SCALING AND INTERMITTENCY IN BURGERS' TURBULENCE

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Abstract: We use the mapping between Burgers' equation and the problem of a directed polymer in a random medium in order to study the fully developped turbulence in the N dimensional forced Burgers' equation. The stirring force corresponds to a quenched (spatio temporal) random potential for the polymer. The properties of the inertial regime are deduced from a study of the directed polymer on length scales smaller than the correlation length of the potential, which is not the regime usually considered in the case of polymers. From this study we propose an Ansatz for the velocity field in the large Reynolds number limit of the forced Burgers' equation in N dimensions, which should become exact in the limit $N \to \infty$. This Ansatz allows us to compute exactly the full probability distribution of the velocity difference $u(r)$ between points separated by a distance r much smaller than the correlation length of the forcing. We find that the moments $< u^q(r) >$ scale as $r^{\zeta(q)}$ with $\zeta(q) \equiv 1$ for all $q \geq 1$ (in particular the $q = 3$ moment agrees with Kolmogorov's scaling $\zeta(3) = 1$). This strong 'intermittency' is related to the large scale singularities of the velocity field, which is concentrated on

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a $N-1$ dimensional froth-like structure, which is turn related to the one-step replica symmetry broken nature of the associated disordered problem. We also discuss the similarities and differences between Burgers turbulence and hydrodynamical turbulence, and we comment on the anomalous tracer fluctuations in a Burgers turbulent field. This replica approach being rather unusual in turbulence problems, we provide all the necessary details of the method.

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1 Introduction

The theory of statistical turbulence is more than fifty years old, but its status is still not fully satisfactory. The simplest and surprisingly robust approach is Kolmogorov's dimensional analysis, which leads to the celebrated $k^{-5/3}$ law for the velocity cascade [1, 2]. However, analytical calculations immediately lead to difficulties: the simplest closure scheme to deal with the non linear term in the Navier-Stokes equation ('DIA') does not reproduce Kolmogorov's scaling [3, 2, 4]. A lot of effort has been devoted, in particular by Kraichnan, to understand why this was so. More refined schemes were proposed to recover the $k^{-5/3}$ law and even to calculate adimensional constants [5, 6, 2], but they are based on uncontrolled assumptions (like the RG approach), or lead to enormous calculations, which are difficult to manipulate and are not fully transparent from a physical point of view [5].

More recently, a tremendous activity has developed on the intermittency (or multifractal) corrections to Kolmogorov's scaling: higher moments of the velocity field do not seem to scale with the one originally predicted by Kolmogorov. Many interesting suggestions have been put forward to describe and explain this feature, starting by Kolomgorov himself [7, 8, 9, 10, 11, 12, 13, 14].

In recent years, non linear partial differential equations with noise have been the focus of quite a number of studies in the context of growing interfaces, with the 'KPZ' equation standing out as a new paradigm [15, 16, 18, 17]. This equation is in fact a variant of Burgers' equation, and has (together with its many siblings) a wide range of applications in different physical contexts [17]. Interestingly, this field is also related (through an appropriate 'Cole-Hopf' mapping) to the physics of disordered systems, in particular elastic strings in random media ('directed polymers'). The Burgers problem in N dimensions is then equivalent to the problem of a directed polymer in $N+1$ dimensions, which is the space time of the original problem. The forcing term in the Burgers equation translates into a random potential for the directed polymer, which is quenched in space time: to each realization of the stirring force in the Burgers language corresponds one sample of a directed polymer. This enables one to adapt techniques originally devised for with spin-glasses and obtain original results on the underlying non-linear equation (or vice-versa) [19, 20, 21, 22, 23, 24]. It is the aim of the present paper to exploit in detail this mapping, which allows us to propose an original *Ansatz* for the velocity field for the randomly stirred Burgers equation, which should become *exact* solution in high dimensions and in the limit of large Reynolds number. Scaling in the inertial range can then be precisely discussed: we obtain in closed form the full probability distribution for velocity differences. In particular, the third moment of the velocity difference grows linearly with distance, i.e *à la* Kolmogorov. In fact, Kolmogorov's dimensional analysis should be directly applicable to Burgers' turbulence. We find however very strong intermittency effects, which we relate in a quantitative manner to the existence of *large scale structures*, in the form of singularities concentrated on time dependent hypersurfaces. Although not exact in finite dimension, we expect that our (variational) description remains qualitatively correct even in one dimension. We also obtain the dynamical evolution of the velocity field: we find that the field is convected away by the largest structures, corresponding to a dynamical exponent equal to $z = 1$ rather than the one obtained from Kolmogorov's scaling, $z_K = 2/3$.

In the bulk of the paper, we primarily focus on the 'dictionary' between Burgers' turbulence and disordered systems and discuss a number of physical points, relegating more technical points to various appendices. We show that Kolmogorov's scaling has a counterpart in the 'directed polymer' language, where it is known as the Larkin-Ovchinnikov scaling. It is however easy to see that this 'naive' scaling cannot hold, as is confirmed by the full calculation. We describe in physical terms the nature of the velocity field, and argue that there should be a large distance regime (beyond the 'injection' scale) characterized by a non-trivial exponent. We briefly discuss the problem of a passive scalar in such a velocity field. In the conclusion, we compare our results to other approximation schemes, and comment on the possible differences with 'true' Navier-Stokes turbulence.

2 From Burgers turbulence to directed polymers: dictionary and dimensional analysis

The problem we shall consider is that of a randomly forced *potential* flow in N dimensions, which follows the Burgers equation [2]:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\, \vec{v} = \nu \nabla^2 \vec{v} + \vec{f}(\vec{x}, t) \tag{2.1}$$

where $\vec{v}$ is minus the gradient of a velocity potential $h(\vec{x}, t)$, and $\vec{f}$ is a randomly fluctuating force, which is also minus the gradient of a potential $\phi(\vec{x}, t)$. $\vec{v}$ and $\vec{x}$ are N-dimensional vectors, and there is no constraint on $\vec{\nabla}.\vec{v}$. Here, we wish to describe the problem of a fluid which is randomly stirred *only at very large lengthscales*. We shall thus take ϕ to be Gaussian, with fluctuations given by:

$$\overline{\phi(\vec{x}, t)\phi(\vec{x}', t')} = \epsilon \Delta^2 N \delta(t - t') \exp - \left[\frac{(\vec{x} - \vec{x}')^2}{2N\Delta^2} \right]. \tag{2.2}$$

Here Δ is the 'injection' length, over which the forcing is roughly constant. The correlation *time* of the forcing was set to zero; a finite (small) correlation time would however not affect the following conclusions. The stirring force

[2] The density of the fluid is taken equal to one.

correlation thus reads:

$$\overline{f^{\mu}(\vec{x},\tau)f^{\nu}(\vec{x}',\tau')} = \epsilon\delta(\tau-\tau')\left[\delta^{\mu\nu} - \frac{(\vec{x}-\vec{x}')^{\mu}(\vec{x}-\vec{x}')^{\nu}}{N\Delta^2}\right]\ \exp -\left[\frac{(\vec{x}-\vec{x}')^2}{2N\Delta^2}\right]. \tag{2.3}$$

As will be clear below, this ensures that the injected energy per unit time, defined as the increase of kinetic energy $\frac{1}{2}\vec{v}^2$, is equal to $\frac{N}{2}\epsilon$ (Note that the dimension of ϵ is $[x^2]/[t^3]$). The dependance of the force correlations on the dimension N has been chosen such as to insure the existence of the large N limit.

The typical velocity at the injection scale is, from dimensional considerations, $v_\Delta \equiv (\epsilon\Delta)^{\frac{1}{3}}$, which allows us to define the Reynolds number as:

$$Re \equiv \frac{v_\Delta \Delta}{\nu} = (\frac{\epsilon\Delta^4}{\nu^3})^{\frac{1}{3}}\ . \tag{2.4}$$

We shall be interested in studying the statistics of the velocity field at large Re.

Let us now use the standard techniques to transform this problem into a directed polymer. Integrating once Eq. (1), with $\vec{v} = -\vec{\nabla}h$ one finds the so-called Kardar-Parisi-Zhang (KPZ) equation:

$$\frac{\partial h(\vec{x},t)}{\partial t} = \frac{1}{2}(\nabla h(\vec{x},t))^2 + \nu\nabla^2 h(\vec{x},t) + \phi(\vec{x},t) \tag{2.5}$$

which describes, in particular, the growth of a surface under a random 'rain' of particles, the flux of which is given by $\phi(\vec{x},t)$. At this stage, the crucial difference with previous work on the KPZ equation is in the correlations of the noise $\phi(\vec{x},t)$. We shall thus be mostly concerned, in the following, with the velocity field statistics at length scales *smaller* than Δ; for length scales larger than Δ, the KPZ scaling prevails. Now, as is well known, the KPZ equation (2.5) can be transformed (through a 'Hopf-Cole' transformation) into a linear problem describing a directed line (polymer) in a random potential. Setting $h(\vec{x},t) = 2\nu \log Z(\vec{x},t)$, one finds that $Z(\vec{x},t)$ obeys the equation

$$\frac{\partial Z(\vec{x},t)}{\partial t} = \nu\nabla^2 Z(\vec{x},t) + \frac{1}{2\nu}\phi(\vec{x},t)Z(\vec{x},t) \tag{2.6}$$

which is the equation for the partition function of an elastic string in a random potential $V(\vec{x},t) = \frac{1}{2\nu}\phi(\vec{x},t)$, subject to the constraint that its end point is fixed at $(\vec{x},t)$. Said differently, the solution of Eq. (2.3) can be written as a path integral (sum over all configurations):

$$Z(\vec{x},t) = \int \rho(\vec{x}_0)d\vec{x}_0 \int_{\vec{x}(t=0)=\vec{x}_0}^{\vec{x}(t)=\vec{x}} d[\vec{x}(\tau)]\exp(-\mathcal{H})\ . \tag{2.7}$$

The hamiltonian $\mathcal{H}$ is given by:

$$\mathcal{H} = \int_0^t d\tau \left[\frac{c}{2}\left(\frac{d\vec{x}}{d\tau}\right)^2 + V(\vec{x}(\tau),\tau)\right]. \tag{2.8}$$

The 'dictionary' between the two problems is the following: The temperature scale of the polymer has been chosen equal to one. Then the elastic modulus of the polymer is

$$c \equiv \frac{1}{2\nu} \ . \tag{2.9}$$

The random potential seen by the polymer V has a gaussian distribution, with a second moment given by:

$$\overline{V(\vec{x},\tau)V(\vec{x}',\tau')} = WN\delta(t-t')\exp - \left[\frac{(\vec{x}-\vec{x}')^2}{2N\Delta^2}\right] . \tag{2.10}$$

where the strength of potential fluctuations, W, is related to the energy density ϵ through:

$$W \equiv \frac{\epsilon\Delta^2}{4\nu^2} \ , \tag{2.11}$$

and the length scale of fluctuations of the potential is equal to Δ, the length-scale at which the Burgers fluid is stirred. The probability distribution $\rho(\vec{x}_0)$ of the initial point of the polymer is related to the initial conditions of the velocity through:

$$\vec{v}(\vec{x}, t=0) = -2\nu\vec{\nabla}\rho(\vec{x}) \ . \tag{2.12}$$

The non-linearity has disappeared from this formulation, and has been replaced by the famous problem of disordered systems, which is to average the logarithm of a partition function (in order to calculate various moments of the velocity field $\vec{v}$). In the next section we shall deal with this problem using the replica trick and a variational method which becomes exact in the limit of very large dimensions N.

Before turning to this calculation, it is useful to give some kind of qualitative (more or less dimensional) analysis of the directed polymer, and to enrich our 'dictionary' by stating its counterpart in the turbulence language. The study of an elastic structure like the directed polymer in presence of a random potential has been discussed in several works recently [25, 21]. In the phase where the disorder is strong, one expects a scaling behaviour of the lateral fluctuations of the polymer described by a wandering exponent ζ:

$$\overline{<(\vec{x}(t)-\vec{x}(t'))^2>} \simeq A|t-t'|^{2\zeta}. \tag{2.13}$$

The thermal fluctuations are irrelevant at large distance and this scaling also holds in the zero temperature limit, or for the disconnected correlation $\overline{(<\vec{x}(t)-\vec{x}(t')>)^2}$. As for the free energy differences for two polymers finishing at point $t, \vec{x}$ and $t', \vec{x}'$, they scale as:

$$\overline{(h(\vec{x},t)-h(\vec{x}',t'))^2} \simeq |\vec{x}-\vec{x}'|^{2\omega/\zeta} g_h\left(\frac{|\vec{x}-\vec{x}'|}{|t-t'|^\zeta}\right) . \tag{2.14}$$

The $t \to t'$ limit then implies that the free energy difference at points $\vec{x}$ and $\vec{x}'$ scales as $|\vec{x}-\vec{x}'|^{2\omega/\zeta}$, and therefore the difference of velocities in Burgers equation should scale as:

$$\overline{\left(\vec{v}(\vec{x},t)-\vec{v}(\vec{x}',t')\right)^2} \simeq |\vec{x}(t)-\vec{x}(t')|^{(2\frac{\omega}{\zeta}-2)} g_v\left(\frac{|\vec{x}(t)-\vec{x}(t')|}{|t-t'|^{\zeta}}\right). \qquad (2.15)$$

If this last scaling form holds, the galilean invariance of Burgers equation implies that the two terms in $\partial\vec{v}/\partial t$ and $(\vec{v}\cdot\vec{\nabla})\,\vec{v}$ should scale in the same way under a rescaling $\vec{x}\to b\vec{x}$ and $t\to b^{1/\zeta}t$, which implies $\omega = 2\zeta-1$ [19, 18]. (Another way to argue about this identity directly on the polymer problem is by observing that the fluctuations of elastic energy in the directed polymer scale as $x^2/t = t^{(2\zeta-1)} = t^{\omega}$). So we are left with only one scaling exponent.

An important point which was discussed in [26, 27] is the existence of two distinct scaling regimes, in the case where the random potential has a large correlation length, Δ. The regime which is most studied for directed polymer is the large time regime where the transverse fluctuations of the polymer are much larger than Δ. But there also exists a short distance regime where the transverse fluctuations of the polymer are smaller than the correlation length of the potential. This is obviously the regime which interests us most for the turbulence problem (although we shall return to the long distance regime later on). This regime holds for time differences shorter than a typical time scale τ^*, defined from:

$$\overline{<\left(\vec{x}(t+\tau^*)-\vec{x}(t)\right)^2>} \simeq \Delta^2\ . \qquad (2.16)$$

One expects to be allowed to linearize the random potential in this slowly varying regime, and to study the much simpler (linear) problem of a polymer with a random force, defined by the Hamiltonian:

$$\mathcal{H} = \int_0^t d\tau\left[\frac{c}{2}\left(\frac{d\vec{x}}{d\tau}\right)^2 - \vec{f}_L(\tau)\cdot\vec{x}(\tau)\right], \qquad (2.17)$$

(where the random force is of order $|f_L| \simeq \sqrt{W}/\Delta$). This random force problem was studied by Larkin and Ovchinikov long time ago [28]. In this regime there is no metastable state and the problem is easily solved for one given sample. Assuming for simplicity periodic boundary conditions, one obtains the Fourier transform of the average polymer's position::

$$<\vec{x}(\omega)> = \frac{\vec{f}(\omega)}{c\ \omega^2}. \qquad (2.18)$$

A dimensional analysis of the Fourier transform would lead to:

$$|<\vec{x}(t)-\vec{x}(t')>|^2 \simeq \frac{W}{c^2\Delta^2}|t-t'|^3, \qquad (2.19)$$

from which one deduces the scaling exponent in this regime, and the value of τ^*:

$$\zeta_K = \frac{3}{2}\ , \quad \omega_K = 2\zeta_K - 1 = 2\ , \quad \tau^* \equiv \left(\frac{c^2\Delta^4}{W}\right)^{1/3} \equiv \frac{\Delta}{v_\Delta} \qquad (2.20)$$

This result looks very nice when translated in terms of turbulence, since it predicts (using (2.15)) that the difference of velocity between two points at distance r will scale like $r^{\omega_K/\zeta_K - 1} = r^{1/3}$, and also that time scales and length scales are related through $t^{\zeta_K} \sim r$, with $\zeta_K = 3/2$ (Richardson diffusion). Note that τ^* is simply the convective time across the injection length Δ.

These results thus precisely reproduce the Kolmogorov scaling, which is here derived from a very simple argument on the directed polymer problem! However, it turns out that this result is wrong, because we must go beyond the linear approximation.

Technically the reason is in the integration which leads from the expression (2.18) of the average position in Fourier space to the scaling expression (2.19). Clearly the corresponding integral over frequencies ω is divergent at small ω, and it turns out that the result is proportional to $(t-t')^2\mathcal{T}$,where $\mathcal{T}$ is the total length of the polymer. One sees that, even if $t - t' < \tau^*$, the scaling depends on time scales which are larger than Δ. Therefore one cannot work out the scaling behaviour at small length and time scales within the linear – random force – approximation. In the polymer problem one must study the case of a random potential, which is a non linear problem with many metastable states [21, 26]. Translated into the turbulence language, these correspond to intermittency effects which are crucial and cannot be neglected. The full solution derived in the next section indeed finds a Kolmogorov scaling, but only for the third moment of the velocity difference $(\vec{v}(\vec{x},t) - \vec{v}(\vec{0},t))^3 \simeq |\vec{x}|$. The other moments do not agree since all the moments $(\vec{v}(\vec{x},t) - \vec{v}(\vec{0},t))^q$ with q larger or equal to 1 scale like $|\vec{x}|$. Furthermore, length and time are related by a *convective* scaling, corresponding to the dynamical exponent $z = \frac{1}{\zeta} = 1$.

3 The replica variational approach

3.1 The replica solution

We now turn to the problem of computing the average 'free energy' $\overline{\log Z(\vec{x},t)}$ of the directed polymer over the random forcing $V(\vec{x},\tau)$. The procedure we use is standard, and we follow closely [21, 26]: we first express $\log Z(\vec{x},t)$ as the 'zero replica' limit $\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n}$, average Z^n which generates an effective attraction between replicas, which we treat using a Gaussian variational Ansatz. The quality of this approximation, and the regimes of dimension and Reynolds number in which it becomes exact, will be discussed in section (3.3).

We proceed as usual by writing the average of Z^n as the partition function of the $n-$ replica Hamiltonian $\mathcal{H}_n$, which reads:

$$\mathcal{H}_n = \frac{1}{2}\sum_{a=1}^{n}\int_0^t d\tau \left[c\left(\frac{d\vec{x}_a}{d\tau}\right)^2 + \mu \vec{x}_a^2 \right]$$

$$-\frac{WN}{2}\sum_{a,b}\int_0^t d\tau \exp\left[-\frac{(\vec{x}_a(\tau)-\vec{x}_b(\tau))^2}{2N\Delta^2}\right] \tag{3.1}$$

where we have added a 'mass' term μ for regularizing intermediate computations, which we will set eventually to zero. We furthermore consider periodic polymers, for which $\vec{x}(t)=\vec{x}(0)$. This is not exactly the same problem as the one with free ends which is in direct correspondence with Burgers's equation. We shall first work out the periodic case, and in section (3.3) we discuss the changes for the free case.

In order to handle the problems of the metastable states, the idea is to use a variational method and approximate $\mathcal{H}_n$ by an effective Gaussian Hamiltonian $\mathcal{H}_v$, which we write in Fourier space as:

$$\mathcal{H}_v \equiv \frac{1}{2}\sum_{a,b}\int \tilde{d\omega}\ \vec{x}_a(-\omega)G_{ab}^{-1}(\omega)\vec{x}_b(\omega) \tag{3.2}$$

where $\int \tilde{d\omega}$ stands for $\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}$. Note that $\mathcal{H}_v$ is isotropic in real space; however its structure in 'replica' space is arbitrary. The trial free-energy obtained with $\mathcal{H}_v$ depends on G_{ab} and reads:

$$\mathcal{F}_v[G] = <\mathcal{H}_n>_v - \frac{N}{2}Tr\log G \tag{3.3}$$

where $<...>_v$ means averaging with the Bolzmann weight associated to the trial Hamiltonian $\mathcal{H}_v$. Remember that the temperature scale has been set to one. The calculation of $<\mathcal{H}_n>_v$ using Gaussian integrals is straightforward and leads to:

$$<\mathcal{H}_n>_v = \frac{N\mathcal{T}}{2}\left(\sum_a\int \tilde{d\omega}(c\omega^2+\mu)G_{aa}(\omega) - W\sum_{ab}\left(1+\frac{B_{ab}}{N\Delta^2}\right)^{-\frac{N}{2}}\right) \tag{3.4}$$

where $\mathcal{T}$ is the total length of the polymer and $B_{ab}\equiv\int \tilde{d\omega}[G_{aa}(\omega)+G_{bb}(\omega)-2G_{ab}(\omega)]$. The variational statement is that $\mathcal{F}_v[G]$ is greater than or equal to the 'true' free-energy. We thus look at the optimal G_{ab} such that $\frac{\partial\mathcal{F}_v[G]}{\partial G_{ab}(\omega)}\equiv 0$. This leads to the following self-consistent equations:

$$[G^{-1}]_{ab}(\omega) = -\frac{W}{\Delta^2}\left(1+\frac{B_{ab}}{N\Delta^2}\right)^{-\frac{N}{2}-1} \quad (a\neq b) \tag{3.5}$$

and

$$G_{aa}(\omega)+\sum_{b\neq a}G_{ab}(\omega)\equiv G_c(\omega)=\frac{1}{\mu+c\omega^2} \tag{3.6}$$

The task is now to solve these equations using some Ansatz on the structure of G_{ab}. This has been discussed in full details in [21]. We keep here to dimensions $N>2$. There are two regimes of Reynolds number, separated by a critical value $Re_c=[2(1-2/N)^{(1-N/2)}]^{1/3}$. For $Re<Re_c$, the solution is a 'replica symmetric' one with $G_{ab}(\omega)=\delta_{ab}G_c(\omega)+G(\omega)$. The two propagators

G_c and G are easily computed. Translated in terms of the velocity, this just corresponds, for an infinite size system (but for a finite injection length Δ) to a vanishing velocity field. For a finite 'box' of length $L = \sqrt{\frac{\nu}{\mu}}$, one finds that $\vec{v} \simeq \nu \frac{(\vec{x}-\vec{x}_0)}{L^2}$, where $\vec{x}_0$ is a random (Gaussian) time dependent variable depending on the forcing history. When one increases the forcing beyond Re_c the correct solution is the so-called 'one step' replica symmetry breaking scheme. This amounts to parametrizing the off-diagonal elements $G_{ab}(\omega)$ with *two* functions $G_0(\omega), G_1(\omega)$, depending on whether the replica indices belong to the same 'block' or to different blocks. The size of these blocks are furthermore parametrized by a number m ranging between 0 and 1, fixed by imposing that $\frac{\partial \mathcal{F}_v[G_0,G_1,m]}{\partial m} = 0$. We refer the reader to [29, 21] for a more detailed discussion of this construction, but will recall below its physical interpretation, on which we shall heavily rely to discuss our results in the turbulence language. The relevant formulae for inverting such matrices are given, for completeness, in Appendix A, together with the basic calculation steps needed to solve Eqs. (3.5-6).

One important final equation is the one fixing m. We find that:

$$(1 - \frac{2m}{N})^{1-N/2} = 2m^3 \frac{\Delta^2 W}{\nu} = \frac{1}{2} m^3 Re^3 \tag{3.7}$$

where we have used the definition of the Reynolds number (Eq. (2.4)). Note that Eq. (3.7) is well behaved in the limit $N \to \infty$, where it becomes $2e^m = (mRe)^3$. We shall see later that when $Re > Re_c$, the breaking of replica symmetry corresponds to a non trivial structuration of the flow into large size structures which are'cells' of size Δ, inside which the velocity is of order $v_\Delta \sim Re \frac{\nu}{\Delta}$ (independently of the box size L). We shall thus identify Re_c as the critical Reynolds number for the onset of turbulence.

3.2 Physical description of the solution

Let us see how this is encoded in the one-step solution, by first looking at *equal time correlations*. Within the Gaussian variational Ansatz, one finds that

$$\mathcal{P}[\{\vec{x}_a(t)\}] = Z^{-1} \sum_\pi \exp\left(-\frac{1}{2} \sum_{a,b} [Q^{-1}]_{\pi(a),\pi(b)} \vec{x}_a(t) \vec{x}_b(t) \right) \tag{3.8}$$

where $Q_{ab} = \int \tilde{d\omega} G_{ab}(\omega)$, and $\sum_\pi$ denotes the sum over all the $n!$ permutations of replica indices. Taking the limit where the mass term μ is zero, we find that (see Appendix A):

$$Q_0 \sim \mu^{N/4-1} \to 0, \quad Q_1 - Q_0 \to \infty, \quad \tilde{Q} - Q_1 = \frac{m\Delta^2}{(1 - \frac{2m}{N})} \tag{3.9}$$

In the very large Reynolds limit, on which we focus now, the parameter m scales as $1/Re$ and one finds $\tilde{Q} - Q_1 \simeq m\Delta^2$.

The physical interpretation of the replica probability distribution Eq. (3.8) has been worked out in [21], and is particularly simple in the present case where Q_0 is zero and $Q_1 - Q_0$ is very large. Remember that one is dealing with a polymer in a random environment. For each 'sample' Ω (i.e for a particular realisation of the disorder $V(\vec{x},t)$) the probability distribution $P_\Omega(\vec{x})$ for the 'end point' of the polymer $\vec{x}$ has a certain shape. The 1 step replica symmetry breaking variational approach assumes that this distribution can be written as a weighted sum of Gaussians:

$$P_\Omega(\vec{x}) = \sum_\alpha W_\alpha \exp\left(-\frac{1}{2\delta}[\vec{x} - \vec{r}_\alpha]^2\right) \tag{3.10}$$

where $\delta \equiv \tilde{Q} - Q_1$, W_α are random weights chosen with a probability proportional to $W^{-1-m}(1-W)^{m-1}$ and $\vec{r}_\alpha$ are uniformly distributed in the 'box' of size L (A more formal description of the solution is presented in Appendix B, together with useful technical details). The W_α and the $\vec{r}_\alpha$ encode the particular features of the sample Ω, and must thus eventually be averaged over.

The above construction was restricted to a certain 'time' (or length of the polymer). Similar considerations also enable us to construct two-time correlation functions, and the result is very simple: it amounts to let the $\vec{r}_\alpha$ acquire a time dependence. More precisely, $\vec{r}_\alpha$ are independent Gaussian, time dependent variables such that (again taking the $\mu = 0$ limit):

$$C(t-t') \equiv \overline{< [\vec{r}_\alpha(t) - \vec{r}_\alpha(t')]^2 >} = \frac{2S}{m} \int \tilde{d\omega} \frac{(1-\cos\omega(t-t'))}{c\omega^2(c\omega^2 + S)} \ , \tag{3.11}$$

where $S = 1/(4c(\tilde{Q} - Q_1)^2)$. Eq. (3.11) shows that a characteristic time scale appears, given by $\sqrt{\frac{c}{S}} = 2^{1/3}\frac{\Delta^2}{\nu Re}$, which is nothing but the 'Larkin-Ovchinnikov' time τ^* encountered above and corresponds to the convective time across the injection length. For $\tau \gg \tau^*$, one finds that $C(\tau) = 2^{\frac{4}{3}}\Delta^2\frac{\tau}{\tau^*}$, whereas for $\tau \ll \tau^*$, one finds $C(\tau) = \frac{1}{2}\Delta^2(\frac{\tau}{\tau^*})^2$.

Eqs. (3.10-11), together with the value of m and δ, are the central results of this paper, from which we shall derive in the next section the very interesting statistical properties of the velocity field, in particular the exact calculation of the full probability distribution of $\vec{v}(\vec{x},t) - \vec{v}(\vec{x}',t)$. Before turning to this computation, we first discuss the validity of this replica solution.

3.3 Discussion of the replica solution

One is not able to solve exactly the directed polymer problem. The variational method to which we have resorted is however known to be a good approximation to the real behaviour of the directed polymer ([21, 30]). An interesting advantage of this approach is that it gives exact answers for the thermodynamics in the limit of a large number of dimensions $N \to \infty$. Technically, as shown in [21], this comes from the fact that this variational Ansatz can be seen as a resummation of the 'Hartree' diagrams in the perturbation theory for the correlation function, which are the only ones surviving in the large N limit. However beyond the thermodynamic potentials, there are some quantities for which this approach fails to give the right answer, even for $N \to \infty$. It is unfortunately the case of the velocity correlation function defined in (2.15), for which our calculation may only become exact in the large N *and* large Reynolds limit. The problem can be understood as follows: the replica calculation is expected to reproduce faithfully the thermodynamical behaviour of the system, by correctly describing the *low-lying* energy states. The weight of these low-lying states is found to be distributed as $\propto W^{-1-m}(1-W)^{m-1}$. Hence the moments $\overline{[P_\Omega(\vec{x})]^\eta}$ which are primarily determined by the low-lying states (i.e. by those with large weights) are expected to be accurate, while those which are sensitive to the $W \to 0$ part of the distribution are not. This is the case when $\eta < m$, in particular for $\overline{\log P_\Omega(\vec{x})}$ which we need in order to calculate the velocity in the Burgers language. Hence, only when $m \to 0$, i.e. at infinite Reynolds number, can the method be reliably used to determine exactly the statistical properties of the velocity field. To summarize this discussion we expect that this approach may be exact only when $N \to \infty$ *and* $m \to 0$. This issue can be studied from the solution by a study of the correlation identities, some of which will be checked in section (4.3). In any case the picture which emerges from the discussion in the next section is an appealing one which seems to be a good approximation even at small N.

Eventually, we want to discuss an important detail. The previous replica calculation was performed, for simplicity, in the case of a periodic polymer, i.e. one for which $x(t) \equiv x(0)$. From the Burgers equation point of view, however, the end point of the 'polymer' must be left free. As argued by Nelson and Vinokur in a different context [31], the difference between points far 'inside' the chain and boundary points is the fact that the statistical weight of the former is given, in quantum mechanical language [32], by the product of *two* propagators, while the end point only needs one. If the ground state wave function of the corresponding quantum mechanical Hamiltonian decays exponentially on a characteristic length scale ℓ, this means that the lateral fluctuations of the polymer's end point decay on length scale ℓ, while the fluctuations for points in the bulk decay on a length scale $\ell/2$. The study of the system with periodic boundary conditions which we performed in Sect.(3.1) only deals with these fluctuations in the bulk. The simple consequence of this analysis for our problem is that formula (3.10) can be used for free end points as well, although the value of δ must be doubled, i.e.:

$$\delta = 2m\Delta^2 \; . \tag{3.12}$$

4 Structure of the velocity field

4.1 Qualitative Arguments

We can now use our 'dictionary' between the two problems, in particular the fact that the velocity field is given by the derivative of the free-energy of the polymer problem with respect to $\vec{x}$. For a given realisation of the forcing, this free-energy is simply the logarithm of the probability distribution $P_\Omega(\vec{x})$ given in Eq. (3.10); one can thus express $\vec{v}$ as:

$$\vec{v} = \frac{2\nu}{\delta} \frac{\sum_\alpha W_\alpha (\vec{x} - \vec{r}_\alpha) e^{-(\vec{x}-\vec{r}_\alpha)^2/2\delta}}{\sum_\alpha W_\alpha e^{-(\vec{x}-\vec{r}_\alpha)^2/2\delta}} \ . \tag{4.1}$$

We shall study the statistical properties of this velocity field at large Reynolds numbers. We recall that from the previous analysis, we have $\delta = 2m\Delta^2 \simeq 2^{4/3}\Delta^2/Re$. Typical snapshots of such a velocity field in $N = 1$ or $N = 2$ dimensions are given in Fig. 1 a,b. Very clearly, a cellular structure appears. Within each cell, the velocity field is radial $\vec{v} \sim \frac{2\nu}{\delta}(\vec{x} - \vec{r}_\alpha)$, with a rapid variation (shocks) across the boundaries of these cells.

This structure can be understood qualitatively from Eq. (4.1), using the important property that the weights W_α have a very broad distribution. More precisely, we know that $W_\alpha = \exp(-f_\alpha)/\sum_\gamma \exp(-f_\gamma)$, where the f_α are independant random variables with an exponential distribution increasing as $\exp(mf)$. For m small, there is a very strong hierarchy between the smallest f appearing in the sum. For instance the gap between the smallest f_α and the next scales as $1/m \propto Re$. So the low lying f_α, which are the only terms contributing to the sum, behave at large Re as $f_\alpha \simeq \hat{f}_\alpha Re$, and the velocity field in this regime takes the form:

$$\vec{v} \ \sim \ \frac{2\nu}{\delta} \ \frac{\sum_\alpha (\vec{x} - \vec{r}_\alpha) \exp\left(Re\left[-\hat{f}_\alpha - \frac{(\vec{x}-\vec{r}_\alpha)^2}{2^{7/3}\Delta^2}\right]\right)}{\sum_\alpha \exp\left(Re\left[-\hat{f}_\alpha - \frac{(\vec{x}-\vec{r}_\alpha)^2}{2^{7/3}\Delta^2}\right]\right)} \ . \tag{4.2}$$

For a given $\vec{x}$ and a large Re, the major contribution to the sum comes from the largest term. This dominant term will suddenly switch from –say– α to β when the space dependent Gaussian factor compensates the weight difference, i.e. when $(\vec{x}^* - \vec{r}_\alpha)^2 \sim (\vec{x}^* - \vec{r}_\beta)^2 + \frac{\delta}{m}$. This leads to a typical size of cells of order $\sqrt{\frac{\delta}{m}} \propto \Delta$, i.e. the injection length itself. The width ℓ of the shock separating two cells is obtained by writing that $\frac{|\vec{x}^* - \vec{r}_\alpha|\ell}{\delta} \sim 1$, i.e. $\ell \propto \frac{\Delta}{Re} \ll \Delta$, which shows that the notion of cells is indeed well defined at large Reynolds. This construction allows one to guess the structure of the probability distribution of the velocity difference between nearby points. Let us present the argument in $N = 1$ dimension (its generalisation to higher dimensions is straightforward), and call x and $x + r$ these two points. The probability that a cell wall is present within the interval $[x, x+r]$ is obviously given by $p = \frac{r}{\Delta}$. In the limit where p is small, the velocity difference $u = v(x+$

$r) - v(x)$ is equal to $\frac{2\nu r}{\delta}$ with probability $1-p$, and of order $\frac{2\nu\Delta}{\delta} \sim \frac{\nu Re}{\Delta} = v_\Delta$ with probability p. Hence:

$$P_r(u) \sim (1-p)\delta\left(u - \frac{2\nu r}{\delta}\right) + p\frac{1}{v_\Delta} f\left(\frac{u}{v_\Delta}\right) \tag{4.3}$$

where $f(.)$ is a certain scaling function. This is true in the limit $Re \to \infty$ where the internal structure of the shocks can be neglected. We will see below that precisely such a form is obtained from an exact calculation. Let us now estimate the various moments of u using expression (4.2):

$$\overline{u^q} \sim (1-p)(\frac{v_\Delta r}{\Delta})^q + p\ A_q v_\Delta^q\ , \tag{4.4}$$

where $A_q = \int du u^q f(u)$. In the limit where $\ell \ll r \ll \Delta$, one finds that for *all* q larger than 1, one has

$$\overline{u^q} = A_q v_\Delta^q \frac{r}{\Delta}\ , \tag{4.5}$$

whereas for $q < 1$, one finds

$$\overline{u^q} \propto v_\Delta^q (\frac{r}{\Delta})^q\ . \tag{4.6}$$

The velocity field is thus strongly intermittent, with a 'multifractal' spectrum given in Fig. 2. Note however that the $q = 3$ moment scales à la Kolmogorov, i.e as in the real turbulence problem. The presence of large scale structures (shocks) forming an $N-$ dimensional froth-like pattern (see Fig. 1-b) is responsible for such a strong intermittency. Large scale structures in 'true' turbulence are similarly thought to be the origin of the experimentally observed intermittency, which is however much milder (see Fig. 2). The deep reason of this difference is probably related to the 'dimension' of the large scale singularities, which is $N-1$ in the present case and only 1 for vortex lines in hydrodynamical turbulence.

This strong intermittency, due to the cell and shock structure, has already been discussed in the one dimensional Burgers turbulence in the decaying (unforced) case ([33, 34, 35]). To the best of our knowledge the forced case has not been studied. Furthermore we are able to derive the exact form of the probability distribution function of velocity differences, to which we now proceed.

4.2 Exact results at infinite Reynolds number

Using the direct evaluation of the moments of the velocity difference from Eq. (4.1), one can establish, after rather long manipulations detailed in Appendix B, the following form for the full distribution of longitudinal velocity differences. Let us first start by the equal time case, and write $u = v^1(\vec{x}) - v^1(\vec{y})$, where the component '1' is along the $\vec{x} - \vec{y}$ axis, and $l = \frac{y^1 - x^1}{\Delta\sqrt{2}}$ (The other components would be treated in a similar manner). The result then reads:

$$P_l(u) = \delta\left(u - l\frac{2\nu}{\delta}\sqrt{\frac{\delta}{m}}\right) e^{-l^2/8}\int_{-\infty}^{\infty} Dh R(h,|l|)$$

$$+|l|\int_{-\infty}^{\infty} dh\int_{-h-|l|/2}^{\infty} Dt\int_{h-|l|/2}^{\infty} Ds\delta\left(u+(t+s)\frac{2\nu}{\delta}\sqrt{\frac{\delta}{m}}\mathrm{sign}(l)\right) R(h,|l|)^2 \ , \tag{4.7}$$

where $R(h,l) = \left[e^{-hl/2}\mathcal{M}_0(h-l/2) + e^{hl/2}\mathcal{M}_0(-h-\lambda/2)\right]^{-1}$ ($\mathcal{M}_0$ is related to the standard error function – see Appendix B), and $Dx \equiv \frac{dx}{\sqrt{2\pi}}e^{\frac{-u^2}{2}}$. Note that Eq. (4.6) has precisely the structure which we guessed using qualitative arguments (Eq. (4.3)). We have tested this formula numerically by generating a one-dimensional velocity field using eq. (4.1), for $Re = 100$, and directly computed the second moment $\overline{u^2}$ as a function of ρ, which we compare in Fig 3 with the exact formula for $Re \to \infty$ obtained by integrating (4.7). For $\frac{\Delta}{Re} \ll \rho \ll \Delta$, corresponding to the 'inertial range', one finds $\overline{u^2} = \frac{16 Re^2\nu^2}{\sqrt{\pi}\Delta^2}\frac{|x-y|}{\Delta}$, or in terms of the 'traditional' energy spectrum $E(k) \equiv k^{N-1}\langle\vec{v}(\vec{k})\vec{v}(-\vec{k})\rangle$, $E(k) \propto k^{-2}$ for all N. Note the saturation for distances much larger than the cell size Δ, on which we shall comment later. We have also computed the weight of the δ peak and compared it to the one obtained analytically, with good agreement again.

We have also extended our analysis to two interesting situations. First of all, one may consider the case of a finite Reynolds number. The calculation of the second moment $\overline{u^2}$ shows that a new length scale appears[3] $\ell_d^{(2)} = \frac{\Delta}{Re}$, separating the above *linear* regime of $\overline{u^2}$ at larger distances, from the *regular*, quadratic behaviour at small length scales:

$$\overline{u^2} = \frac{\epsilon}{2\nu}r^2; \qquad r < \ell_d^{(2)} \tag{4.8}$$

which is the standard result (up to the numerical prefactor) obtained in turbulence for length scales smaller than the dissipation length (see e.g. [36]). Note that the 'dissipation length' ℓ_d depends on the moment of the velocity one wishes to calculate: this is another consequence of intermittency. Matching the regular behaviour (4.7) with (4.4) or (4.6) suggests that the q^{th} order dissipation length scales as $\ell_d^{(q)} = \Delta Re^{\frac{q}{2-2q}}$; only for $q = 3$ does one recover the usual Kolmogorov dissipation length $\ell_d^{(3)} = \frac{\Delta}{Re^{3/4}}$.

From our Ansatz for the velocity field (4.1), and the fact that $\vec{r}_\alpha(t)$ evolve according to (3.11), one can see that the full velocity correlation function has the following scaling form (in the limit $\frac{\Delta}{Re} \ll r \ll \Delta$):

$$\overline{[\vec{v}(\vec{x}+\vec{r},t) - \vec{v}(\vec{x},t')]^2} = r\ g\left(\frac{r}{\sqrt{2C(t'-t)}}\right) \tag{4.9}$$

[3]Note that $\ell_d^{(2)}$ is the width of the shocks ℓ introduced above

where $C(\tau)$ is defined in Eq. (3.11) and $g(.)$ is a certain scaling function. We have obtained the precise form of g in the case where t and t' are well 'inside' the polymer, but did not attempt to compute it for 'end points' – where additional numerical factors would appear (see the discussion at the end of section 3.3).

In the limit $t-t' \ll \tau^*$ (see Eq. (3.11)), $\sqrt{2C(t-t')} = v_\Delta|t-t'|$: Eq. (4.9) then means that fluctuations travel in a ballistic way with a 'velocity' fixed by the injection scale. For larger time scales, the dynamics recovers a diffusive character, since $\sqrt{2C(t-t')} \propto \Delta\sqrt{\frac{|t-t'|}{\tau^*}}$. Note however that the effective diffusion constant $\nu_{\mathrm{t}urb.} = \frac{\Delta^2}{\tau^*} \propto Re\ \nu$ is enormously enhanced compared to its 'bare' value ν – as is the case for usual turbulence. From Eq. (4.9) in the limit $r \to 0$, one finds that for coinciding points, the velocity difference grows with time as:

$$\overline{[\vec{v}(\vec{x},t) - \vec{v}(\vec{x},t')]^2} \propto \sqrt{2C(t-t')} \tag{4.10}$$

4.3 'Sum-Rules' and the large Reynolds limit

There are a number of 'sum-rules' that the correct solution of the Burgers equation should satisfy, allowing us to test our prediction for the structure of the velocity field, as given by Eq. (4.1). The first and most interesting one physically is the 'energy conservation', i.e.

$$\frac{1}{2}\frac{\overline{\partial \vec{v}^2}}{\partial t} = 0 = -\overline{\vec{v}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v}} + \nu\ \overline{\vec{v}\cdot\Delta\vec{v}} + \overline{\vec{v}\cdot\vec{f}} \tag{4.11}$$

The last term of this expression is the injected energy, and is equal to $+\frac{N}{2}\epsilon$. The two first terms represent the 'dissipated' energy – notice that contrarily to the incompressible Navier-Stokes case, this dissipation is non local (i.e. it cannot be expressed in terms of velocity derivatives only). We find (see Appendix C, Eq. C.11), that the total dissipation is given by $-\frac{N}{2}\epsilon(1-m)$. Hence we find that the energy conservation is indeed exactly satisfied in the limit $m \to 0$ ($Re \to \infty$) *at fixed* ϵ (i.e. for $\Delta \to \infty$ or $\nu \to 0$). Interestingly, in this limit, this sum rule is true independently of N. Eq. (4.11) can also be interpreted slightly differently. Suppose that at $t=0$ the forcing is switched off. The subsequent evolution of the energy density is then given by the dissipative terms, i.e. $N\Delta^{2/3}\frac{d\epsilon^{2/3}}{dt} \simeq -\frac{N\epsilon}{2}$, leading to $\epsilon \sim \frac{\Delta^2}{t^3}$, or else for the velocity scale $v_\Delta(t) \sim \frac{\Delta}{t}$.

We have also checked that $\overline{\frac{\partial(\vec{v}^2)^2}{\partial t}} = 0$ in the limit $Re \to \infty$ for all N, while $\overline{\frac{\partial \nabla\cdot\vec{v}}{\partial t}} = 0$ identically for all N and Re. The calculation of velocity correlations for non coinciding points, such as $\frac{\partial}{\partial t}\overline{\vec{v}(\vec{x})\vec{v}(\vec{y})}$, are much more intricate, and are currently under investigation.

Hence these direct checks of certain (local) correlation identities suggest that our Ansatz might be exact for large Re.

5 Discussion

There are quite a number of points worth discussing under the light of the previous results, before commenting on the similarities and differences with the 'real' turbulence problem.

5.1 Burgers turbulence in small dimensions

As stated above, our results are *a priori* only exact for large dimensions and Reynolds number, the method we have used becoming a variational approximation in finite dimensions. By comparison with other problems where this approach was used, however, one expects that our solution describes faithfully the physical situation, even for relatively small values of N. Furthermore, all the sum-rules which we have checked so far (Section 4.3) are satisfied for *any* N in the limit $Re \to \infty$. Hence, we believe that the forced Burgers equation in say $N = 3$ dimensions will produce a cellular arrangement of the flow pattern, as described above. It would be extremely interesting to test this prediction numerically, as well as to measure the intermittency corrections – which might be weaker in finite dimensions than the ones obtained within our Ansatz.

For $N \leq 2$, one however knows from previous studies that the structure of the replica solution changes: instead of a 'one step' breaking, a full continuous breaking scheme is needed. This raises the interesting question of knowing whether such a scheme could describe a more complicated behaviour of the moments of the velocity field than the one encountered above. We do not have a complete answer to this question, but it seems that the solution obtained for $N \leq 2$ does not lead to a qualitatively different picture, the reason being that all the length scales appearing in recursive construction of the sample dependent measure P_Ω [21] (i.e. the width of the Gaussians), as well as the 'm' parameters describing the weight distributions, are all of the same order of magnitude. Thus, forced Burgers turbulence in one dimension is expected to look very much like the picture shown above (Fig 1-a), as indeed is well known from studies of 'decaying' turbulence from random initial conditions [33, 34, 35].

More work on this aspect would certainly be interesting, in particular to study a variant of the present problem for $N = 1$, where f – rather that ϕ – is a random noise (see [37, 38]). In this case also, a full breaking of replica symetry is needed to describe the velocity field.

5.2 Long-distance scaling of the velocity field

As discussed in Section 2, the main difference between this work and previous studies on the directed polymer problem lies in the regime of length scales. Usually, in the directed polymer problem, one assumes that the correlation length of the potential Δ is very small, and one is interested in the long-distance $x \gg \Delta$ scaling behaviour of – say, the free-energy, where non trivial exponents appear. On the other hand, in line with most studies of the turbulence problem, we focused on the contrary on the *small* scales $x \ll \Delta$ regime, where an inertial range appears, characterized by an energy cascade $E(k) \propto k^{-2}$. The existence of these two regimes for the Burgers equation suggests that a similar situation may also occur in 'real' turbulence, where a non trivial scaling regime could exist for $x \gg \Delta$, characterized by a velocity correlation converging towards its asymptotic value as a power law:

$$\overline{[\vec{v}(\vec{x}+\vec{r})-\vec{v}(\vec{x})]^2} \simeq 2\overline{\vec{v}(\vec{x})^2} - \frac{B}{r^{2(1-\frac{\omega}{\zeta})}} + \ldots \qquad r \gg \Delta \tag{5.1}$$

or equivalently, by a new exponent for the energy spectrum: $E(k) \propto k^{1-2\frac{\omega}{\zeta}}$, where ω and ζ are the generalisation of the exponents defined for the polymer problem in Eqs. (2.13-14), corresponding now to the usual large scale situation $r \gg \Delta$ studied in directed polymers. Numerical results on directed polymers in 3+1 dimensions [17] give $\zeta \sim 0.6$, $\omega = 2\zeta - 1 \sim 0.2$. It would be very interesting to analyze experimental data beyond the injection length Δ along these lines.

5.3 Passive scalar dispersion

A subject of recent debate is the behaviour of a passive scalar in turbulent flows, which also shows experimentally 'anomalous' density fluctuations – in particular exponential tails [39]. Although a detailed study is beyond the scope of the present paper, it is interesting to discuss qualitatively this problem for Burgers' turbulence. Let us suppose that the diffusing scalar obeys a Langevin equation of the form

$$\frac{d\vec{x}}{dt} = \vec{v}(\vec{x},t) + \vec{\eta}(t) \tag{5.2}$$

where $\vec{v}(\vec{x},t)$ obeys the forced Burgers equation Eq. (2.1), and $\eta(t)$ describes the molecular diffusion, with diffusion constant equal to D. In the 'adiabatic' limit where the evolution of the velocity field can be assumed to be small compared to the equilibration time (i.e $D \gg v_\Delta \Delta$), the tracer concentration $\theta(\vec{x},t)$ is simply given by the Boltzman equilibrium:

$$\theta(\vec{x},t) \propto \exp -\frac{h(\vec{x},t)}{D} \tag{5.3}$$

where $h(\vec{x},t)$ is the velocity potential introduced in Eq. (2.5). Using $h(\vec{x},t) = 2\nu \log P_\Omega(\vec{x},t) + \text{cst.}$, one finds that:

$$\theta(\vec{x},t) \propto [P_\Omega(\vec{x},t)]^{-\frac{2\nu}{D}} \tag{5.4}$$

Now the full distribution function of P_Ω can be exactly calculated, and is found to be a totally asymmetric Lévy distribution L_μ with index $\mu = m$. This in turn allows one to obtain the full distribution of θ, as

$$\mathcal{P}(\theta) = \theta^{-1-\frac{D}{2\nu}} L_m(\theta^{-\frac{D}{2\nu}}) \tag{5.5}$$

Hence, we find that the *tail* of the tracer density distribution is a *power-law* for small θ, $\mathcal{P}(\theta) \sim \theta^{-1+\frac{Dm}{2\nu}}$, which becomes broader and broader as the Reynolds number increases ($m \propto Re^{-1}$). For large θ, $\mathcal{P}(\theta) \sim \theta^{\frac{Dm}{2\nu}-1} \exp\left[-\mathrm{cst}\theta^{\frac{Dm}{2\nu}}\right]$. Note however that our adiabatic assumption breaks down precisely when $\frac{Dm}{\nu} < 1$. It would be interesting to understand the nature of the fluctuations of θ in the large Reynolds regime.

Hence we have argued that the fluctuations of tracer concentration are strongly anomalous in Burgers's turbulence. In more physical terms, Eq. (5.2) shows that the tracer is convected towards the shock regions, where the concentration piles up. High density 'sheets' of particles spontaneously form, revealing the 'froth-like' structure of the flow. This is similar to what happens for the Burgers equation with random initial conditions (but no forcing) [40], and might be relevant for astrophysical applications [41].

Finally, it must be noted that the temporal dispersion of tracers is *convective* at short times, since it essentially follows the velocity field which evolves according to Eq.(3.11). This is, as noted above, in contrast with Richardson's diffusion, which states that $x \simeq t^{\frac{3}{2}}$. Convective dispersion is however rather commonly observed (see [42]).

5.4 Turbulence versus 'Burgulence'

The differences between Burgers' turbulence (coined 'Burgulence' by Fournier and Frisch in [33]) and hydrodynamical turbulence have been discussed many times. The most important one is the absence of vortex line singularities in a potential flow, which are thought to play an important role in turbulence [43]. In our case, singularities are concentrated on $N-1$ structures, giving rise to much larger 'intermittency' corrections.

Second, the energy is only dissipated by viscosity at small length scales in turbulence, while there is an additional dissipation term in Burgulence (which is, as discussed above, non-local). This might be another important difference although, as shown above, it does not prevent the existence of a well-defined inertial range where the energy spectrum follows a k^{-2} decay.

A third difference invoked by Kraichnan is that in fluids, the incompressibility condition is maintained by the underlying pressure field, which would play an important role in the dynamics of the fluctuations and be at the origin of the Richardson-Kolmogorov scaling $x \sim t^{3/2}$, instead of the convection law $x \sim v_\Delta t$ found here. On the other hand, the Burgers and incompressible Navier-Stokes equations look very similar, in particular from a dimensional analysis point of view, which is at the heart of Kolmogorov's argument; furthermore, the third moment of the velocity difference indeed scale in the same way (at least for large N).

On a technical level, the simplest closure scheme for turbulence is Kraichnan's DIA, which leads *both* for the Navier-Stokes equation and for the Burgers' equation, to a convective dynamics $x \sim v_\Delta t$ and to a $k^{-3/2}$ energy spectrum. However, DIA's extension to Lagrangian coordinates (LDIA) was argued by Kraichnan [5] to reproduce exactly Kolmogorov's scaling for the Navier-Stokes equation, but not for the Burgers case, where results similar to some of ours are obtained [4]. It would be interesting to investigate the precise relation between the two seemingly very different approaches, and in particular to compare LDIA to our results.

5.5 Conclusions

We have thus used a method inspired from spin-glasses to investigate a toy model of turbulence, and to propose an Ansatz for the structure of the velocity field, which should become exact in high dimensions and when the Reynolds number is also large. We find that beyond a critical Reynolds number, there exists a well defined inertial range where the energy spectrum decreases as k^{-2}. The scaling variable is $\frac{x}{v_\Delta t}$, where v_Δ is the velocity at the injection scale Δ. The third moment of the velocity difference scales linearly with distance, à la Kolmogorov, but strong intermittent corrections come into play, due to the presence of shocks localized on a froth-like, cellular pattern. The full distribution for the velocity difference is obtained exactly. Interesting scaling results are also argued to hold at scales *larger* than the injection scale, and we suggest that experimental data on grid turbulence could be analyzed accordingly. Our results are presumably qualitatively correct in low dimension; numerical simulations would be welcome. We have discussed qualitatively the passive scalar problem and we have found power-law tails in the concentration distribution, reflecting the localisation of the tracers near the shocks.

From a technical point of view, it would be interesting to understand the precise relation between the present approach and Kraichnan's Lagrangian DIA, which gives the same scaling as the ones obtained here. From a different point of view, one could also generalize this work to the case of a space-correlated forcing term $f(\vec{x}, t)$ with a *power-law* correlation function. This is the starting point of the renormalisation group (RG) analysis of turbulence: the exponent describing the decay of the power-law function is chosen as to reproduce Kolmogorov's scaling and an RG procedure [6] is applied to obtain adimensional prefactors (the Kolmogorov constant). The model studied here could provide an interesting benchmark to discuss the validity of such a procedure. In fact it can be used to test all the various approximate methods which have been introduced in the study of fully developped turbulence.

Finally, our Ansatz for the velocity field [Eq(4.1)], which is inspired from our 'replica' approach to the problem, has some interesting mathematical properties (see in particular Appendices B and C). It could be fruitful to generalize Eq. (4.1) to describe rotational flows.

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[4]Although the situation investigated by Kraichnan was that of 'decaying' turbulence; the intermittent corrections were furthermore not discussed

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Appendix A: Solution of the saddle-point equations.

The starting point of the one-step replica symmetry breaking calculation is the expression of the free-energy in terms of $\tilde{G}(\omega), G_0(\omega), G_1(\omega)$ and m. Using the expression (AII.11) given in [21] for the trace of the logarithm of a one-step Parisi matrix, we find that:

$$\mathcal{F} = \frac{1}{2}\int \tilde{d\omega}(c\omega^2+\mu)\tilde{G}(\omega) + \frac{W}{2}\{(1-m)(1+\frac{B_1}{N\Delta^2})^{-N/2} + m(1+\frac{B_0}{N\Delta^2})^{-N/2}\}$$
$$-\frac{1}{2}\int \tilde{d\omega}\ \Big[\frac{1}{m}\log(\tilde{G}(\omega)-mG_0(\omega)-(1-m)G_1(\omega))$$
$$+\frac{G_0(\omega)}{\tilde{G}(\omega)-mG_0(\omega)-(1-m)G_1(\omega)} - \frac{1-m}{m}\log(\tilde{G}(\omega)-G_1(\omega))\Big] \quad (A.1)$$

where $B_i \equiv 2\int \tilde{d\omega}[\tilde{G}(\omega)-G_i(\omega)]$. (We have set the temperature to $T=1$). Differentating $\mathcal{F}$ with respect to $\tilde{G}(\omega), G_0(\omega), G_1(\omega)$ yields equations (3.5-6) specialized to the one-step solution, with

$$G_c(\omega) \equiv \tilde{G}(\omega) - mG_0(\omega) - (1-m)G_1(\omega) = \frac{1}{[G^{-1}]_c} = \frac{1}{(\mu+c\omega^2)} \quad (A.2),$$

$$G_0(\omega) = -\frac{[G^{-1}]_0}{(\mu+c\omega^2)^2} \quad (A.3)$$

and

$$G_1(\omega) - G_0(\omega) = \frac{1}{(\mu+c\omega^2)}\frac{S}{m(\mu+c\omega^2+S)} \quad (A.4)$$

where we have introduced $S \equiv m([G^{-1}]_0 - [G^{-1}]_1)$. Using the definitions of B_0, B_1, we obtain the equations:

$$B_1 - B_0 = -\frac{2}{m}\int \tilde{d\omega}\frac{1}{\mu+c\omega^2}\frac{S}{\mu+c\omega^2+S} \quad (A.5)$$

and, using (A.2)

$$mB_0 + (1-m)B_1 = 2\int \tilde{d\omega}[\tilde{G}(\omega) - mG_0(\omega) - (1-m)G_1(\omega)] = 2\int \tilde{d\omega}\frac{1}{\mu+c\omega^2} \quad (A.6)$$

from which we can deduce B_0 and B_1 as a function of S. Inserting these values into Eq. (3.5) then leads, in the limit $\mu \to 0$, to:

$$[G^{-1}]_0 \sim 0; \qquad S = -m[G^{-1}]_1 = \frac{mW}{\Delta^2}[1+\frac{1}{N\Delta^2\sqrt{Sc}}]^{-1-N/2} \quad (A.7)$$

Differentiating now (A.1) with respect to m, after a few manipulations, leads to

$$\sqrt{\frac{S}{c}} = 2m^2 W[1 + \frac{1}{N\Delta^2\sqrt{Sc}}]^{-N/2} \qquad (A.8)$$

Eqs (A.7-8) allows us to obtain both S and m as:

$$4m^3 W\Delta^2 c = (1 - \frac{2m}{N})^{1-N/2} \qquad (A.9)$$

and

$$S = \frac{(1-\frac{2m}{N})^2}{4m^2\Delta^4 c} \qquad (A.10)$$

Using Eqs (A2-7), we thus obtain in the limit $\mu \to 0$:

$$Q_c = \frac{1}{2\sqrt{\mu c}}, \quad Q_1 - Q_0 = \frac{1}{2m\sqrt{\mu c}}, \quad \tilde{Q} - Q_1 = \frac{1}{2\sqrt{Sc}} \qquad (A.11)$$

and $Q_0 \propto \mu^{\frac{N}{4}-1}$. From this, we also obtain the inverse R of the matrix Q as:

$$R_c = \frac{1}{Q_c} = 2\sqrt{\mu c}, \quad R_0 = -\frac{Q_0}{Q_c^2} \propto \mu^{\frac{N}{4}},$$

$$R_1 - R_0 = \frac{Q_0 - Q_1}{Q_c[Q_c + m(Q_0 - Q_1)]} = \frac{2\sqrt{Sc}}{m} \qquad (A.12)$$

Appendix B: Probability distribution of the velocity difference

In this Appendix we compute the probability distribution function (pdf) of the difference of velocities between two points $\vec{x}$ and $\vec{y}$, projected onto $\vec{x}-\vec{y}$, at large Reynolds numbers. We shall proceed in three steps: first we show how to compute the first few moments for general Reynolds numbers. Then we show how the expression simplifies for large Re. This simplification is such that we can extract in this large Re limit all moments and deduce from it the pdf.

Our starting point is the result from the replica computation, which provides the following random process to build up the velocity distribution: at a given time, the velocity field is given by:

$$\vec{v}(\vec{x}) = \frac{2\nu}{\delta} \frac{\sum_\alpha W_\alpha (\vec{x}-\vec{r}_\alpha) e^{-(\vec{x}-\vec{r}_\alpha)^2/2\delta}}{\sum_\alpha W_\alpha e^{-(\vec{x}-\vec{r}_\alpha)^2/2\delta}} \tag{B.1}$$

The sum over α goes from 1 to M. The points $\vec{r}_\alpha$ are uniformly distributed in the volume $[-L/2, L/2]^d$. The weights W_α are random numbers drawn as follows: First one chooses M "energies" f_α at random, they are identically distributed independent random variables with a probability distribution:

$$\mathcal{P}(f) = m e^{m(f-f_c)} \theta(f_c - f) \tag{B.2}$$

Then the weights are given by:

$$W_\alpha = \frac{e^{-f_\alpha}}{\sum_\beta e^{-f_\beta}} \tag{B.3}$$

We shall let M, L, and f_c go to infinity together, keeping the density of states Me^{-mf_c}/L fixed. The $\vec{r}_\alpha$ and f_α are uncorrelated. We shall denote respectively by $\mathbf{E_r}$ and $\mathbf{E_f}$ the expectation values with respect to these two sets of random variables.

Let us first evaluate the second moment. We take $\vec{x}-\vec{y}$ along the first axis, and compute the longitudinal correlation:

$$\overline{v_1(x)v_1(y)} = \left(\frac{2\nu}{\delta}\right)^2$$

$$\mathbf{E_{r,f}} \left(\frac{\sum_{\alpha,\beta} \mathbf{e}^{-\mathbf{f}_\alpha - \mathbf{f}_\beta} (\vec{\mathbf{x}} - \vec{\mathbf{r}}_\alpha)_\mathbf{1} (\vec{\mathbf{y}} - \vec{\mathbf{r}}_\beta)_\mathbf{1} \mathbf{e}^{-[(\vec{\mathbf{x}}-\vec{\mathbf{r}}_\alpha)^\mathbf{2} + (\vec{\mathbf{y}}-\vec{\mathbf{r}}_\beta)^\mathbf{2}]/\mathbf{2}\delta}}{\sum_{\alpha,\beta} \mathbf{e}^{-\mathbf{f}_\alpha - \mathbf{f}_\beta} \mathbf{e}^{-[(\vec{\mathbf{x}}-\vec{\mathbf{r}}_\alpha)^\mathbf{2} + (\vec{\mathbf{y}}-\vec{\mathbf{r}}_\beta)^\mathbf{2}]/\mathbf{2}\delta}} \right). \tag{B.4}$$

We shall compute separately the two contributions to this correlation which come from the terms $\alpha = \beta$ and $\alpha \neq \beta$ in the numerator of (B.4). We thus write:

$$\overline{v_1(x)v_1(y)} \equiv \left(\frac{2\nu}{\delta}\right)^2 (g_{11} + g_{12}), \tag{B.5}$$

where g_{11} and g_{12} respectively contain the terms $\alpha = \beta$ and $\alpha \neq \beta$ in (B.4).

To compute g_{11} we write the denominator of (B.4) in integral form, which gives:

$$g_{11} = M\mathbf{E}_{\mathbf{r},\mathbf{f}} \int_{\mathbf{0}}^{\infty} \mathbf{d}\lambda \int_{\mathbf{0}}^{\infty} \mathbf{d}\mu \mathbf{e}^{-\mathbf{2f}_\alpha} (\mathbf{x}-\vec{\mathbf{r}}_\alpha)_\mathbf{1}(\mathbf{y}-\vec{\mathbf{r}}_\alpha)_\mathbf{1} \mathbf{e}^{-[(\mathbf{x}-\vec{\mathbf{r}}_\alpha)^\mathbf{2}+(\mathbf{y}-\vec{\mathbf{r}}_\alpha)^\mathbf{2}]/\mathbf{2}\delta}$$

$$\exp\left[e^{-f_\alpha}\left(\lambda e^{-(x-\vec{r}_\alpha)^2/2\delta} + \mu e^{-(y-\vec{r}_\alpha)^2/2\delta}\right)\right]$$

$$\prod_{\beta(\neq\alpha)} \left(\exp\left[e^{-f_\beta}\left(\lambda e^{-(x-\vec{r}_\beta)^2/2\delta} + \mu e^{-(y-\vec{r}_\beta)^2/2\delta}\right)\right]\right) \qquad (B.6)$$

As for the mean over the free energies, we shall use repeatedly in this Appendix the following formula, valid for large f_c:

$$\int_{-\infty}^{f_c} df m e^{m(f-f_c)}\ e^{-kf-Ae^{-f}} = \begin{cases} me^{-mf_c}A^{m-k}\Gamma(k-m), & \text{if } k\geq 1; \\ 1-e^{-mf_c}A^m\Gamma(1-m), & \text{if } k=0. \end{cases} \qquad (B.7)$$

After taking this average over the free energie, (B.6) becomes:

$$g_{11} = M\mathbf{E}_\mathbf{r} \int_{\mathbf{0}}^{\infty} \mathbf{d}\lambda \int_{\mathbf{0}}^{\infty} \mathbf{d}\mu \quad (\vec{\mathbf{x}}-\vec{\mathbf{r}}_\alpha)_\mathbf{1}(\vec{\mathbf{y}}-\vec{\mathbf{r}}_\alpha)_\mathbf{1} \mathbf{e}^{-[(\vec{\mathbf{x}}-\vec{\mathbf{r}}_\alpha)^\mathbf{2}+(\vec{\mathbf{y}}-\vec{\mathbf{r}}_\alpha)^\mathbf{2}]/\mathbf{2}\delta}$$

$$me^{-mf_c}\Gamma(2-m)\left(\lambda e^{-(\vec{x}-\vec{r}_\alpha)^2/2\delta} + \mu e^{-(\vec{y}-\vec{r}_\alpha)^2/2\delta}\right)^{m-2}$$

$$\prod_{\beta(\neq\alpha)} \left[1-e^{-mf_c}\Gamma(1-m)\left(\lambda e^{-(\vec{x}-\vec{r}_\beta)^2/2\delta} + \mu e^{-(\vec{y}-\vec{r}_\beta)^2/2\delta}\right)^m\right] \qquad (B.8)$$

We now average over the values of $\vec{r}_\alpha$ and $\vec{r}_\beta$, with a uniform measure in a box of size L^N. In the limit of large L, M, f_c at fixed density we get:

$$g_{11} = m\Gamma(2-m)\frac{Me^{-mf_c}}{L}\int_0^\infty d\lambda \int_0^\infty d\mu$$

$$\int d\vec{r}(\vec{x}-\vec{r})_1(\vec{y}-\vec{r})_1 e^{-[(\vec{x}-\vec{r})^2+(\vec{y}-\vec{r})^2]/2\delta}\left(\lambda e^{-(\vec{x}-\vec{r})^2/2\delta} + \mu e^{-(\vec{y}-\vec{r})^2/2\delta}\right)^{m-2}$$

$$\exp\left[-\frac{Me^{-mf_c}}{L}\Gamma(1-m)\int d\vec{r}\left(\lambda e^{-(\vec{x}-\vec{r})^2/2\delta} + \mu e^{-(\vec{y}-\vec{r})^2/2\delta}\right)^m\right] \qquad (B.9)$$

It is convenient to rewrite $\mu \to \lambda\mu'$, and integrate over λ, which gives:

$$g_{11} = (1-m)\int_0^\infty d\mu$$

$$\left[\frac{\int dr(x-r)(y-r)e^{-[(x-r)^2+(y-r)^2]/2\delta}\left(e^{-(x-r)^2/2\delta} + \mu e^{-(y-r)^2/2\delta}\right)^{m-2}}{\int dr\left(e^{-(x-r)^2/2\delta} + \mu e^{-(y-r)^2/2\delta}\right)^m}\right]. \qquad (B.10)$$

(In this expression the $\int dr$ is over a single r variable, namely the component of $\vec{r}$ in the direction of $\vec{x}-\vec{y}$, the other components have already been integrated out, their contributions cancel between the numerator and the denominator. We denote $x=\vec{x}_1$, and similarly $y=\vec{y}_1$)

It is straightforward to perform the same steps for the second contribution g_{12} in (B.5). One finds:

$$g_{12} = m \int_0^\infty d\mu \quad \left[\int dr (x-r) e^{-(x-r)^2/2\delta} \left(e^{-(x-r)^2/2\delta} + \mu e^{-(y-r)^2/2\delta} \right)^{m-1} \right]$$

$$\left[\int dr (y-r) e^{-(y-r)^2/2\delta} \left(e^{-(x-r)^2/2\delta} + \mu e^{-(y-r)^2/2\delta} \right)^{m-1} \right]$$

$$\left[\int dr \left(e^{-(x-r)^2/2\delta} + \mu e^{-(y-r)^2/2\delta} \right)^m \right]^{-2} . \qquad (B.11)$$

It should be clear that these techniques allow to write the velocity correlations of low order in a relatively closed form (this means reduced to some finite dimensional integrals, where all the averages over r and f have been taken care of, as well as the limits $M \to \infty$, $L \to \infty$ and $f_c \to \infty$). However the expressions are complicated enough, especially when one goes to high moments, and we have not found a closed form for the pdf in general.

Fortunately the situation simplifies in the limit of large Reynolds numbers. Let us first work out the expression of the second moment in this $Re \to \infty$ limit. We know from the replica solution that in this limit the width δ scales as $\delta \simeq 2m\Delta^2$, where Δ is the scale at which energy is injected, and m, the breakpoint in Parisi's RSB solution, behaves as $m \simeq 2^{1/3}/Re$. Hereafter we shall use m instead of Re, and we want to understand the small m limit of the velocity pdf.

We proceed and first work out the small m limit of the two point correlation. In g_{11} we change variable to $x = \tilde{x}\sqrt{\delta/m}$, $y = \tilde{y}\sqrt{\delta/m}$; We shall take the $m \to 0$ (large Re) limit keeping $\tilde{x}$, and $\tilde{y}$ fixed. The algebraic distance between the two points is measured by $l = (\tilde{x} - \tilde{y}) = (\vec{x} - \vec{y})_1/(\Delta\sqrt{2})$. We also change the dummy integration variables in (B.10) from r to $z = r\sqrt{\delta/m} - (\tilde{x} + \tilde{y})/2$, and from μ to $h = -m\ln(\mu)/|l|$. This gives:

$$g_{11} = (1-m) \left(\frac{\delta}{m^2} \right) |l| \int_{-\infty}^\infty dh e^{-h|l|/m} \frac{I_{1,1}(h)}{I_{0,0}(h)} , \qquad (B.12)$$

where we have introduced the functions (defined for integer k_1, k_2):

$$I_{k_1,k_2}(h) \equiv \int \frac{dz}{\sqrt{2\pi}} (l/2 - z)^{k_1} (-l/2 - z)^{k_2}$$

$$e^{-z^2/2 + lz(k_1-k_2)/2m} \left(e^{lz/2m} + e^{-h|l|/m - lz/2m} \right)^{m-(k_1+k_2)} . \qquad (B.13)$$

Their small m limit is easily worked out by a saddle point integration. The result for $k_1 \geq 1, k_2 \geq 1$ is:

$$I_{k_1,k_2}(h) \sim_{m\to 0} \frac{m}{|l|} \mathrm{sign}(l)^{k_1+k_2} \ e^{h|l|(k_2/m - 1/2)} \ \frac{e^{-h^2/2}}{\sqrt{2\pi}}$$

$$\left(\frac{|l|}{2} + h \right)^{k_1} \left(-\frac{|l|}{2} + h \right)^{k_2} \frac{\Gamma(k_1)\Gamma(k_2)}{\Gamma(k_1+k_2)} \qquad (B.14)$$

While the results for $k_1 k_2 = 0$ read:

$$I_{k_1,0}(h) \sim_{m\to 0} \; e^{l^2/8}\,\mathrm{sign}(l)^{k_1}\; \mathcal{M}_{k_1}\left(-h - \frac{|l|}{2}\right),$$

$$I_{0,k_2}(h) \sim_{m\to 0} \; e^{l^2/8}\,\mathrm{sign}(l)^{k_2}\; \mathcal{M}_{k_1}\left(h - \frac{|l|}{2}\right) e^{h|l|(k_2/m-1)},$$

$$I_{0,0}(h) \sim_{m\to 0} \; e^{l^2/8-h|l|/2}\; \left[e^{-h|l|/2}\mathcal{M}_0(h-\frac{|l|}{2}) + e^{h|l|/2}\mathcal{M}_0(-h-\frac{|l|}{2})\right]. \qquad (B.15)$$

In these equations we have used the definition:

$$\mathcal{M}_k(x) \equiv \int_x^\infty \frac{dz}{\sqrt{2\pi}} z^k e^{-z^2/2}. \qquad (B.16)$$

Therefore we get for the $m \to 0$ limit of g_{11}:

$$g_{11} = \frac{\delta}{m}\sqrt{\frac{2}{\pi}}\int_0^\infty dh e^{-h^2/2}$$

$$\frac{h^2 - l^2/4}{e^{l^2/8}\left[e^{-h|l|/2}\mathcal{M}_0(h-\frac{|l|}{2}) + e^{h|l|/2}\mathcal{M}_0(-h-\frac{|l|}{2})\right]}\,. \qquad (B.17)$$

The small m limit of the second contribution to this second moment can be worked out with the same technique. One gets:

$$g_{12} = \frac{\delta}{m}\frac{-l}{\pi}\int_0^\infty dh e^{-h^2}$$

$$\frac{1}{e^{l^2/4}\left[e^{-h|l|/2}\mathcal{M}_0(h-\frac{|l|}{2}) + e^{h|l|/2}\mathcal{M}_0(-h-\frac{|l|}{2})\right]^2}\,. \qquad (B.18)$$

We remember that from (B.5) the velocity two point correlations between points at distance $l\Delta\sqrt{2}$ equals $(2\nu/\delta)^2(g_{11}+g_{22})$. It is therefore finite when $m \to 0$. It is interesting to work out its small l limit, which gives the correlation between two points whose distance is small with respect to the injection scale Δ but large with respect to the dissipation scale (since we have taken the limit $Re \to \infty$ -or $m \to 0$- first):

$$< v_1(x) v_1(y) > \simeq -\frac{8\nu^2}{\sqrt{\pi} m^2 \Delta^2}\,\frac{x-y}{\Delta} \qquad (B.19)$$

We now turn to the computation of higher order moments. We are interested in arbitrary moments of the velocity difference $\vec{v}(\vec{x}) - \vec{v}(\vec{y})$ projected onto the direction of $\vec{x}-\vec{y}$ (chosen as the '1' direction. The starting point is analogous to (B.4):

$$\overline{v_1(\vec{x})^p v_1(\vec{y})^{p'}} = (\frac{2\nu}{\delta})^{p+p'}\mathbf{E}_{\mathbf{r},\mathbf{f}} \sum_{\alpha_\mathbf{1},\ldots,\alpha_\mathbf{p}}\sum_{\beta_\mathbf{1},\ldots,\beta_\mathbf{p}} \mathbf{e}^{-(\mathbf{f}_{\alpha_\mathbf{1}}+\ldots+\mathbf{f}_{\alpha_\mathbf{p}}+\mathbf{f}_{\beta_\mathbf{1}}+\ldots+\mathbf{f}_{\beta_{\mathbf{p}'}})}$$

$$(\vec{x}-\vec{r}_{\alpha_1})_1 ... (\vec{x}-\vec{r}_{\alpha_p})_1 \exp\left(-\frac{(\vec{x}-\vec{r}_{\alpha_1})^2+...+(\vec{x}-\vec{r}_{\alpha_p})^2}{2}\right)$$
$$(\vec{y}-\vec{r}_{\beta_1})_1 ... (\vec{y}-\vec{r}_{\beta_{p'}})_1 \exp\left(-\frac{(\vec{y}-\vec{r}_{\beta_1})^2+...+(\vec{y}-\vec{r}_{\beta_{p'}})^2}{2}\right)$$
$$\left[\sum_\alpha e^{-f_\alpha-(\vec{x}-\vec{r}_\alpha)^2/2\delta}\right]^{-p}\left[\sum_\beta e^{-f_\beta-(\vec{y}-\vec{r}_\beta)^2/2\delta}\right]^{-p'} \qquad (B.20)$$

We shall proceed as for the second moment. The first step is to exponentiate the denominator in (B.20) using

$$\frac{1}{A^p B^{p'}} = \int_0^\infty d\lambda \int_0^\infty d\mu \frac{\lambda^{p-1}}{\Gamma(p)}\frac{\mu^{p'-1}}{\Gamma(p')} e^{-(\lambda A+\mu B)} \ . \qquad (B.21)$$

In order to perform the averages over the values of the free energies f_α and the positions r_α in (B.20) we need to distinguish how many indices in $\alpha_1,...,\alpha_p,\beta_1,...,\beta_{p'}$ are distinct one from another. Let us suppose that there appear in this sequence k different indices, which we call $\gamma_1,...,\gamma_k$. We shall call q_j the number of indices in $\alpha_1,...,\alpha_p$ which are equal to γ_j, and q'_j the number of indices in $\beta_1,...,\beta_{p'}$ which are equal to γ_j. Such a configuration of indices, characterized by the number $k\in\{1,2,...,p+p'\}$, and the sequences $q_1,...,q_k,q'_1,...q'_k$ (such that $q_1+...+q_k=p$, $q'_1+...+q'_k=p'$ and for every j: $q_j+q'_j\geq 1$) appears a certain number of times, which we call $\mathcal{C}^{(k)}_{q_1,...,q_k,q'_1,...q'_k}$, in the sum over $\alpha_1,...,\alpha_p,\beta_1,...,\beta_p$ of (B.20). For each configuration of indices, we just do the same transformations as for the second moment. We shall not repeat them here, and just give the final result which generalizes (B.12):

$$\overline{v_1(x)^p v_1(y)^{p'}} = (\frac{2\nu}{\delta})^{p+p'}(\frac{\delta}{m})^{(p+p')/2}\frac{1}{m}\int_{-\infty}^\infty dh \frac{e^{-p'h|l|/m}}{\Gamma(p)\Gamma(p')}\sum_{k=1}^{p+p'}\Gamma(k)m^{k-1}$$
$$\sum_{q_1,...,q_k,q'_1,...q'_k}\mathcal{C}^{(k)}_{q_1,...,q_k,q'_1,...q'_k}\prod_{j=1}^k\left[\frac{\Gamma(q_j+q'_j-m)I_{q_j,q'_j}(h)}{I_{0,0}(h)}\right] \ . \qquad (B.22)$$

Let us now work out the leading behaviour of this expression in the small m (large Re) limit. From the integrals $I_{q_j,q'_j}(h)$ we get a factor $\exp(h|l|\sum_j q'_j/m)$ which exactly compensates the explicit $e^{-p'h|l|/m}$. Using $\delta=2m\Delta^2$, we find that the contribution to (B.22) from a given value of k scales as $(\nu/\delta)^{p+p'}(\delta/m)^{-(p+p')/2}m^{a(k)}$. A close look at the behaviours of the integrals $I_{q,q'}$ shows that $a(k)$ is zero for $k=1,2$, and it is strictly positive for $k\geq 3$. So we can neglect the terms with $k\geq 3$ in (5.5). The term $k=1$ is easily worked out. The only allowed configuration of indices has $q_1=p$ and $q'_1=p'$, and its degeneracy is $\mathcal{C}^{(1)}_{p,p'}=1$. For the $k=2$ term we keep only the leading terms which correspond to $q_1=0$, $q_2=p$, $q'_1=p'$, $q'_2=0$ or to $q_1=p$, $q_2=0$, $q'_1=0$, $q'_2=p'$, which also have a degeneracy 1 (*** a verifier ***). Altogether the leading contribution to this moment at large Re is:

$$\overline{v_1(x)^p v_1(y)^{p'}} = (\frac{2\nu}{\delta})^{p+p'}(\frac{\delta}{m})^{(p+p')/2}\mathrm{sign}(l)^{p+p'}\int dh$$
$$\left[\left(\frac{|l|}{2}+h\right)^p\left(-\frac{|l|}{2}+h\right)^{p'}\frac{e^{-(h^2/2+l^2/8)}}{\sqrt{2\pi}}R(h,|l|)\right.$$
$$\left.+(-1)^p|l|\mathcal{M}_p(-h-\frac{|l|}{2})\mathcal{M}_{p'}(h-\frac{|l|}{2})R(h,|l|)^2\right]\ . \qquad (B.23)$$

It is a simple exercise to sum these moments and get the qth moment of the longitudinal velocity difference:

$$\overline{(v_1(x)-v_1(y))^q} = (\frac{2\nu}{\delta})^q(\frac{\delta}{m})^{q/2}\mathrm{sign}(l)^q\int dh \quad \left[|l|^q\frac{e^{-(h^2/2+l^2/8)}}{\sqrt{2\pi}}R(h,|l|)\right.$$
$$\left.+(-1)^q|l|\int_{-h-|l|/2}^{\infty}\frac{dt}{\sqrt{2\pi}}e^{-t^2/2}\int_{h-|l|/2}^{\infty}\frac{ds}{\sqrt{2\pi}}e^{-s^2/2}(t+s)^q R(h,|l|)^2\right]\ , \quad (B.24)$$

where:

$$R(h,l)\equiv\left[e^{-hl/2}\mathcal{M}_0(h-l/2)+e^{hl/2}\mathcal{M}_0(-h-l/2)\right]^{-1} \qquad (B.25)$$

Under this form the moments can be inverted and we obtain the explicit form of the pdf $P(u)$ of the velocity difference $u\equiv v_1(x)-v_1(y)$ between two points at distance $x-y=l\Delta\sqrt{2}$ at large Re:

$$P(u)=\delta\left(u-l\frac{2\nu}{\delta}\sqrt{\frac{\delta}{m}}\right)\ e^{-l^2/8}\int_{-\infty}^{\infty}\frac{dh}{\sqrt{2\pi}}e^{-h^2/2}R(h,|l|)$$
$$+|l|\int_{-\infty}^{\infty}dh\int_{-h-|l|/2}^{\infty}\frac{dt}{\sqrt{2\pi}}e^{-t^2/2}\int_{h-|l|/2}^{\infty}\frac{ds}{\sqrt{2\pi}}e^{-s^2/2}$$
$$\delta\left(u+(t+s)\frac{2\nu}{\delta}\sqrt{\frac{\delta}{m}}\mathrm{sign}(l)\right)\ R(h,|l|)^2\ . \qquad (B.26)$$

Appendix C: Correlation identities

In this Appendix we check that the solution we have found does satisfy some necessary identities of correlation functions, in the limit of large Reynolds numbers. We shall perform the explicit check in the case of the "energy" balance. From Burgers' equation one finds:

$$\frac{1}{2}\frac{\partial}{\partial t}\overline{(v^\mu v^\mu)} = A + B + C \ ,$$

$$A \equiv -\frac{1}{2}\ \overline{v^\mu \partial^\mu (v^\rho v^\rho)} \ \ B \equiv \nu \ \overline{v^\mu \partial^\mu \partial^\rho v^\rho} \ \ C \equiv \overline{f^\rho v^\rho} \ , \tag{C.1}$$

with a convention of summation on the repeated vector indices μ, ρ from 1 to N which will be used in this whole Appendix. The overbar denotes the expectation value with respect to various realizations of the force $\vec{f}$. We shall compute successively the three contributions A, B (from our solution), and C (from a direct computation of the energy injected). In the end we shall check that their sum vanishes at large Reynolds number, as implied by the stationnarity of the forced flow.

We use the Ansatz for the velocity given in (4.1) and developped in Appendix B. It will be useful to introduce somewhat more compact notations and define:

$$u^\mu \equiv x^\mu - r^\mu \tag{C.2}$$

where r^μ is a random variable which takes value r^μ_α with a probability $W_\alpha \exp(-(\vec{x} - \vec{r}_\alpha)/2\delta)$. We shall denote as before by brackets the expectation values with respect to this process with fixed W_α and $\vec{r}_\alpha$, while the averaging over W_α and $\vec{r}_\alpha$ (corresponding to the overline in (C.1)) will be denoted as in Appendix B by $E_{r,f}$. (In terms of the directed polymer, $< O >$ denotes a thermal average, and the overline denotes an average over quenched disorder). With these definitions we have:

$$v^\mu = \frac{2\nu}{\delta} < u^\mu >$$

$$\partial^\rho v^\mu = \frac{2\nu}{\delta}\left(\delta^{\rho\mu} - \frac{1}{\delta}(< u^\rho u^\mu > - < u^\rho >< u^\mu >)\right) \ . \tag{C.3}$$

which leads to the following expression of A:

$$A = \left(\frac{2\nu}{\delta}\right)^3 E_{r,f}\Big[-< u^\mu >< u^\mu > + \frac{1}{\delta}< u^\rho >< u^\mu >< u^\rho u^\mu >$$
$$-\frac{1}{\delta}< u^\rho >< u^\mu >< u^\rho >< u^\mu > \Big] \ . \tag{C.4}$$

The same steps give the expression for B:

$$B = \left(\frac{2\nu}{\delta}\right)^3 \frac{1}{\delta} E_{r,f} \left[-< u^\rho >< u^\mu > (< u^\rho u^\mu > - < u^\rho >< u^\mu >) \right.$$

$$\left. + \frac{1}{2} < u^\rho >< u^\mu u^\mu u^\rho > - \frac{1}{2} < u^\rho >< u^\mu u^\mu >< u^\rho > \right] . \qquad (C.5)$$

The partial cancellation leaves:

$$A + B = \left(\frac{2\nu}{\delta}\right)^3 E_{r,f} \left[-< u^\mu >< u^\mu > + \frac{1}{2\delta} < u^\rho >< u^\mu u^\mu u^\rho > \right.$$

$$\left. - \frac{1}{2\delta} < u^\rho >< u^\mu u^\mu >< u^\rho > \right] \qquad (C.6)$$

We now evaluate each of the terms in this expression . The first term is given by:

$$E_{r,f}(< u^\mu >< u^\mu >) = E_{r,f} \left[\frac{\sum_{\alpha,\beta} W_\alpha W_\beta (\vec{x} - \vec{r}_\alpha)^\mu (\vec{x} - \vec{r}_\beta)^\mu e^{-\frac{(\vec{x}-\vec{r}_\alpha)^2 + (\vec{x}-\vec{r}_\beta)^2}{2\delta}}}{\left(\sum_\gamma W_\gamma e^{-(\vec{x}-\vec{r}_\gamma)^2/2\delta}\right)^2} \right] \qquad (C.7)$$

Using the technique of Appendix B, one finds after some work that the only non vanishing contribution comes from the $\alpha = \beta$ term in (C.7) and gives:

$$E_{r,f}(< u^\mu >< u^\mu >) = \frac{1-m}{m} \; \delta \; N \; . \qquad (C.8)$$

The same technique can be applied to each term in (C.6). One obtains:

$$E_{r,f}(< u^\rho >< u^\mu u^\mu u^\rho >) = \frac{1-m}{m^2} \; \delta^2 \; (N^2 + 2N) \qquad (C.9)$$

and:

$$E_{r,f}(< u^\rho >< u^\mu u^\mu >< u^\rho >) = \frac{1-m}{m^2} \; \delta^2 \; (N^2 + (2-m)N) \qquad (C.10)$$

The sum gives in the end:

$$A + B = -\frac{1}{2} \left(\frac{2\nu}{\delta}\right)^3 \; \frac{1-m}{m} \; \delta \; N \qquad (C.11)$$

One should notice that the terms of order N^2 vanish automatically due to the structure of the velocity field (4.1), and independently from the value of m.

We now proceed to the evaluation of the last term, C, in the correlation identity (C.1). This point needs a little care because one must be more precise about the correct prescription of the forcing term, whether it is of Ito or Stratanovitch type. To settle this issue in a pedestrian but safe way, we have discretized the time in units of $\tau_0 \ll 1$. The Hopf Cole mapping can be carried out in this case, and we deduce from this computation:

$$C = \frac{1}{2}\overline{f^\mu f^\mu} = \frac{1}{2}N\epsilon \qquad (C.12)$$

Using the value (3.12) of δ: $\delta = 2m\Delta^2$, the relation (3.7) between m and the Reynolds number: $(1-\frac{2m}{N})^{1-N/2} = \frac{1}{2}m^3 Re^3$, together with the definition (2.4) of Re: $Re^3 = \epsilon\Delta^4/\nu^3$, we finally get:

$$\lim_{Re\to\infty} A + B + C = 0 \qquad (\epsilon \ \text{fixed}) \qquad (C.13)$$

which establishes the correlation identity for the energy balance (C.1).

Using the same techniques, we have also checked that the correlation identities corresponding to $\frac{\partial}{\partial t}\overline{\vec{\nabla}.\vec{v}} = 0$ and $\frac{\partial}{\partial t}\overline{((\vec{v})^2)^2} = 0$ also hold *for any* N in the limit of large Re.

Appendix D: Replica computation of the velocity two points correlation

The aim of this Appendix is to compute the two point correlation $\overline{v_(x,t)v(y,t)}$ directly from the replica method of sect.3. This will provide a check that the intermediate physical representation (B.1) is safe. In the course of this computation we shall also establish useful replica identities which can be of wider interest. To keep the computations simple we consider only the one dimensional case, $N = 1$. We start from the result (3.8) for the partition function for n replicas of the polymer arriving at time t at the points $\vec{x}_a, a = 1, ..., n$:

$$\overline{Z(x_1,t)...Z(x_n,t)} = c\sum_{\pi} \exp\left(\frac{1}{2}\sum_{a,b} R_{\pi(a)\pi(b)} x_a(t) x_b(t)\right) \qquad (D.1)$$

We shall compute $\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}}$ which can be written as:

$$\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}} = c\sum_{\{\sigma\}}{}'$$

$$\exp\left(\frac{1}{2}\sum_{a,b} R_{ab}\left[\frac{1-\sigma_a}{2}x + \frac{1+\sigma_a}{2}y\right]\left[\frac{1-\sigma_b}{2}x + \frac{1+\sigma_b}{2}y\right]\right) \qquad (D.2)$$

where the $\sum'_{\{\sigma\}}$ is over n Ising spins $\sigma_a = \pm 1$, with the constraint that $\sum_a \sigma_a = 0$.

As an intermediate step to this computation, we first compute the action of $\sum'_{\{\sigma\}}$ onto a polynomial in the spins. Let $a_1, ..., a_k$ be k different replica indices. We define:

$$A_k \equiv \sum_{\{\sigma\}}{}' \sigma_{a_1}..\sigma_{a_k} \ . \qquad (D.3)$$

Clearly $A_1 = 0$. As for A_2, one can use $0 = \sum_{a,b}\sum'_{\{\sigma\}} \sigma_a\sigma_b = n + n(n-1)A_2$ to deduce $A_2 = 1/(1-n)$. The general result can be deduced from an iteration of the above procedure:

$$A_k = 0 \qquad \text{if k is odd}$$

$$A_k = \frac{1}{1-n}\frac{3}{3-n}...\frac{k-1}{k-1-n} \quad \text{if k is even} \qquad (D.4)$$

and this can be written in the form:

$$A_k = \frac{2^{n+1}}{B(-n/2,-n/2)}\int_{-\infty}^{\infty} dy\ (cosh(y))^n/2(tanh(y))^k \qquad (D.5)$$

This expression is well defined for $n < 0$, and can be continued analytically to positive n. B is Euler beta function [44]. From this expression of A_k one can deduce the generating function:

$$g[h_1...h_n] \equiv \sum_{\{\sigma\}}' \exp\left(\sum_a h_a\sigma_a\right) = (\prod_a cosh(h_a)).$$

$$\sum_{k=0}^{n} A_k \sum_{a_1<...<a_k} tanh(h_{a_1})...tanh(h_{a_k})$$
$$= \frac{2^{n+1}}{B(-n/2,-n/2)} \int_{-\infty}^{\infty} dy \prod_{a=1}^{n} cosh(y+h_a) \qquad (D.6)$$

Formula (D.6) is a replica identity which may turn out useful in other contexts. Here we will use it to compute $\overline{v(x)v(y)}$. Starting from (D.2), and using the fact that $\sum_b R_{ab} = 0$ (derived in Appendix A), we have:

$$\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}} = c\sum_{\{\sigma\}}' \exp\left(\frac{\alpha}{2}\sum_{a,b} R_{ab}\sigma_a\sigma_b\right) \qquad (D.7)$$

where α is defined as:

$$\alpha = \frac{(x-y)^2}{4} \qquad (D.8)$$

We call as usual $\tilde{r}, r_1, r_0$ the various elements of the hierarchical R_{ab} matrix. Using the fact that $r_0 = 0$, we get from (D.6):

$$\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}} = \frac{2^{n+1}}{B(-n/2,-n/2)} \exp\left(\alpha\frac{n}{2}(\tilde{r}-r_1)\right)$$
$$\int_{-\infty}^{\infty} dh_0 \left[\int Dh \cosh^m(h_0 + h\sqrt{\alpha r_1})\right]^{n/m}$$
$$= \frac{2^{n+1}}{B(-n/2,-n/2)} \exp\left(\alpha\frac{n}{2}(\tilde{r}-r_1+mr_1)\right)$$
$$\int_{-\infty}^{\infty} dh_0 e^{nh_0} \left[\int Dh \left(1+e^{-2h_0-2h\sqrt{\alpha r_1})-2m\alpha r_1}\right)^m\right]^{n/m} \qquad (D.9)$$

where $Dh \equiv dh/\sqrt{2\pi}\exp(-h^2/2)$. Changing variables to $\mu = \exp(-2h_0)$ and $h = m\sqrt{r_1}(z-x)\mathrm{sign}(x-y)$, and using the fact that $\tilde{r} - r_1 + mr_1 = 0$, we derive:

$$\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}} = \frac{2^n}{B(-n/2,-n/2)} \int_0^{\infty} \frac{d\mu}{\mu}\mu^{-n/2}$$
$$\left[\frac{m\sqrt{r_1}}{2\pi}\int dz \left(e^{-mr_1(z-x)^2/2} + \mu e^{-mr_1(z-y)^2/2}\right)^m\right]^{n/m} . \qquad (D.10)$$

Having computed $\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}}$, the velocity correlation is easily deduced as:

$$\overline{v(x)v(y)} = 4\nu^2\overline{\left(\frac{Z'(x)}{Z(x)}\frac{Z'(y)}{Z(y)}\right)} = \lim_{n\to 0}\frac{16\nu^2}{n^2}\frac{\partial^2}{\partial x\partial y}\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}} \quad (D.11)$$

It is straightfoward to check that this replica computation agrees with the direct physical space computation of Appendix B. More precisely we recover the expression (B.5),(B.10),(B.11) for the correlation, using the fact that $mr_1\delta = 1$.

Figure Captions

Fig. 1: Typical snapshots of the velocity field as given by Eq. (4.1), with $m = \frac{1}{Re} = 10^{-2}$, in one dimension $N = 1$ (Fig. 1-a), or two dimensions $N = 2$ (Fig 1-b, where we have plotted in grey levels the modulus of the velocity field).

Fig 2: Comparison of the numerically determined velocity correlation function (directly from Eq. (4.1)), and our analytical formula, obtained after integrating Eq. (4.7), multiplied by u^2. Note the linear regime at small $x-y$.

Fig 3. Sketch of the 'multifractal' spectrum $\zeta(q)$, giving the r dependence of the q th moment of the velocity field, both for the forced Burgers' equation (triangles) and hydrodynamical turbulence (circles - see, e.g. [13]). Note that $\zeta(3) = 1$ for both models (Kolmogorov's scaling). Intermittency corrections (i.e. the departure of $\zeta(q)$ from $\frac{q}{3}$, as given by the dashed line) are much stronger in Burgers' turbulence: this is due to the fact that singularities are concentrated on $N-1$ dimensional structures, rather than on vorticity tubes.

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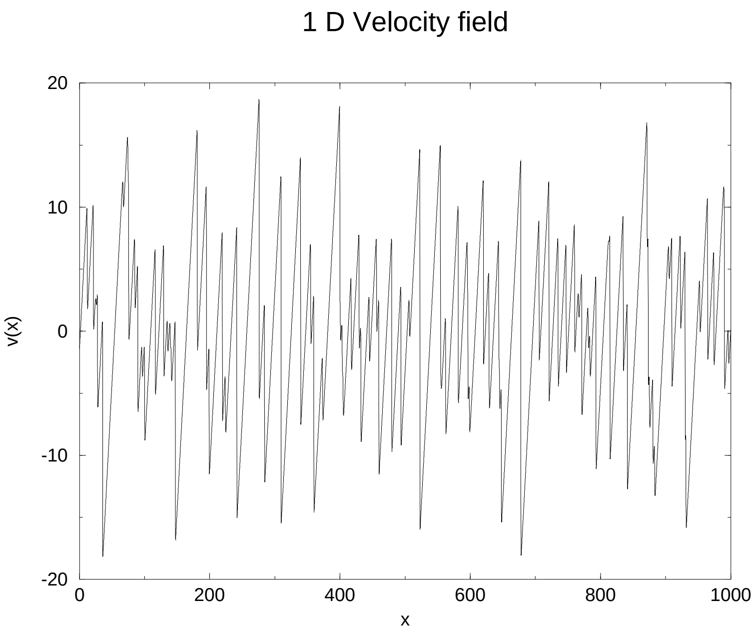
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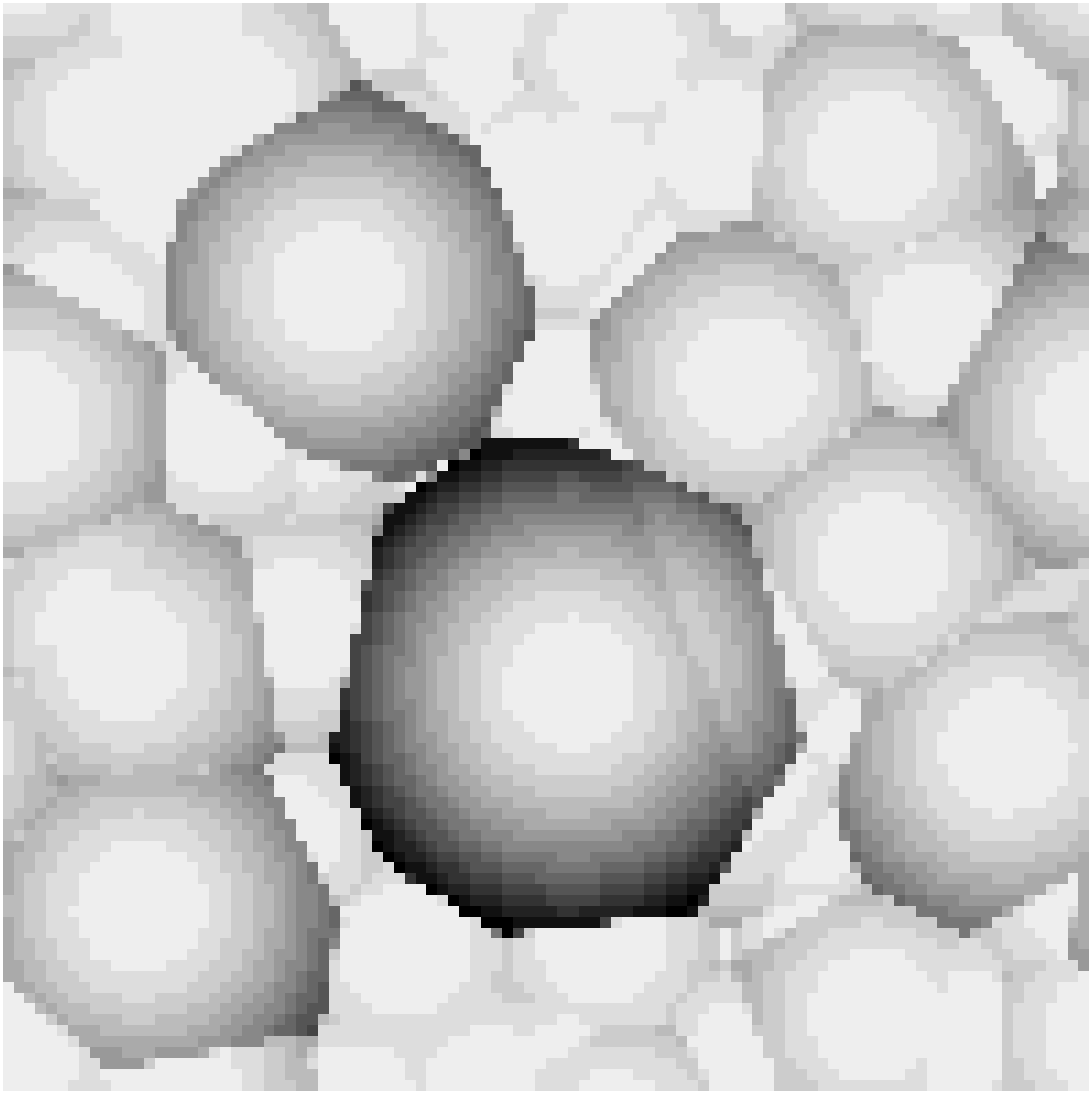

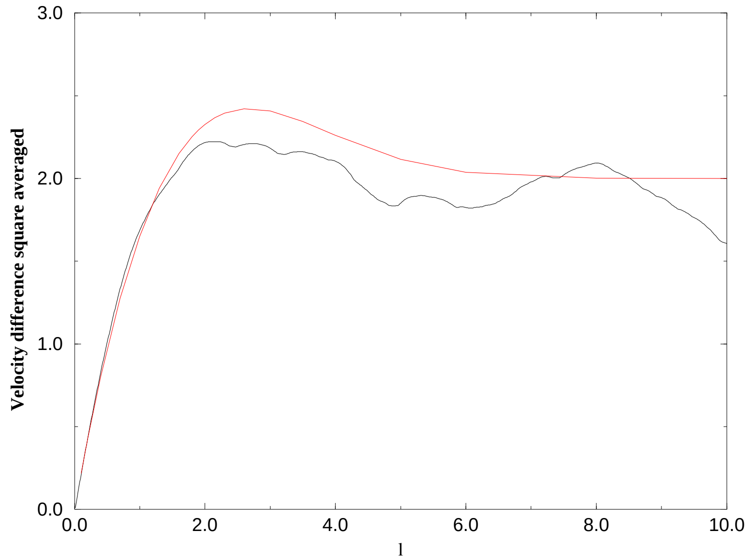
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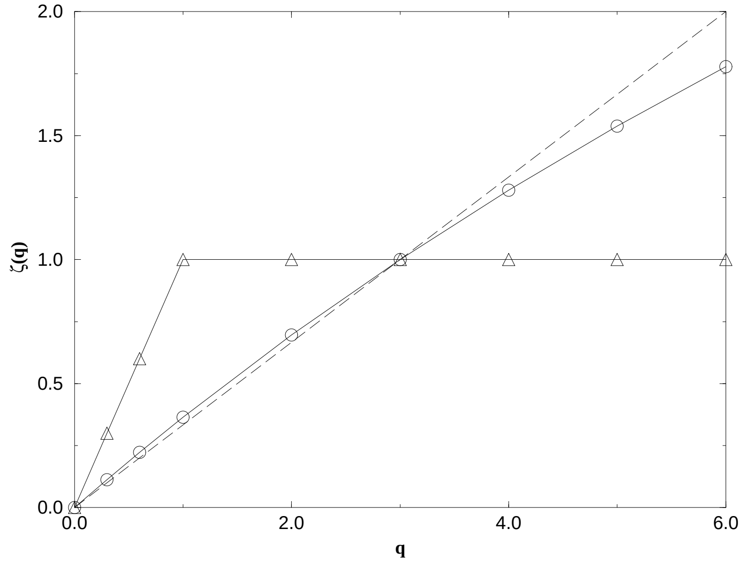
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