existence of a second order transition to a glassy phase, as we will show in the next section. Introducing the parameter

$$\tilde{z} := \frac{\tilde{\lambda}}{2\sqrt{\tilde{t}}}\,, \tag{3.7}$$

The study of the stability of the replica symmetric solution will show that, at zero temperature, the replicant model exhibits a phase transition to a glassy phase when a crosses a certain value a_c. The replica symmetry breaking which occurs in the glassy phase ($a < a_c$) implies the breakdown of the ergodicity of the system: when a becomes small, the evolution of the system depends strongly on the initial conditions, and in general we will not be able to make any precise prediction on the equilibrium state of the system.

From the biological point of view, the glassy phase is the unstable one: in the high a phase, a single equilibrium state exists, and the system is able to recover its equilibrium configuration after any external change of the concentrations of its elements; on the contrary, in the glassy phase, the same perturbation can change drastically the final configuration of the system, if it is led to a different ergodic region of the phase space. Here however we study only the properties of the statics associated to Hamiltonian (1.2) and we do not consider the dynamics of a system leading to this equilibrium distribution.

2. The Random Replicant Model: analytical solution

Now we derive the expression for the quenched free energy density of the random replicant model. In this and the next section, we follow closely [5]. The evolution of the system is ruled by the Hamiltonian (1.2); averaging over all the possible choices of the $\{J_{ij}\}$, the quenched free energy of the system is given by

$$-\beta N f = \int \prod_{i<j} \mathrm{d}J_{ij}\ P(J_{ij})\ \ln \sum_{[\boldsymbol{x}]} \exp\left(-\beta \mathcal{H}_J[\boldsymbol{x}]\right). \tag{2.1}$$

To compute (2.1), we use the replica method [7 – 12] introducing a set $\{\lambda_\alpha,\ \alpha = 1,\ldots,n\}$ of Lagrange multipliers which ensure the normalization condition (1.1) in each of the n replicas. With standard calculations [13], we arrive at the following expression for f:

$$-\beta f = \lim_{n\to 0^+} \max_{\boldsymbol{Q},\boldsymbol{\lambda}} \left[-\frac{1}{n}\sum_{\alpha\gamma} Q_{\alpha\gamma}^2 + \frac{1}{n}\sum_{\alpha}\lambda_\alpha + \frac{1}{n}\ln \mathrm{Tr}_n \exp \mathrm{L}(\boldsymbol{Q},\boldsymbol{\lambda},\boldsymbol{x}) \right], \tag{2.2}$$

where

$$\mathrm{L}(\boldsymbol{Q},\boldsymbol{\lambda},\boldsymbol{x}) := -\beta a \sum_\alpha x_\alpha^2 - \sum_\alpha \lambda_\alpha x_\alpha + \beta J \sum_{\alpha\gamma} Q_{\alpha\gamma} x_\alpha x_\gamma\ ; \tag{2.3}$$

$\boldsymbol{Q}$ and $\boldsymbol{\lambda}$ are respectively an $n\times n$ matrix and an n-dimensional vector; $\{x_\alpha,\ \alpha = 1,\ldots,n\}$ is a new set of real positive variables, and Tr_n denotes the integral over all possible values of the x_α's.

From (2.2) and (2.3) we find that the stationarity equations for f are:

$$\begin{cases} Q_{\alpha\gamma} = \dfrac{\beta J}{2}\,\dfrac{\mathrm{Tr}_n\, x_\alpha x_\gamma \exp \mathrm{L}(\boldsymbol{Q},\boldsymbol{\lambda},\boldsymbol{x})}{\mathrm{Tr}_n \exp \mathrm{L}(\boldsymbol{Q},\boldsymbol{\lambda},\boldsymbol{x})} & \forall \alpha,\gamma = 1,\ldots,n; \\[2ex] 1 = \dfrac{\mathrm{Tr}_n\, x_\alpha \exp \mathrm{L}(\boldsymbol{Q},\boldsymbol{\lambda},\boldsymbol{x})}{\mathrm{Tr}_n \exp \mathrm{L}(\boldsymbol{Q},\boldsymbol{\lambda},\boldsymbol{x})} & \forall \alpha = 1,\ldots,n. \end{cases} \tag{2.4}$$

The remaining sections are devoted to the study of the solutions of these stationarity conditions.

1. Introduction

The replicant models study the coevolution of sets of interacting species able to reproduce themselves: they have a huge number of applications in biologic and optimization problems [1 – 4]. In this paper we study a non-deterministic evolution: we consider a system of replicants which evolve with random interactions.

The model, introduced by S. Diederich and M. Opper in [5], is defined as follows. Given N species, let x_i/N be the concentration of the i-th family in the system. The real variables $\{x_i \in \mathbb{R},\ i = 1, \ldots, N\}$ are then subject to the constraints

$$\sum_{i=1}^{N} x_i = N, \qquad\qquad x_i \geq 0 \quad \forall i = 1, \ldots, N. \tag{1.1}$$

The interactions between different species are described through a fitness functional $F_J(\boldsymbol{x})$ that must be maximized at equilibrium. Typically, F_J is chosen as a quadratic function of the concentrations, that is equivalent to take into account only pair interactions between the species:

$$F_J(\boldsymbol{x}) = -\mathcal{H}_J[\boldsymbol{x}] = -\sum_{i<j=1}^{N} J_{ij}\; x_i\; x_j - a \sum_{i=1}^{N} x_i^2, \tag{1.2}$$

where the parameters $\{J_{ij}\}$ are chosen at random from the Gaussian probability distribution

$$P(J_{ij}) = \sqrt{\frac{N}{\pi J^2}}\, \exp\left(-\frac{N J_{ij}^2}{J^2} \right) \tag{1.3}$$

like in the Sherrington-Kirkpatrick model of spin glasses [6, 7]. The control parameter a has the aim of limiting the growth and the supremacy of one single species: for big values of a, the growth of all the species is strongly limited by the factor $a x_i^2$; in that case, the random interactions become negligible and the equilibrium configuration is

$$x_i^{\mathrm{eq}} \simeq 1 \qquad \forall i = 1, \ldots, N, \qquad (a \gg J) \tag{1.4}$$

almost independently from the interactions between the species. Instead, for small values of a, the pair interactions play a central rôle and a few species prevail among the others. Analytically, this model differs from the SK spin glass in that we impose the constraint (1.1): the spins are then allowed to take any real value, but the total magnetization is fixed.

In section 2 we show how it is possible to solve the random replicant model within the replica formalism. In sections 3 and 4, we analyze the replica symmetric solution and its stability and in section 5 we perform the first step of the hierarchical replica symmetry breaking. The biological applications of the results are found in the limit $T \to 0^+$ because the fitness functional F_J introduced in the last section is, a minus sign apart, the low temperature limit of the free energy.

Replica Symmetry Breaking
in the Random Replicant Model

P Biscari[(1)] & G Parisi[(2)]

[(1)] Dip. Matematica, Università di Pisa, Via Buonarroti 2, 56127 Pisa, Italy
biscari@vaxsns.sns.it
[(2)] Dip. Fisica, Università di Roma "La Sapienza", P.le A. Moro, 00187 Roma, Italy
parisi@vxrm70.roma1.infn.it

Abstract. We study the statistical mechanics of a model describing the coevolution of species interacting in a random way. We find that at high competition replica symmetry is broken. We solve the model in the approximation of one step replica symmetry breaking and we compare our findings with accurate numerical simulations.

Short title: *RSB in the Random Replicant Model*

PACS numbers: 75.50.Lk, 75.40.Gb, 64.60.Ht

Submitted to: *Journal of Physics A: Mathematical and General*

Date: 1 February 1995