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Monte Carlo simulations of 4d simplicial quantum gravity[1]

Bernd Brügmann

Max-Planck-Institut für Physik,
Föhringer Ring 6, 80805 München, Germany
bruegman@mppmu.mpg.de

Enzo Marinari

Dipartimento di Fisica and Sezione Infn, Università di Cagliari,
Via Ospedale 92, 09100 Cagliari, Italy
marinari@ca.infn.it

Abstract

Dynamical triangulations of four-dimensional Euclidean quantum gravity give rise to an interesting, numerically accessible model of quantum gravity. We give a simple introduction to the model and discuss two particularly important issues. One is that contrary to recent claims there is strong analytical and numerical evidence for the existence of an exponential bound that makes the partition function well-defined. The other is that there may be an ambiguity in the choice of the measure of the discrete model which could even lead to the existence of different universality classes.

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1 Introduction

Dynamically triangulated random surfaces (DTRS) [1] play an important role in the efforts to develop a coherent description of quantum gravity. The (euclidean) space-time is approximated by a d-dimensional simplicial triangulation, where the link length is constant, equal to 1, but the connectivity matrix is a dynamical variable.

The most important advances have been obtained in two-dimensional quantum gravity, where DTRS are simplicial triangulations of a $2d$ manifolds. The analytic success of matrix models, which can be for example exactly solved in the case of pure $2d$ gravity [2], has strongly encouraged this approach. The results obtained in the triangulated approach and in the continuum lead to consistent predictions for correlation functions and critical exponents.

Dynamical triangulations are also potentially relevant in four dimensions. One can hope that a sensible, non-perturbative definition of the quantum gravity theory can be obtained in some *scaling* limit of the theory of $4d$ hyper-tetrahedra. This approach has much in common with Regge calculus, where the connectivity is fixed but the functional integration runs over the link lengths. The underlying principle is clearly very similar, and one could say that DTRS have the status of an improved Regge calculus. The fact that the coordination number can vary in the DTRS makes it easier to describe a situation in which long spikes play an important role.

We face the usual problem inherent in discretizing a theory, i.e. the discretization scheme can break some of the continuous symmetries, which will have to be recovered in the continuum limit (if there is one). Indeed, Wilson lattice gauge theories have taught us an important lesson. The fact that gauge invariance is exactly conserved in the lattice theory, for all values of the lattice spacing a, is in that case crucial: it would have been very difficult to establish firm numerical results if one would have had to care about the presence of non gauge-invariant corrections, which would disappear only in the $a \to 0$ limit. In the case of quantum gravity, diffeomorphism invariance plays such a crucial role, and DTRS are diffeomorphism invariant by construction, at least on the space of piecewise flat manifolds. Hence part of the difficulties Regge calculus has in forgetting about the lattice structure are eliminated a priori in the DTRS lattice approach. The results of [3] actually show that the conventional application of Regge calculus to quantum gravity in two dimensions fails to reproduce the analytical results, although a more sophisticated approach may still succeed [3].

There are two more important points to stress. The first one is that in the DTRS approach in $3d$ and $4d$, as opposed to $2d$, we can try to make sense out of the pure Einstein action, without, for example, curvature squared terms. Even though the partition function formally diverges, at fixed volume the local curvature is bounded both from below and from above. Therefore we can study the theory at fixed (or better quasi-fixed, see later) volume, and look for the existence of a stable fixed point in the large volume limit. A second order phase transition with diverging correlation lengths, in the statistical mechanics language, would allow us to define a continuum limit which is universal and is not influenced by the details of the underlying discrete lattice structure. Precisely this scenario constitutes one of the best hopes we have to find a consistent quantum theory of gravity. It could be a way to give a non-perturbative definition of euclidean quantum gravity based on the Einstein action.

On the other hand, we believe that there are at least four potentially deep issues in $4d$ simplicial quantum gravity on which the final success of this model hinges. The first one is related to the unrecognizability of 4-manifolds, which may invalidate the whole Monte Carlo approach. The second one is the question whether there exists an exponential bound for the partition function such that the model is well-defined. This question is likely to have been settled in the affirmative recently. Third, it is quite unclear what role the unknown measure of the path integral approach to quantum gravity plays in the triangularized model, and it may well be that there are different universality classes. Finally, since Euclidean general relativity cannot in general be extended to the physical, Lorentzian sector, one may wonder what the appropriate physical observables in the theory are.

Our contribution to this volume will be to exemplify concrete numerical problems in the four-dimensional case. In section 2, we introduce the model and Monte Carlo simulations. We comment on the issue of ergodicity and state what the simplest observables are that we evaluate. In section 3, we introduce the problem of the exponential bound and describe a detailed analysis of the numerical simulations addressing this problem. In section 4, we present some preliminary results about the universality structure for a one-parameter family of measures that contains the trivial measure as a special case, which is the one considered in all the other investigations. We close with a summary in section 5.

2 Simplicial quantum gravity in four dimensions

2.1 The Model

The model we consider is based on the four-dimensional Euclidean Einstein-Hilbert action,

$$S_E[g] = \lambda \int d^4x\sqrt{g} - \frac{1}{G}\int d^4x\sqrt{g}R(g), \tag{1}$$

where λ is the cosmological constant, G is the gravitational constant, g is the determinant of the metric, and $R(g)$ its scalar curvature.

As discussed in [4, 5], if one considers only manifolds which are simplicial complexes with S^4 topology and defines the metric by the condition that all edges have length 1, then the volume integral V and the net scalar curvature R can be replaced by

$$V \equiv \int d^4x\sqrt{g} \quad \leftrightarrow \quad N_4[T], \tag{2}$$

$$R \equiv \int d^4x\sqrt{g} \quad \leftrightarrow \quad \frac{2\pi}{\alpha}N_2[T] - 10N_4[T], \tag{3}$$

where $N_i[T]$ denotes the number of i-simplices of the triangulation T and α is derived from the condition that for approximately flat triangulations the curvature vanishes, $\alpha = arccos(1/4) \approx 1.318$. The action takes a very elegant and simple form [4, 5],

$$S_E[T] = k_4N_4[T] - k_2N_2[T], \tag{4}$$

where the coupling constants are

$$k_4 = \lambda + 10/G, \quad k_2 = \frac{2\pi}{\alpha G}. \tag{5}$$

Notice that (4) is the most general action of the type $S_E = \sum_i N_i$ in four dimensions. Euler's relation for S^4 and the Dehn-Sommerville relations leave only two of $N_0, \ldots, N_4$ independent. The number of vertices N_0 in the simplicial complex for example is

$$N_0 = \frac{1}{2}N_2 - N_4 + 2. \tag{6}$$

In addition, there are inequalities between the N_i. Denote by $o(a)$ the order of vertex a, i.e. the number of four-simplices that contain a. For the average order of simplices we have

$$5 \leq \frac{1}{N_0} \sum_a o(a) = \frac{5N_4}{N_0} < \infty, \tag{7}$$

which implies

$$2 < \frac{N_2}{N_4} < 4. \tag{8}$$

Hence the average curvature is asymmetrically bounded from below and above, $-0.614 < R/N_4 < 12.0$, which is a reflection of the fact that the choice of S^4 topology restricts the choice of possible metrics.

The purpose of the above discretization is to make the path integral well-defined,

$$\int \mathcal{D}g \, e^{-S_E[g]} \leftrightarrow \sum_T e^{-S_E[T]}, \tag{9}$$

where the integration over all metrics is replaced by a summation over all triangulations with S_4 topology. By itself, this replacement is not sufficient to define a finite partition function, rather there are conditions on the coupling constants as we will discuss in section 2. Notice also that although triangulations allow us in principle to perform a summation over different topologies, this sum is known to diverge badly.

In conclusion, we study a rather simple looking model for Euclidean quantum gravity defined by a grand canonical partition function $Z(k_4, k_2)$ and a canonical partition function $Z(N_4, k_2)$,

$$\begin{aligned} Z(k_4, k_2) &= \sum_{N_4} e^{-k_4 N_4} Z(N_4, k_2), && (10) \\ Z(N_4, k_2) &= \sum_{T:|T|=N_4} e^{k_2 N_2[T]}, && (11) \end{aligned}$$

where we have split the sum over all triangulations of S^4 into a sum over all possible volumes (equal to the number of 4-simplices N_4) and a sum over all triangulations T with volume $|T|$ equal to N_4.

The existence of a non-trivial continuum limit of the theory would manifest itself as a critical point, with a diverging correlation length and non-trivial critical exponents. A diverging correlation length allows physical observables to forget about the original discrete nature of the model. The issue of the existence of such a limit, and of the features which would characterize such a continuum theory is the main point of the discussion we are summarizing here.

2.2 Monte Carlo simulations of the model

The Monte Carlo evaluation of the partition function (10) is largely standard once an ergodic random walk through the space of triangulations has been defined (see e.g. [6] for a detailed description). Notice that while for dynamical triangulations the action of gravity has become a very simple linear function of two global numbers, the non-trivial part of the theory is represented by the complexity of the space of triangulations. This is reflected in the ergodicity problem of the random walk: while there does exist a simple set of five local moves that is ergodic in the space of all triangulations of compact four-manifolds [7], it is not finitely ergodic [8].

Let us first introduce these elementary moves and then comment on the ergodicity problem. Denoting a four-simplex by its five vertices,

$$abcde \quad \leftrightarrow \quad Aabcd, Aabde, Aacde, Abcde, \tag{12}$$

$$abcdA, abcdB \leftrightarrow ABabc, ABabd, ABacd, \tag{13}$$

$$abcAB, abcAC, abcBC \rightarrow ABCab, ABCac, ABCbc, \tag{14}$$

where $a, b, \ldots$ and $A, B, \ldots$ are the vertices which are common to all four-simplices on the left and right-hand sides, respectively. Notice the regular structure which involves permuting all indices of one type on one side keeping the others fixed and doing the opposite on the other side. Move i is the exchange of i-simplices for appropriate $(4-i)$-simplices. For example, move 0 as given by going from right to left in (12) removes the vertex A common to precisely five 4-simplices creating one new 4-simplex. The two dimensional analog of (12) is adding and removing a vertex inside a triangle.

There are two restrictions on when a move may be performed which ensure that the simplicial complex does not change topology. First of all, move i is only possible if the order of the simplex which is to be removed is $5-i$. For move 0, for example, a vertex can only be removed if it is of order 5, since otherwise its neighboring vertices do not form a four-simplex which always has five vertices (of order 3 in two dimensions). And second, a move is not allowed if it creates simplices which are already present. For example, move 3 — left to right in (13) — introduces an new link AB that, if already part of the simplicial complex, would lead to overlapping four-volumes.

The computer challenge posed by dynamical triangulations is to implement a data structure for the simplicial complex which allows efficient updating under the moves (12)–(14), and which in particular is not static (e.g. [6]). A novel approach to speed up thermalization is called baby-universe surgery [9]. This approach works very well in the elongated phase of the theory (see section 2.3), showing its validity at the critical point and in the crumpled phase might require further study.

Even though the elementary moves introduced above are ergodic, they are not finitely ergodic [8]. This is directly related to a result by Markov, stating that most simplicial four-manifolds are not algorithmically distinguishable. It is not known whether this is the case for S^4. There obvious might be a serious problem for the Monte Carlo simulations if the random walk does not reach physically relevant portions of the space of triangulations. There are numerical attempts to detect the non-distinguishability in Monte Carlo simulations for S^4 without any indication of such [10], but the same is true for S^5 which is known to be non-distinguishable [11]. Regarding these attempts it may well be that the systems studied are far too small or the method of study is not appropriate. The physical relevance of this is quite open. It could very well be that the measure is strongly peaked close to *distinguishable* manifolds, and that the problem does not arise with finite measure in the continuum limit.

A technical problem related to the elementary moves is that no set of elementary moves is known that leave N_4 invariant. In the simulations fixed N_4 is approximated by allowing N_4 to vary in a certain range which is a small fraction of N_4 but as large as feasible to approach ergodicity.

2.3 Basic results

Of interest are the expectation values of observables depending a priori on both k_2 and k_4. When simulating the system described by the partition function $Z(k_4, k_2)$ with variable volume, one finds that there exists a line $k_4^c(k_2)$ in the plane of the coupling constants such that for any k_2, if $k_4 > k_4^c$ the volume is driven towards zero, and if $k_4 < k_4^c$, the volume goes to infinity. The larger the deviation from k_4^c the faster the trend. For these reasons $k_4^c(k_2)$ is often called the critical line (although it has nothing to do with statistical criticality). Whether one can prove the existence of the critical line is the subject of section 3.

A typical simulation is then performed for given k_2 and a fixed volume N_4^0, and $k_4 = k_4^c(k_2)$ is determined dynamically during the run. The same technical trick that allows one to keep the volume near a given value stabilizes the balance between drifting towards zero or infinite volume: one adds an artificial 'potential well' to the action that is dynamically adjusted to be centered at k_4^c.

On the critical line one then looks for a second order phase transition, since this is where a continuum theory might be defined. Indeed, the data are consistent with a continuous phase transition near $k_2^c \approx 1.1$. In [12] the claim is made that the smooth nature of the transition has been proved. One distinguishes two phases of the model, the elongated phase where the Haussdorff dimension approaches 2, and the crumpled phase where the Haussdorff dimension approaches infinity.

How to measure this dimension is somewhat tricky, but it would be directly related to an effective physical dimension, e.g. like the one experienced by a test particle. One of the appealing features of simplicial quantum gravity is that in this way one could derive the number of physical dimensions, and in fact at the critical point the dimension is observed to be close to four (e.g. 4.6 in [14] for the diffusion equation of a heavy test particle).

Finding good physical observables in simplicial quantum gravity is an important problem. Notice that because the quantized metric field is not an observable of quantum gravity (because it is not diffeomorphism invariant), one cannot sensibly study correlation functions based on the configuration variables themselves. Integration of a density over the manifold leads to a diffeomorphism invariant quantity, which is the case for example for the total curvature R, (3). In integrated form one can also study objects like the curvature-curvature correlation [12, 13].

There are many interesting observations about simplicial quantum gravity that could be discussed at this point, but let us now focus on two particular topics as promised in the introduction.

3 Is the partition function well-defined?

As we have explained in section 2, there are many indications that the partition function $Z(k_4, k_2)$ defines a sensible discrete model for quantum gravity. However, in [19] Catterall, Kogut and Renken put forward the claim that the partition function is actually ill-defined since based on their new numerical data it is not exponentially bounded in the large volume limit. This prompted a more careful examination of the partition function, both numerically and analytically, with the net outcome that an exponential is very likely to exist after all.

Let us pose the problem. Finiteness of the partition function $Z(k_4, k_2)$ defined in (10) is related to the existence of an exponential bound for $Z(N_4, k_2)$ as follows. Suppose there exists an exponential bound for the canonical partition function,

$$Z(N_4, k_2) \sim e^{k_4^c(k_2) N_4} , \tag{15}$$

for large N_4 and some constant $k_4^c(k_2)$. Then the partition function $Z(k_4, k_2)$ is finite for $k_4 > k_4^c(k_2)$ and divergent for $k_4 \leq k_4^c(k_2)$.

The question of the existence of an exponential bound for the canonical partition function is directly related to the asymptotic behavior of the number of triangulations for a given volume, $\mathcal{N}(N_4)$, which might grow as fast as $(5N_4)!$. Since $2N_4 < N_2 < 4N_4$,

$$\mathcal{N}(N_4) \leq Z(N_4, k_2) \leq e^{4k_2 N_4} \mathcal{N}(N_4) \qquad \text{if } k_2 \geq 0, \tag{16}$$

$$Z(N_4, k_2) < e^{2k_2 N_4} \mathcal{N}(N_4) < \mathcal{N}(N_4) \qquad \text{if } k_2 < 0. \tag{17}$$

Hence the existence of an exponential bound on $\mathcal{N}(N_4)$ implies the same for the canonical partition function for arbitrary k_2, and if there exists an exponential bound on the canonical partition for a single value $k_2 \geq 0$ then it exists for all k_2.

To summarize what is known analytically, in two dimensions there is the classical result of [15] giving an explicit exponential bound on the number of triangulations. In three dimensions, there does not yet exist a proof and the situation is quite subtle, e.g. [16]. In [17], a novel method for counting minimal geodesic ball coverings of two- and four-dimensional manifolds is developed, which is based on certain finiteness theorems about the number of homeomorphism types for geometrically bounded manifolds. While there are strong plausibility arguments that counting such balls gives also an upper bound on the number of triangulations, the issue is not completely settled. For example, the construction does not apply directly to piecewise-linear manifolds as we are considering here, although a heuristic connection can be made. But it seems likely that an analytical proof of the existence of the exponential bound can be constructed.

On the numerical side, there is strong numerical evidence for the existence of an exponential bound [18]. As already mentioned, in four dimensions the numerical data of the initial investigations did not show any inconsistency related to the absence of an exponential bound. If an exponential bound $\exp aN_4$ to the canonical partition function exists, then

$$k_4^c(N_4) = a + bN_4^{-\alpha}, \tag{18}$$

where $N_4^{-\alpha}$ represents a natural polynomial correction to the exponential. If instead of an exponential bound only a factorial bound holds, then one expects

$$k_4^c(N_4) = a + b \log N_4. \tag{19}$$

In figure (1) we show a typical plot of k_4^c versus $\log N_4$ up to $N_4 = 128k$ at $k_2 = 0$ based on [18].

In [19], data for $k_2 = 0, 0.25, 0.5$ were presented for volumes up to 32k simplices, and since a logarithmic fit as in (19) seemed reasonable, absence of an exponential bound was claimed. In [20] and [21], the accuracy of the data was improved upon somewhat, but it was also noted that a small power in (18) may fit the available data about as well as a logarithm. The authors of [20] prefer the logarithmic fit, while in [21] for volumes up to 64k simplices a power law with $\alpha = 1/4$ seems to better accommodate the data.

The data of [19, 20, 21] suggested that one required more data at larger volumes to decide whether the data favors a logarithmic or a power law fit. We have been able to get reliable data up to a volume of 128k [18]. (For the reader unfamiliar with the computationable effort required, for the largest volume we used 30000 sweeps that took four months on a shared IBM/RISC workstation.) Since there is not enough data to determine α reliably, setting $\alpha = 1/4$ serves well enough to distinguish the exponential from the factorial fit.

The result is that when superimposing our new points to the fits of ref. [21] they fall very well on the power fit (quite far indeed from the logarithmic divergence prediction) obtained from smaller triangulations.

We have fitted our data for $k_2 = 0$ with the two forms (18) and (19), by setting the power $\alpha = \frac{1}{4}$. They are both two parameter fits. Figure (1) is quite eloquent about the success of the two fits. The result is

$$\begin{aligned} k_4^{c(log)} &= 0.864 + .0277 \ln N_4 \ , && (20) \\ k_4^{c(power)} &= 1.252 - 1.317 N_4^{-\frac{1}{4}} \ . && (21) \end{aligned}$$

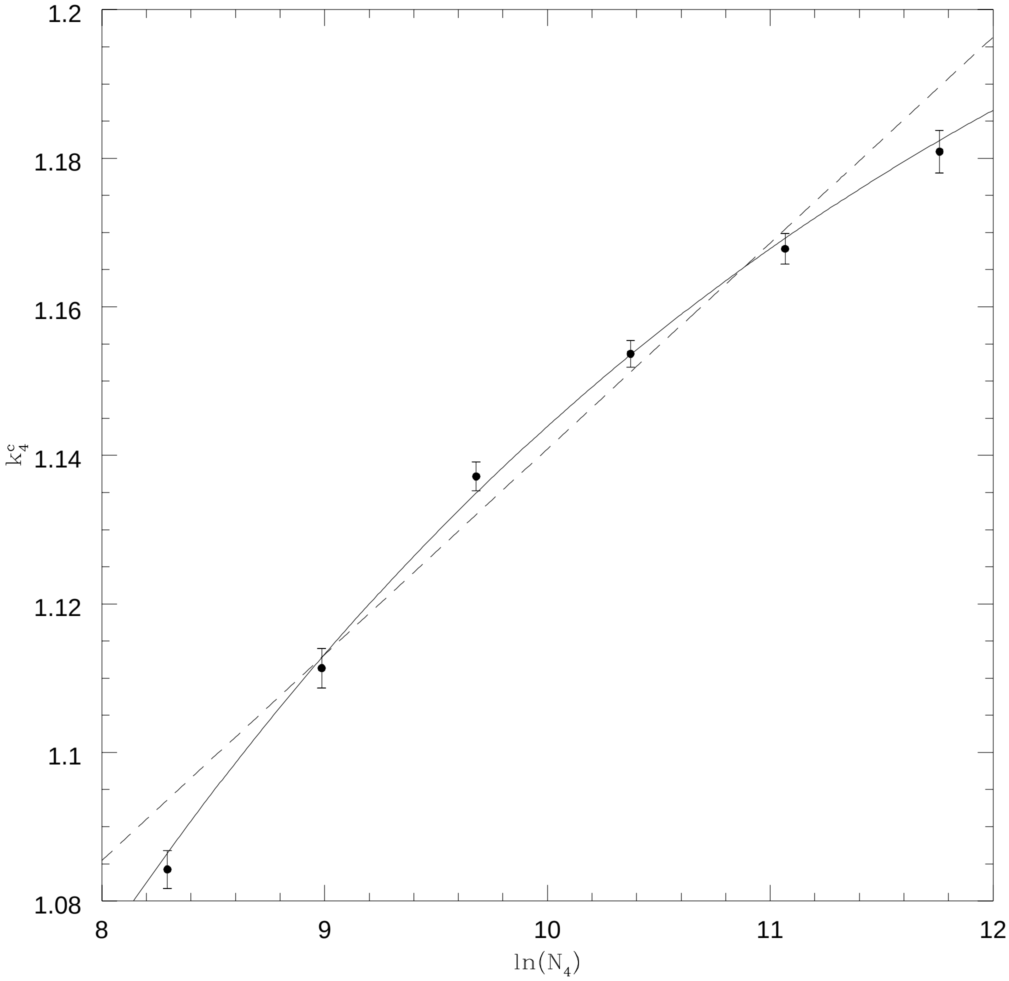

Figure 1: k_4^c versus $\ln(N_4)$ for $k_2 = 0$. Values for N_4 are 4000, 8000, 16000, 32000, 64000 and 128000. With the dashed line we give our best logarithmic fit, with the solid line we give our best fit to a converging power, with $\alpha = .25$. Both fits have two free parameters. The χ^2 of the power fit is ten times better than the one of the logarithmic fit.

The power fit has a value of the χ^2 which is ten times better than the logarithmic one. We have also tried 3 parameter fits. In the power fit we have left the power as a free parameter, while in the logarithmic fit we have added a volume scale term N_4^0, as in $\ln(N_4 - N_4^0)$. Both fits improve quite a lot, but the power fit stays far superior to the logarithmic fit (the χ^2 ratio is now 3). While such a power fit (where the best power is now $.36 \pm .04$) matches perfectly the data points, the logarithmic fit is still not totally congruent to the data (we get N_4^0 of order 3000, that is a reasonable scale for the transient behavior). We are not very confident in playing with many parameters, since the allowed corrections are of many different functional forms, and it is clear that with 6 data points they cannot be distinguished. We just take the results of the 3 parameter fits as further evidence that the power fit is superior to the logarithmic fit. Let us also note that indeed the best preferred power is surely not too small.

What about the consistency of the numerical data? The first observation about the data in figure (1) should really be that there is a remarkable agreement in the data from four independent computer implementations considering that the underlying algorithms are somewhat similar but not identical. In fact, notice that even the data from [19] that lead to the claim about the absence of an exponential bound curves away from a straight line in the same way the other data sets do.

The conclusion we draw is that *the fits of the numerical data largely favor the existence of an exponential bound at $k_2 = 0$ over the presence of a factorial bound.*

Having analyzed in detail the situation for $k_2 = 0$, we now turn to *generic* values of the coupling k_2. In theory, the existence of an exponential bound for any one value of $k_2 \geq 0$ implies existence for all the others. But as is well known, but has not been discussed in detail in this context, there is an important practical difference between the phases for k_2 below and above the critical value $k_2^c \approx 1.1$. For large positive k_2 the simplicial complex is in an elongated phase with an intrinsic dimension close to two, while for negative k_2 the intrinsic dimension diverges to infinity and the simplicial complex becomes extremely crumpled. One of the most intriguing and attractive features of simplicial quantum gravity is that at k_2^c the intrinsic dimension is close to four [14, 22] (for simplicity we ignore here the problem of giving the best definition of the intrinsic dimensionality of the system).

The point is that the two phases are not only different, but there is a genuine asymmetry. Note that at k_2^c the intrinsic system size for $N_4 = 10,000$ is of the order of $(10,000)^{1/4} = 10$, while at $k_2 = 0$ it is $(10,000)^{1/10} \approx 2.5$. Therefore, what constitutes a *large volume* that guarantees the absence of finite size effects depends very sensitively on the value of k_2 [6]. For example, the asymmetry in the susceptibility present in these systems may be due to such effects.

With regard to the discussion of the exponential bound one should therefore consider the whole k_2 range. Such data already exist in [4, 6] and were improved upon near the transition in [20] but were not considered in [19, 21]. For concreteness we show in figure (2) a plot of $\lambda^c(k_2)$ versus $\lambda_0 \sim 1/G$ for $N_4 = 4k, 8k, 16k$ based on [6], which for our purpose is better suited than the more accurate data of [20] since figure (2) extends to extreme values of k_2. The constants are defined by the relations

$$k_2 = 2\pi\lambda_0 \; , \quad k_4 = \lambda + 10\alpha\lambda_0 \; . \tag{22}$$

There is a definite volume dependence for $k_2 < k_2^c$ while above the transition no volume effect is discernible. The linear transformation from k_2 and k_4 to the cosmological constant λ is useful for magnifying the volume dependence which is invisible in this range of coupling constants for $k_4^c(k_2)$ [6]. This is discussed in [20], but even when explicitly looking for a small volume dependence for k_2 clearly above k_2^c, none is found. In this region the plot analogous to figure (1) appears to be a perfectly horizontal straight line, i.e. there are no detectable polynomial corrections to the exponential bound.

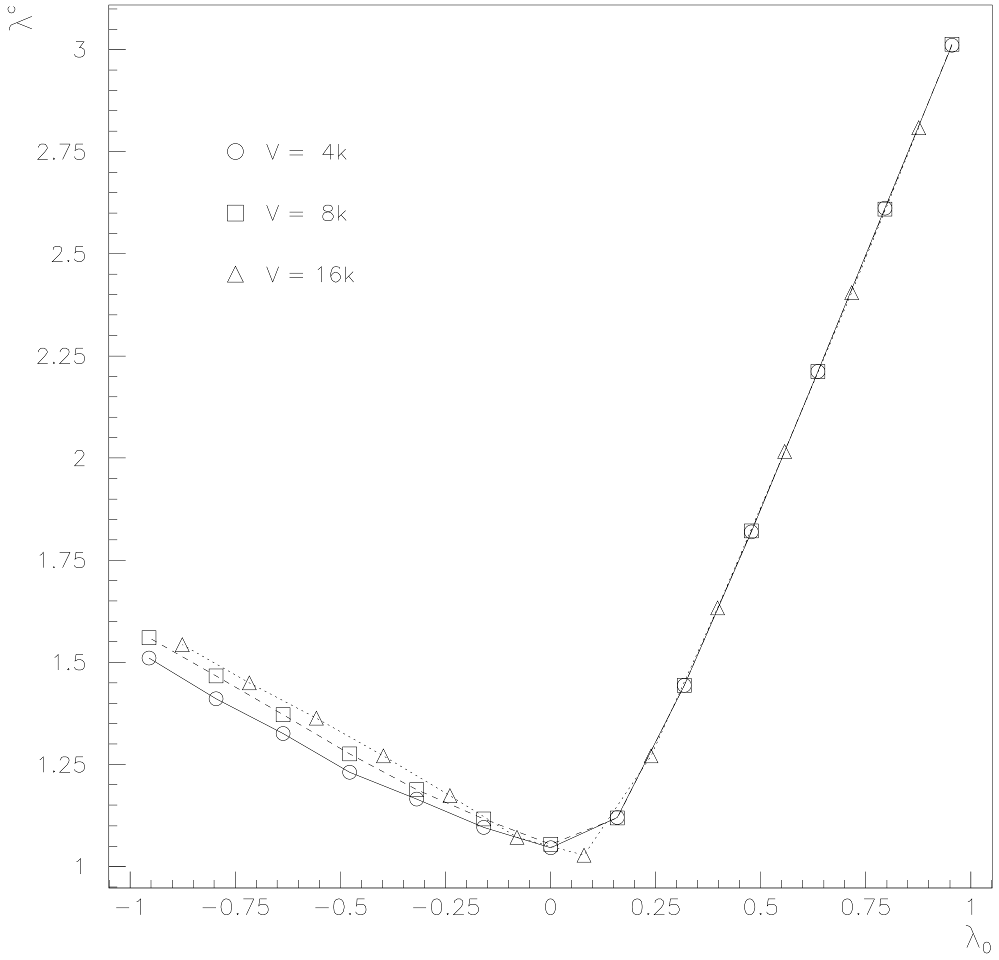

Figure 2: λ^c versus λ_0. Indications of a phase transition are found near $\lambda_0 = 0.18$.

The discussion can be taken one step further by noticing that the critical value k_2^c of k_2 moves to larger values with increasing volumes [14, 20]. For larger volumes at $k_2 = 0$ the finite size effects become even more pronounced (internal dimension up to 50). Given that for extreme values of k_2 the simplicial complex freezes and $k_4^c(k_2)$ becomes a perfect straight line with different slopes, the shift in k_2^c keeping the part $k_2 > k_2^c$ in figure (2) fixed translates directly into (part of) the volume dependence in the range $k_2 < k_2^c$. When this effect constitutes the significant part of the volume dependence (for large enough volume), then the volume dependence of the critical value k_2^c can be estimated by the volume dependence of k_4^c for a small enough but fixed value of k_2. In particular, if there is no exponential bound, then $k_2^c \to \infty$ with $N_4 \to \infty$.

It is instructive to examine the condition for the critical line in the Monte Carlo simulations (here we follow [6, 23]). Consider the ergodic random walk in the space of triangulations of S^4 consisting of the five standard moves, where for the move of type i an i-simplex is replaced by a $(4-i)$-simplex. On the critical line, the average volume N_4 is constant, and therefore N_2 must also be constant since it is bounded. This means that the average variations $\overline{\delta N_j}$ must vanish,

$$\overline{\delta N_j} \sim \sum_{i=0}^{4} \Delta N_j(i) p_i = 0, \tag{23}$$

where p_i is the probability with which a move of type i is performed on average, and $\Delta N_j(i)$ is the change in N_j due to that move. Since the moves are independent, we obtain

$$p_0 = p_4, \quad p_1 = p_3, \tag{24}$$

on the critical line.

Since the action is linear in N_2 and N_4, and since the moves are local, we can be more specific about the conditions on the p_i. The p_i can be chosen to be

$$p_i = [e^{-\Delta S(i)}]\, p_i^{geo} \ . \tag{25}$$

The bracket is the Metropolis weight, its key feature being that it depends only on the type of move and not on the N_i or the triangulation in general. While the action looks quite trivial, all the non-trivialities are hidden in the probability p_i^{geo} for a move to be allowed by the geometric constraints on the triangulation. (Detailed balance is incorporated in the way the moves are chosen. Potentially, there is a factor of order $O(1/N_4)$.)

Therefore (24) is equivalent to

$$k_4^c = \frac{5}{2} k_2 - \ln p_0^{geo}, \tag{26}$$

$$k_4^c = 2k_2 - \ln \frac{p_1^{geo}}{p_3^{geo}}, \tag{27}$$

where we have used that $p_4^{geo} = 1$. The question of the existence of an exponential bound has therefore been translated into the question whether there exist appropriate bounds on the $p_i \equiv p_i(k_4, k_2, N_4)$ which are independent of N_4.

First of all, $p_0 \leq 1$ implies that k_4^c is bounded from below by $2.5k_2$. The hard part is to find a suitable lower bound on p_0, for example, and although it may be possible to do so by some more detailed analysis of the space of triangulations, we do not have a conclusive argument. Notice that since moves of type 4 are always allowed, we have that $p_0 > 0$. However, a naive counting of possible moves of type 0 and 4 around a fixed background triangulation gives $p_0 \sim 1/N_4$, which would be the divergent scenario, but the same kind of counting would also make 2d divergent. The counting is, of course, difficult because moves of type 1, 2, and 3 may change the geometric constraints.

Coming from the numerical side, it is quite suggestive that e.g. the data for $N_4 = 4k$ in figure (2) corresponds to $k_4^c(k_2 \geq 4.0) = 2.497k_2 + c$ and $k_4^c(k_2 \leq -4.0) = 2.002k_2 + d$. This means that in the extreme k_2 regions the relevant geometric probabilities must be independent of k_2 (combining (26) with (27) gives a factor of $\exp(k_2/2)$ for the opposite side).

Considering the general structure of the phase diagram, the volume dependence can also be understood on the level of the random walk as follows. It is the moves of type 4 that drive the system into the elongated phase ($\Delta N_2(4)/\Delta N_4(4) = 2.5$), while moves of type 1 drive to the crumpled phase ($\Delta N_2(3)/\Delta N_4(3) = 2.0$). Depending on k_2 the random walk is driven towards one of the bounds in $2 < N_2/N_4 < 4$. One of the two possible phases, the elongated phase, is therefore characterized by low order vertices, and the average order does not depend on N_4 since a maximal elongation can be obtained for a rather small number of simplices. Hence p_0^{geo}, which is the ratio of the number of vertices of minimal order to the number of all vertices, is expected to be independent of N_4 in the elongated phase. On the other hand, in the crumpled region the average order of vertices is driven towards large values, and the average order will grow with N_4. Hence p_0^{geo}, which is defined by the low order tail of the vertex order distribution, goes to zero with N_4 in the crumpled phase. Equation (26) gives the corresponding volume dependence of k_4^c.

A very attractive possibility is that sophisticated methods to count triangulations like [17] may also allow to estimate the basic probabilities p_i^{geo}. The necessary extension is to counting triangulations subject to constraints, for example that there is a certain number of i-simplices of minimal order that therefore can be removed. If this can be done, there could be analytical predictions for the Monte Carlo random walk.

In conclusion, when looking for evidence for an exponential bound in the numerical data of simplicial quantum gravity in four dimensions, one should take the whole range of k_2 into account. If one insists on looking in the crumpled phase at $k_2 = 0$, the numerical data strongly support the validity of an exponential bound.

4 Is there a measure ambiguity?

In the transition from the continuum path integral to the triangularized model in (9) we tacitly assumed that each triangulation carries equal weight. Strictly speaking there should be a symmetry factor which is conventionally ignored since it presumably has a negligible effect. What we want to stress in this section is that the assumption of uniform weight (modulo symmetries) is quite a serious matter since it is completely open whether there are different universality classes for simplicial quantum gravity in four dimensions. In fact, we want to discuss some data obtained for a one-parameter family of non-uniform measures that seem to show such non-universality.

We have selected not one but a family of measures in order to investigate the influence of the measure in a rather general setting. Our choice is guided by diffeomorphism invariance of the measure [24] but ignores more sophisticated arguments like BRST invariance. We have studied, as a function of n, a measure contribution of the form

$$\prod_x g^{n/2} , \tag{28}$$

i.e. in the triangulated theory $S_E[T]$ is replaced by $S[T] = S_E[T] + S_M[T]$, where

$$S_M = -n \sum_a \log \frac{o(a)}{5} . \tag{29}$$

The sum runs over all 0-simplices (sites) of the manifold, and $o(a)$ is the number of 4-simplices which include the site a. We considered n in the interval from -5 to 5. The case

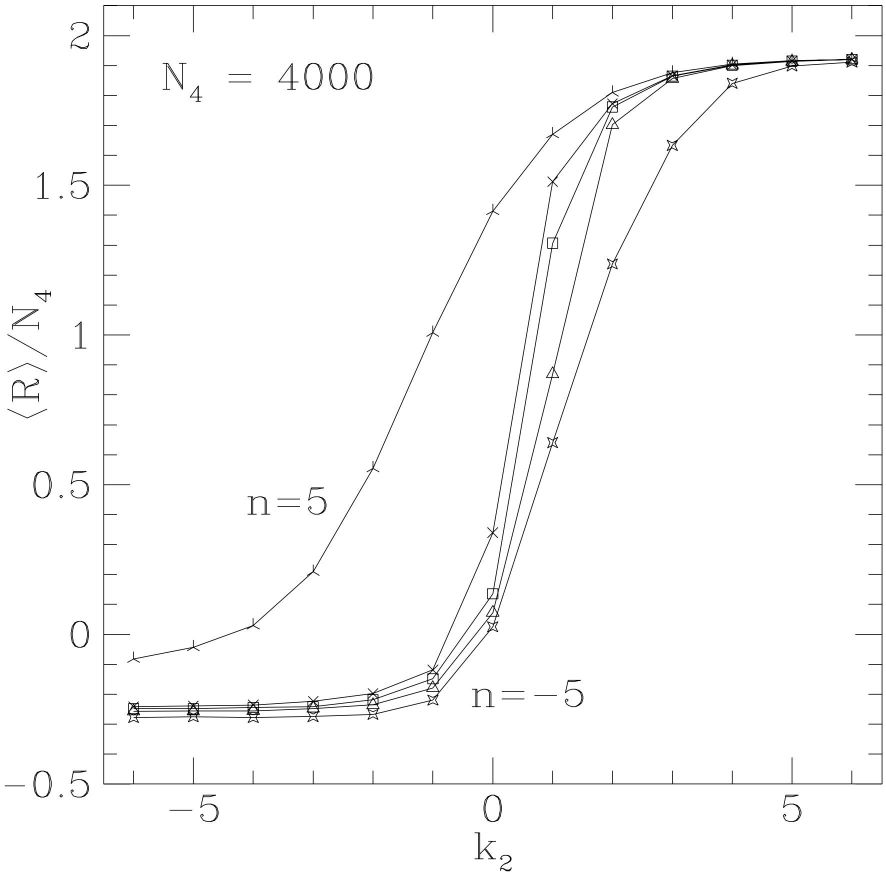

Figure 3: Expectation value of R/V for $n = -5, -1, 0, 1, 5$. The curves are ordered from the smallest n at the bottom upwards.

$n = 0$ repeats simulations with the trivial, uniform measure, which can be compared with previous results.

In two-dimensional dynamical triangulations, it has been already observed that on finite lattices, as expected, the phase diagram depends on the measure coupling (e.g. [25]). However, in this case the measure term is naively an irrelevant operator, and once an extrinsic curvature term has been added it has been shown that the phase diagram [26] is insensitive to this term in the large lattice limit (also at the crossover point).

In the 4d case the phase diagram is also expected to depend on n (if one sends n to $\pm\infty$, then the measure term would dominate the action). It is an interesting question to ask whether for some reasonable values of n one obtains different universality classes. The conventional choice of uniform measure is valid if this is not the case.

Let us summarize our results. We find that the measure factor plays an important role, and that the critical behavior does depend on n. Varying n not only changes non-universal quantities like the value of the critical coupling, but there are indications that it changes the actual critical behavior.

In figure (3) we plot the average curvature R/V for $V \equiv N_4 = 4000$ as a function of the coupling k_2 for different values of n, $n = -5$ for the lowest curve, then $n = -1$, 0, 1 and $n = 5$ for the upper curve. In figure (4) we plot the average distance (in the internal space) of two 4-simplices. We count the minimum number of steps from 4-simplex to 4-simplex across 3-simplex faces that connect a pair of 4-simplices and average over all 4-simplices and random manifolds.

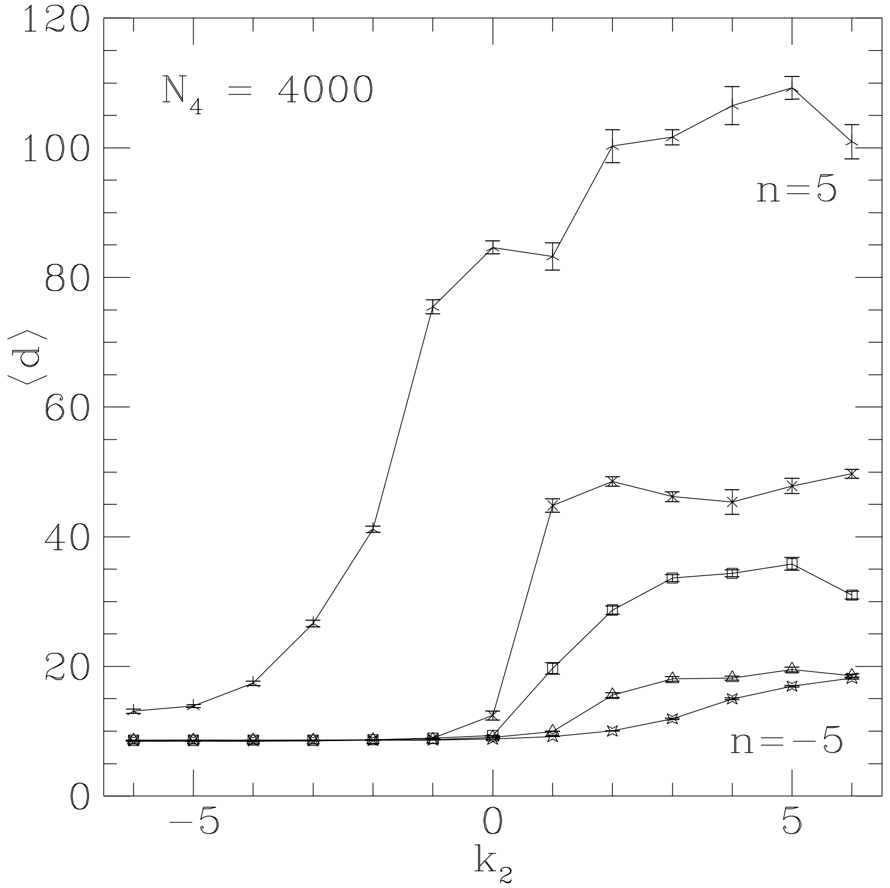

Figure 4: Expectation value of d for $n = -5, -1, 0, 1, 5$, $N_4 = 4000$; curves ordered as in figure (3).

Both figures show that the measure operator has a pronounced effect. Increasing the coupling of the measure term leads to a continuous, monotonous deformation of the curves. Notice that the curves are not just shifted. In the case of R/V, the singularity seems stronger for $n \simeq 0$, where the jump in R/V is quite sharp. The distance d has a sharper jump for $n = 1$, where it seems to jump from one constant value to another constant. Smaller values of n show a slower increase in d.

For large absolute values of n, especially for $n = -5$, the plots show a weaker singularity. The profile of R/V hints less at a sharp jump than the former cases, and the distance increases very smoothly from a critical value of k_2, $k_2^c(n)$, on. When n increases to the value of 5 the system seems to loose criticality on an absolute scale. Its behavior through the crossover is quite smooth.

A critical value of k_2^c can be defined, for example, as the point where the distance value starts to change. But for the $n = 5$ case the transition point is not very clear. Let us note that such a value of k_2^c changes its sign as a function of n.

Our conclusion is that the measure term has a strong effect, which seems difficult to reabsorb in a simple renormalization of the critical coupling. Always keeping in mind that a precise finite size study is required before making quantitative statements, we believe there are two basic possibilities. The first possibility is that there is only one universality class, and that all the theories we have studied do asymptotically show the same critical behavior. In this case the rate of approach to the continuum limit is strongly influenced by n. We will select the theory with faster convergence to the continuum.

The second possibility (which is the most interesting one) is that the measure factor changes the universality class. Our results, albeit preliminary, seem to hint in this direction. In this case we could have a critical value of n_c, and transitions belonging to different universality classes. This is a very appealing scenario, and here the lattice theory could make its own original contribution. It could be possible to pick out the correct measure, on the lattice, by requiring a particular expectation value and scaling behavior of some physical observable. Such a prescription would be a powerful tool, turning the discrete version of the theory from a source of indetermination into a completely determined scheme.

5 Conclusions

It is not easy to summarize such an evolving scenario. Things look good, and interesting.

The number of triangulation does not seem to increase in a pathological way, and the exponential bound seems to be valid. This is the first evidence that makes our hope of finding interesting phenomena stronger.

Different models based on different choices of the lattice measure have quite different behaviors. One will have to investigate in more detail if all the lattice theories will have the same continuum behavior or if we are finding a more complex phase diagram.

Finally, the new results of [12] seem to fortify the hope that a critical theory could be generated at one point of the phase diagram. Simplicial quantum gravity looks like a promising field, definitely worth of further investigations.

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