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More on the exponential bound of four dimensional simplicial quantum gravity

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Abstract

A crucial requirement for the standard interpretation of Monte Carlo simulations of simplicial quantum gravity is the existence of an exponential bound that makes the partition function well-defined. We present numerical data favoring the existence of an exponential bound, and we argue that the more limited data sets on which recently opposing claims were based are also consistent with the existence of an exponential bound.

1 Introduction

Theories that we hope could be connected to Euclidean quantum gravity can be investigated through Monte Carlo simulations of discrete models, one of the most popular approaches being dynamical triangulations of simplicial quantum gravity. Here we consider the four dimensional theory without matter but with a cosmological constant for fixed topology $\mathcal{M} = S^4$ [1, 2, 3, 4]. The most attractive feature of this model is that there is evidence for a second order phase transition and therefore a continuum limit may exist.

In this language the Einstein-Hilbert action is very simple. It depends only on the total number N_i of i-simplices contained in the triangulation T:

$$S[T] = k_4 N_4[T] - k_2 N_2[T] \ . \tag{1}$$

The coupling constants k_4 and k_2 are directly related to the cosmological constant λ and Newton's constant G,

$$k_4 = \lambda - \frac{10}{G} \ , \quad k_2 = \frac{2\pi}{\alpha G} \ , \tag{2}$$

where $\alpha = \arccos(\frac{1}{4})$. The grand canonical partition function of the theory is

$$\begin{aligned} Z(k_4, k_2) &= \sum_T e^{-S[T]} \\ &= \sum_{N_4} e^{-k_4 N_4} \sum_{T:|T|=N_4} e^{k_2 N_2[T]} \ , \end{aligned} \tag{3}$$

where we have split the sum over all triangulations of $\mathcal{M}$ into a sum over all possible volumes (equal to the number of 4-simplices N_4) and a sum over all triangulations T with volume $|T|$ equal to N_4, i.e. the second sum gives the canonical partition function $Z(N_4, k_2)$.

The question that we want to address in this note, and which has been prompted by [5] and further discussed in [6, 7], is whether the partition function of such a model is actually well-defined, i.e. finite. Suppose that there exists an exponential bound for the canonical partition function,

$$Z(N_4, k_2) \sim e^{k_4^c(k_2) N_4} \ , \tag{4}$$

for large N_4 and some constant $k_4^c(k_2)$. Then the partition function $Z(k_4, k_2)$ is finite for $k_4 > k_4^c(k_2)$ and divergent for $k_4 \leq k_4^c(k_2)$.

The question of the existence of an exponential bound for the canonical partition function is directly related to the asymptotic behavior of the number of triangulations for a given volume, $\mathcal{N}(N_4)$, which might grow as fast as $(5N_4)!$. Since $2N_4 < N_2 < 4N_4$,

$$\mathcal{N}(N_4) \leq Z(N_4, k_2) \leq e^{4k_2 N_4} \mathcal{N}(N_4) \qquad \text{if } k_2 \geq 0, \tag{5}$$

$$Z(N_4, k_2) < e^{2k_2 N_4} \mathcal{N}(N_4) < \mathcal{N}(N_4) \qquad \text{if } k_2 < 0. \tag{6}$$

Hence the existence of an exponential bound on $\mathcal{N}(N_4)$ implies the same for the canonical partition function for arbitrary k_2, and if there exists an exponential bound on the canonical partition for a single value $k_2 \geq 0$ then it exists for all k_2.

In the next section we discuss how the absence or presence of an exponential bound manifests itself in the numerical simulations, but first let us summarize the history of the subject. Until recently the existence of an exponential bound for the number of triangulations for large volumes was considered very probable. The bound can be rigorously proven in two dimensions [8]. In three dimensions there is strong numerical evidence for its existence [9], and in four dimensions the numerical data still seemed reasonably consistent with that claim (e.g. [1, 2, 3, 4]).

However recently the authors of [5] have claimed that their more detailed examination of new numerical data shows the absence of an exponential bound. (In [5] the coupling constants are $\kappa_0 = 2k_2$ and $\kappa_4 = k_4 - 2k_2$.) Their main points are:

1. In $2d$ everything is fine, and the bound is manifest.

2. In $3d$ $k_4^c(N_4)$ has strong power corrections to a (to be proved) asymptotic constant behavior. One does not see the asymptotic behavior, but the transient behavior is power like.

3. In $4d$ $k_4^c(N_4)$ is not constant, but changes far less than in $3d$. The authors of [5] assume this means that in this case there is a logarithmic divergence of k_4^c (while the behavior in $3d$, far more abrupt, was assumed to be convergent) that implies a violation of the exponential bound.

4. The situation in $d \neq 4$ is mainly discussed for $k_2 = 0$, while for $4d$ data is presented also for $k_2 = 0.25$ and 0.50, all lower than $k_2^c \approx 1.1$, based on the idea that proving existence of the exponential bound at one value of $k_2 \geq 0$ is sufficient for proving that the partition function is convergent everywhere.

After [5], two papers have discussed the issue in further detail, both observing that a logarithmic and a small power fit are reasonably consistent with their data. The authors of ref. [6] lean towards the logarithmic scaling, and they propose for this case an interesting, potentially still well-defined scenario, on which we will comment below. Ref. [7] argues in favor of a power law approach to an exponential bound, proposing the ansatz of a leading power $\alpha = \frac{1}{4}$. They fit the old and some new data that look quite consistent with the ansatz.

Here we present data for $d = 4$ and $k_2 = 0$ for the largest system size yet, and conclude that the extension to larger systems allows one to decide that there does exist an exponential bound — if one is willing to accept as a compelling evidence the fact that the ratio of χ^2 of the factorial fit to χ^2 of the exponential fit is 10. We discuss the strength and the implications of such a result.

Points (1) through (4) remind us that there is an important issue to be clarified, but also make the numerically oriented physicist quite perplex. It is well known indeed that measuring logarithmic corrections is very difficult (for a classical study see for example [10]). So in the case of point (2) it is impossible to exclude that there could be a logarithmic divergence, underlying a transient behavior dominated by power law corrections. The main point in all the following will be that we are discussing a transient region, with an overlap of different possible corrections, which could all be present at the same time.

The same kind of perplexities arise about point (3). When numerical data taken in some limited range of the parameters are compatible with a logarithmic fit they can also be fitted by a power law with a small exponent. What one can achieve is putting a bound on the allowed value of the power, and in our case we will show that indeed a convergent power dependence has to be preferred over a logarithmic scaling.

We will also discuss in more detail the different k_2 regimes (the crumpled and the elongated phases), but already the evidence contained in [3] makes the appreciation of point (4) quite delicate. Indeed, figure 2 of [3] (also [1], and later figure 2 of [6], to be compared) shows that in the two different phases $k_4^c(N_4)$ is behaving in a very different way. In the elongated phase $k_4^c(N_4)$ does not depend at all on the volume, while the residual dependence is all in the crumpled phase. It is possible that because of strong finite size effects in the crumpled phase, reliable numerical data about the large volume limit can only be obtained above the apparent phase transition for $k_2 > k_2^c \approx 1.1$.

Finally, in [11] hints are given toward the fact that in $4d$ the number of triangulations could be exponentially bounded. In three dimensions, there does not yet exist a proof (e.g. [12]). Independent of any analytical arguments, however (and, as far as we know and understand, we cannot yet rely on a rigorous proof), one should arrive at the best possible understanding of the numerical data. This point we will address below, stressing again that we are talking here about the nature of a transient region, where different corrections of unknown form may conspire to make the picture difficult to disentangle.

2 Results for $k_2 = 0$

When simulating the system described by the partition function $Z(k_4, k_2)$ (3), with variable volume, one finds that there exists a line $k_4^c(k_2)$ in the plane of the coupling constants such that for any k_2, if k_4 is larger than k_4^c the volume tends towards zero (towards six to be precise), and if k_4 is smaller than k_4^c the volume goes to infinity. The larger the deviation from k_4^c the faster is the trend. For these reasons $k_4^c(k_2)$ is often called the critical line (although it has nothing to do with statistical criticality [4]). This critical line at finite volumes represents the behavior one expects from the discussion of the exponential bound in the introduction.

A typical Monte Carlo simulation is performed close to a fixed volume N_4 and for fixed k_2, while k_4 is kept close to k_4^c which is determined dynamically during the simulation. One then looks for a phase transition determined by some $k_2 = k_2^c$ in the limit of large volume. How $k_4^c(k_2)$ is measured is a purely technical question, and the particular algorithm we use is described in detail in [3].

In figure 1 we show k_4^c versus $\ln(N_4)$ for $k_2 = 0$ $(1/G = 0)$. Here we are mainly discussing the point $k_2 = 0$ because this is where in [5] the strongest argument against an exponential bound is made. If instead of an exponential bound only a factorial bound holds, then one expects

$$k_4^c(N_4) = a + b \ln N_4 \ . \tag{7}$$

If an exponential bound e^{aN_4} to the canonical partition function exists, then

$$k_4^c(N_4) = a + bN_4^{-\alpha} \ , \tag{8}$$

where $N_4^{-\alpha}$ represents a natural polynomial correction to the exponential. One can argue for $\alpha = 1/4$ [7], which allows a very nice fit to the data. Since there is not enough data to determine α reliably, setting $\alpha = 1/4$ serves well enough to distinguish the exponential from the factorial fit.

In [5], a straight line (we draw our best logarithmic fit with dashes in figure 1) has been chosen as the best fit to data points between $N_4 = 1k$ and $32k$, corresponding to the absence of an exponential bound due to factorial growth. In [7], a polynomial fit for $\alpha = 1/4$ is preferred for N_4 from 4k to 64k, representing the existence of an exponential bound which is approached only for still larger volumes. Volumes of $1k$ and $2k$ 4-simplices have not been included in the fits since they are likely to suffer from strong finite size effects. The data of [5, 6, 7] suggest that we require more data at larger volumes for a reasonable estimate of the asymptotic behavior.

It is just a matter of computer time to take data at larger volumes — more so than a matter of computer memory, and we estimate that our implementation uses only about one fifth the memory of the implementation of [7]. We have been able to get reliable data up to a volume of $128k$ (50000 sweeps, counting N_4 moves actually performed, for

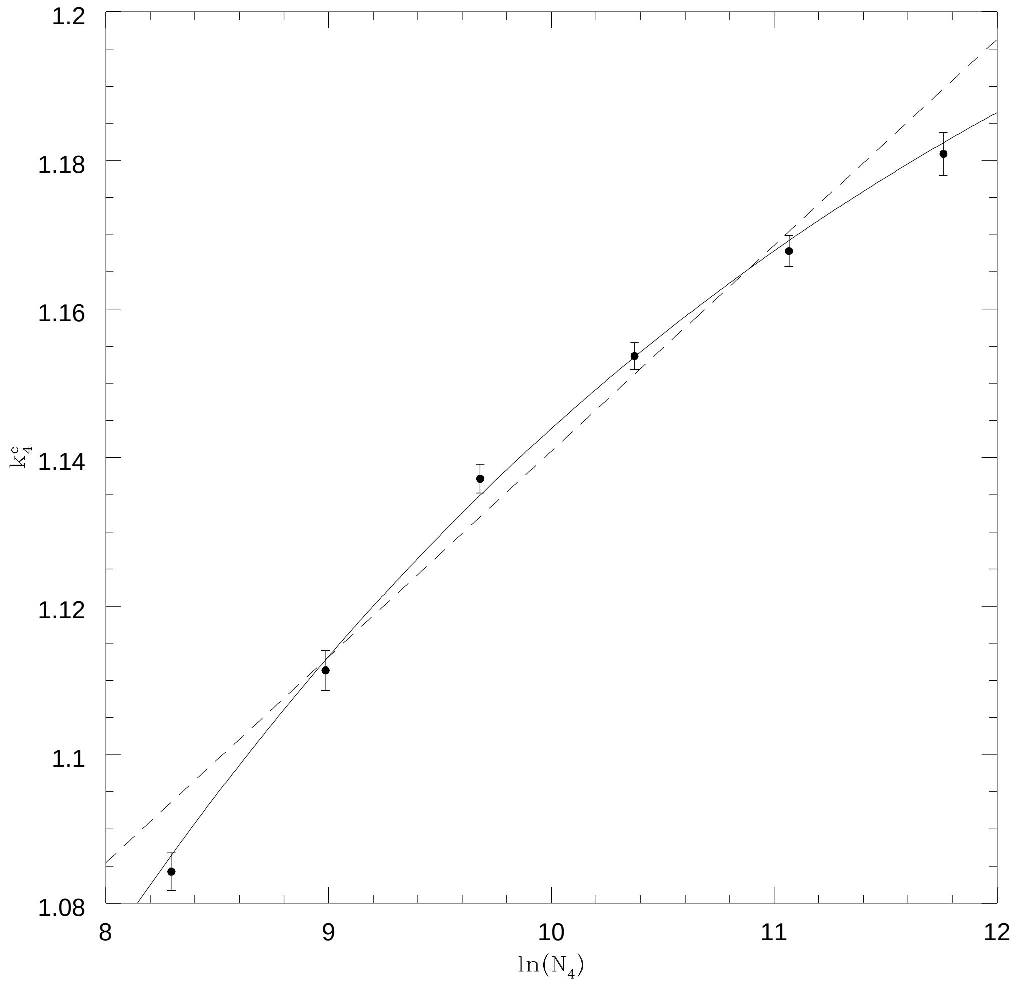

Figure 1: k_4^c versus $\ln(N_4)$ for $k_2 = 0$. Values for N_4 are 4000, 8000, 16000, 32000, 64000 and 128000. With the dashed line we give our best logarithmic fit, with the solid line we give our best fit to a converging power, with $\alpha = .25$. Both fits have two free parameters. The χ^2 of the power fit is ten times better than the one of the logarithmic fit.

$N_4 = 4k$ through $32k$, about 30000 sweeps for $N_4 = 64k$ and $128k$, 10000 sweeps are discarded for thermalization, the largest volume taking four months on a (shared) IBM/RISC workstation, autocorrelation time was on the order of 50 sweeps).

It is remarkable that when superimposing our new points to the fits of ref. [7] they fall very well on the power fit (quite far indeed from the logarithmic divergence prediction) obtained from smaller triangulations.

We have fitted our data for $k_2 = 0$ with the two forms (7) and (8), by setting the power $\alpha = \frac{1}{4}$. They are both two parameter fits. Figure 1 is quite eloquent about the success of the two fits. The result is

$$k_4^{c(log)} = 0.864 + .0277 \ln N_4 \ , \tag{9}$$

$$k_4^{c(power)} = 1.252 - 1.317 N_4^{-\frac{1}{4}} \ . \tag{10}$$

The power fit has a value of the χ^2 which is ten times better than the logarithmic one. We have also tried 3 parameter fits. In the power fit we have left the power as a free parameter, while in the logarithmic fit we have added a volume scale term N_4^0, as in $\ln(N_4 - N_4^0)$. Both fits improve quite a lot, but the power fit stays far superior to the logarithmic fit (the χ^2 ratio is now 3). While such a power fit (where the best power is now $.36 \pm .04$) matches perfectly the data points, the logarithmic fit is still not totally congruent to the data (we get N_4^0 of order 3000, that is a reasonable scale for the transient behavior). We are not very confident in playing with many parameters, since the allowed corrections are of many different functional forms, and it is clear that with 6 data points they cannot be distinguished. We just take the results of the 3 parameter fits as further evidence that the power fit is superior to the logarithmic fit. Let us also note that indeed the best preferred power is surely not too small.

The conclusion we draw is that *the fits of the numerical data largely favor the existence of an exponential bound at $k_2 = 0$ over the presence of a factorial bound.*

3 Discussion

What about the consistency of the numerical data? The first observation about the data in figure 1 should really be that there is a remarkable agreement in the data from four independent computer implementations considering that the underlying algorithms are somewhat similar but not identical. In fact, notice that even the data from [5] that lead to the claim about the absence of an exponential bound curves away from a straight line in the same way the other data sets do.

Having analyzed in detail the situation for $k_2 = 0$, we now turn to *generic* values of the coupling k_2. In theory, the existence of an exponential bound for any one value of $k_2 \geq 0$ implies existence for all the others. But as is well known, but has not been discussed in detail in this context, there is an important practical difference between the phases for k_2 below and above the critical value $k_2^c \approx 1.1$. For large positive k_2 the simplicial complex is in an elongated phase with an intrinsic dimension close to two, while for negative k_2 the intrinsic dimension diverges to infinity and the simplicial complex becomes extremely crumpled. One of the most intriguing and attractive features of simplicial quantum gravity is that at k_2^c the intrinsic dimension is close to four [1, 13] (for simplicity we ignore here the problem of giving the best definition of the intrinsic dimensionality of the system).

The point is that the two phases are not only different, but there is a genuine asymmetry. Note that at k_2^c the intrinsic system size for $N_4 = 10,000$ is of the order of $(10,000)^{1/4} = 10$, while at $k_2 = 0$ it is $(10,000)^{1/10} \approx 2.5$. Therefore, what constitutes a *large volume* that guarantees the absence of finite size effects depends very sensitively on the value of k_2 [3]. For example, the asymmetry in the susceptibility present in these systems may be due to such effects.

With regard to the discussion of the exponential bound one should therefore consider the whole k_2 range. Such data already exists in [1, 3] and were improved upon near the transition in [6] but were not considered in [5, 7]. For concreteness we show in figure 2 a plot of $\lambda^c(k_2)$ versus $\lambda_0 \sim 1/G$ for $N_4 = 4k, 8k, 16k$ based on [3], which for our purpose is better suited than the more accurate data of [6] since figure 2 extends to extreme values of k_2. The constants are defined by the relations

$$k_2 = 2\pi\lambda_0 \ , \quad k_4 = \lambda + 10\alpha\lambda_0 \ . \tag{11}$$

There is a definite volume dependence for $k_2 < k_2^c$ while above the transition no volume effect is discernible. The linear transformation from k_2 and k_4 to the cosmological constant λ is useful for magnifying the volume dependence which is invisible in this range of coupling constants for $k_4^c(k_2)$ [3]. This is discussed [6], but even when explicitly looking for a small volume dependence for k_2 clearly above k_2^c, none is found. In this region the plot analogous to figure 1 appears to be a perfectly horizontal straight line, i.e. there are no detectable polynomial corrections to the exponential bound.

The discussion can be taken one step further by noticing that the critical value k_2^c of k_2 moves to larger values with increasing volumes [1, 6]. For larger volumes at $k_2 = 0$ the finite size effects become even more pronounced (internal dimension up to 50). Given that for extreme values of k_2 the simplicial complex freezes and $k_4^c(k_2)$ becomes a perfect straight line with different slopes, the shift in k_2^c keeping the part $k_2 > k_2^c$ in figure 2 fixed translates directly into (part of) the volume dependence in the range $k_2 < k_2^c$. When this effect constitutes the significant part of the volume dependence (for large enough volume), then the volume dependence of the critical value k_2^c can be estimated by the volume dependence of k_4^c for a small enough but fixed value of k_2. In particular, if there is no exponential bound, then $k_2^c \to \infty$ with $N_4 \to \infty$.

It is instructive to examine the condition for the critical line in the Monte Carlo simulations (here we follow [3, 4]). Consider the ergodic random walk in the space of triangulations of S^4 consisting of the five standard moves, where for the move of type i an i-simplex is replaced by a $(4-i)$-simplex. On the critical line, the average volume N_4 is constant, and therefore N_2 must also be constant since it is bounded. This means that the average variations $\overline{\delta N_j}$ must vanish,

$$\overline{\delta N_j} \sim \sum_{i=0}^{4} \Delta N_j(i) p_i = 0, \tag{12}$$

where p_i is the probability with which a move of type i is performed on average, and $\Delta N_j(i)$ is the change in N_j due to that move. Since the moves are independent, we obtain

$$p_0 = p_4, \quad p_1 = p_3, \tag{13}$$

on the critical line.

Since the action is linear in N_2 and N_4, and since the moves are local, we can be more specific about the conditions on the p_i. The p_i can be chosen to be

$$p_i = [e^{-\Delta S(i)}]\, p_i^{geo} \ . \tag{14}$$

The bracket is the Metropolis weight, its key feature being that it depends only on the type of move and not on the N_i or the triangulation in general. While the action looks quite trivial, all the non-trivialities are hidden in the probability p_i^{geo} for a move to be allowed

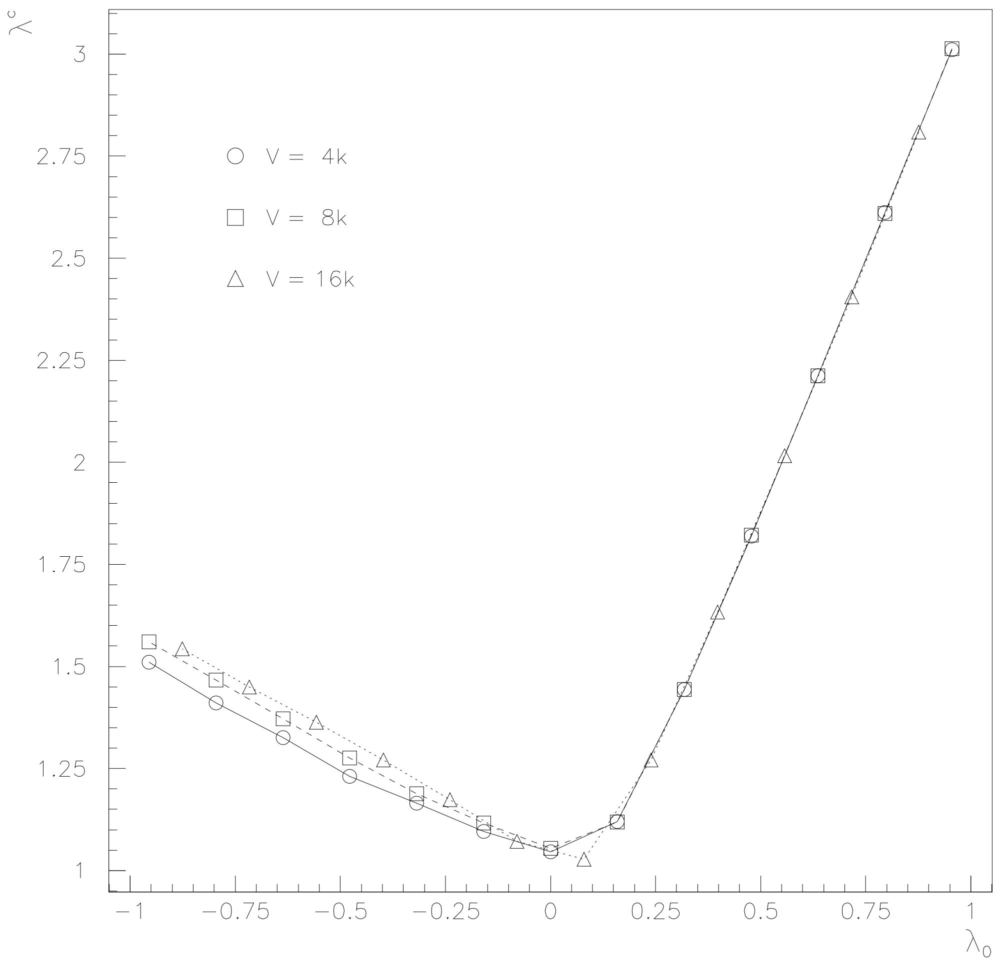

Figure 2: λ^c versus λ_0. Indications of a phase transition are found near $\lambda_0 = 0.18$.

by the geometric constraints on the triangulation. (Detailed balance is incorporated in the way the moves are chosen. Potentially, there is a factor of order $O(1/N_4)$.)

Therefore (13) is equivalent to

$$k_4^c = \frac{5}{2}k_2 - \ln p_0^{geo}, \qquad (15)$$

$$k_4^c = 2k_2 - \ln \frac{p_1^{geo}}{p_3^{geo}}, \qquad (16)$$

where we have used that $p_4^{geo} = 1$. The question of the existence of an exponential bound has therefore been translated into the question whether there exist appropriate bounds on the $p_i \equiv p_i(k_4, k_2, N_4)$ which are independent of N_4.

First of all, $p_0 \leq 1$ implies that k_4^c is bounded from below by $2.5k_2$. The hard part is to find a suitable lower bound on p_0, for example, and although it may be possible to do so by some more detailed analysis of the space of triangulations, we do not have a conclusive argument. Notice that since moves of type 4 are always allowed, we have that $p_0 > 0$. However, a naive counting of possible moves of type 0 and 4 around a fixed background triangulation gives $p_0 \sim 1/N_4$, which would be the divergent scenario, but the same kind of counting would also make 2d divergent. The counting is, of course, difficult because moves of type 1, 2, and 3 may change the geometric constraints.

Coming from the numerical side, it is quite suggestive that e.g. the data for $N_4 = 4k$ in figure 2 corresponds to $k_4^c(k_2 \geq 4.0) = 2.497k_2 + c$ and $k_4^c(k_2 \leq -4.0) = 2.002k_2 + d$. This means that in the extreme k_2 regions the relevant geometric probabilities must be independent of k_2 (combining (15) with (16) gives a factor of $\exp(k_2/2)$ for the opposite side).

Considering the general structure of the phase diagram, the volume dependence can also be understood on the level of the random walk as follows. It is the moves of type 4 that drive the system into the elongated phase ($\Delta N_2(4)/\Delta N_4(4) = 2.5$), while moves of type 1 drive to the crumpled phase ($\Delta N_2(3)/\Delta N_4(3) = 2.0$). Depending on k_2 the random walk is driven towards one of the bounds in $2 < N_2/N_4 < 4$. One of the two possible phases, the elongated phase, is therefore characterized by low order vertices, and the average order does not depend on N_4 since a maximal elongation can be obtained for a rather small number of simplices. Hence p_0^{geo}, which is the ratio of the number of vertices of minimal order to the number of all vertices, is expected to be independent of N_4 in the elongated phase. On the other hand, in the crumpled region the average order of vertices is driven towards large values, and the average order will grow with N_4. Hence p_0^{geo}, which is defined by the low order tail of the vertex order distribution, goes to zero with N_4 in the crumpled phase. Equation (15) gives the corresponding volume dependence of k_4^c.

In conclusion, when looking for evidence for an exponential bound in the numerical data of simplicial quantum gravity in four dimensions, one should take the whole range of k_2 into account. If one insists on looking in the crumpled phase at $k_2 = 0$, the numerical data strongly support the validity of an exponential bound.

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