WEAK-ERGODICITY BREAKING IN MEAN-FIELD SPIN-GLASS MODELS

L. F. Cugliandolo and J. Kurchan

Dipartimento di Fisica, Università di Roma I, *La Sapienza*,

I-00185 Roma,

Italy

INFN Sezione di Roma I, Roma, Italy

Talk given at
Fifth International Workshop on Disordered Systems
Andalo Trento
Italy
February, 1994

The past decade has witnessed a great deal of work on the theoretical modeling of spin-glasses. The efforts have been mainly concentrated on the study of the equilibrium properties of mean-field and low-dimensional models.

Thanks to the the replica method there is at present a good understanding of the Gibbs-Boltzmann measure in mean-field models [1]. The phase-space turns out to have a very complicated landscape with many hierarchically organised states. These states are separated by barriers diverging with the size of the system and hence implying ergodicity breaking.

The implementation of the replica approach for low-dimensional models has proven to be a hard problem and few analytical results are at present available for them. Whether the picture found from the replica analysis of mean-field models can be applied to realistic systems is still an open question; the nature of their ordered state and phase transition remain controversial [2, 3]. Indeed, it is sometimes argued that mean-field models are too unrealistic and cannot be accepted as candidates to describe real spin glasses.

However, the experimental evidence suggests that spin-glass physics in the low temperature phase is essentially dynamical: they continue to evolve long after thermalization at a subcritical temperature and their dynamical properties depend on their history throughout the experimental time. These effects are known as *aging phenomena* and are widely observed in many other disordered systems. Evidently, the relevance of a model consists in its ability to capture these phenomena.

Besides, the experiments suggest that the dynamics of spin-glasses has similar features to those found in usual glasses [4]. One may then hope to apply the developments in the (perhaps simpler) area of spin-glasses to other glassy systems.

Though several phenomenological approaches to aging phenomena in spin-glasses have been made [5], a fully microscopic description is still lacking. It was recently noticed that mean-field models, though schematic, do possess a rich dynamical phenomenology resembling, at least qualitatively, the experimental observations [6]. Furthermore, it turns out that one can obtain analytical results for their long-time dynamics.

In what follows we shall describe the long-time dynamical behaviour of two representative examples: the p-spin spherical model defined by

$$H_J[s] = -\sum_{i_1<\ldots<i_p}^{N} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} + z(t)\left(\sum_{i=1}^{N} s_i^2 - N\right) , \qquad (1)$$

and the Sherrington-Kirkpatrick (soft-spin) model

$$H_J[s] = -\sum_{i_1<i_j}^{N} J_{ij} s_i s_j + u \left(\sum_{i=1}^{N} (s_i^2 - 1)^2 \right) . \quad (2)$$

The last terms enforce the spin measure (spherically constrained and soft spin, respectively), and the quenched disorder J has in both cases zero mean and variance $\overline{J}^2 = p!/(4N^{(p-1)})$ and $\overline{J}^2 = 1/(2N)$, respectively.

The analytical study of the long-time relaxation of these models reveals that both models have aging phenomena, but with rather different characteristics [6, 7]. Their behaviour can be interpreted roughly on the lines introduced by Bouchaud [8].

1 Geometry of phase-space

As a first approach to the dynamical problem it is convenient to have an understanding of the low-temperature structure of phase-space. We have at hand two tools: the replica solution and the Thouless-Anderson-Palmer (TAP) approach. The former gives us very detailed information of the lowest-lying ('equilibrium') states which contribute to the Gibbs-Boltzmann measure, while the latter gives us an idea of what the landscape looks like for higher free-energies. Both approaches reveal that the two models we are considering have some important qualitative differences, as the dynamics also shows.

In the case of the p-spin spherical model, one level of replica symmetry breaking is exact. The equilibrium states are separated by $O(N)$ barriers (the difference in energy and free-energy between them being $O(1)$) [11]. The 'size' of the equilibrium states is q_{EA} while the overlap between different states is $q_0 = 0$ [12].

The TAP equations for this model have solutions for a free-energy range (F_{1rsb}, F_{thres}), corresponding to an energy range (E_{1rsb}, E_{thres}). The threshold energy E_{thres} (free-energy F_{thres}) is greater than the equilibrium energy E_{1rsb} (free-energy F_{1rsb}) by a (temperature and p dependent) $O(N)$ value. Under the threshold the local minima of the TAP free-energy are separated by $O(N)$ barriers while above the threshold there are no minima at all [11]. The free-energy Hessian of a solution has eigenvalues larger than a certain

λ_{min} that depends on the energy of the solution. For sub-threshold solutions λ_{min} is positive-definite, its value decreases with increasing free-energy until it vanishes at the threshold. As the temperature is changed the solutions neither merge nor bifurcate, and their free energy changes smoothly.

In the case of the SK model, the equilibrium states are also organised in an ultrametric fashion, but with all possible distances between them allowed. The barriers between equilibrium states are believed to be of only $O(N^\alpha)$ with $\alpha \sim 1/3$. The solutions of the TAP equations tend to split as the temperature is lowered in a second-order fashion. The Hessian of the solutions has a spectrum going down to zero [13], so that there is the possibility that the barriers between neighbouring high free-energy TAP solutions be finite. For this model there seems to be no high free-energy threshold below which all TAP solutions are separated by infinite barriers and above which there are no solutions, as happens in the p-spin spherical model.

Fig. 1 shows schematically this structure of states and barriers.

2 Asymptotic out of equilibrium regime

Let us now turn to the dynamics of an infinite system. We take the thermodynamic limit $N \to \infty$ from the outset [1], large times henceforth mean $t \to \infty$ *after* $N \to \infty$. Under this hypothesis there may be unsurmountable barriers. We consider a process in which the system evolves from an initial configuration chosen at random.

Taking into account the structure of the phase-space, three possibilities for the long-time dynamics can be expected:

A. The system reaches the Gibbs-Boltzmann distribution in a characteristic time t_{eq}.

B. The system equilibrates within a separate ergodic component. The distribution is a Gibbs-Boltzmann distribution restricted to that sector of phase-space, and it does not evolve after a characteristic time t_{eq}.

C. The system does not reach a time-independent Gibbs-Boltzmann distribution within *any fixed sector* of phase-space at any time. There is no t_{eq}

[1] This is the main difference between this dynamical approach and the one proposed by Sompolinsky [14].

such that for $t > t_{eq}$ the distribution has stabilised.

Case A corresponds to an ordinary equilibration process, thermodynamical and long-time dynamical calculations coincide. The equilibrium theorems hold for times larger than t_{eq}.

Case B corresponds to falling in a high-lying stable state and a purely statical calculation does not necessarily yield the right values for order parameters and free-energy. However, from the dynamical point of view this situation is not very different from the situation A: after the time t_{eq} the system is for all practical (dynamical) purposes as if in equilibrium. Time-homogeneity and the fluctuation-dissipation theorems (FDT) hold for any two times $t_1 > t_2 > t_{eq}$.

Case C is what we find in mean-field spin-glass models. As the system evolves, the dynamical free-energy density decreases, and the system finds an ever decreasing portion of the landscape where it can move. While at high temperatures this process ultimately leads the system to the equilibrium state, at low temperatures the geometry of the region of phase-space available at a certain time becomes more and more complicated, and the system is slowed down and does not reach an equilibrium state in finite times. However, it does not remain permanently confined in any finite region within which it is in local equilibrium.

For instance, in the scenario of Bouchaud [8], one considers a phase-space with 'traps' separated by finite barriers with a wide distribution of lifetimes. Traps visited at long times tend to have long lifetimes, for purely probabilistic reasons. The system does not reach the true states (those with infinite lifetime) in finite times, hence the name '*weak* ergodicity breaking'.

Interestingly enough, once the thermodynamic limit has been taken there are no external parameters in the dynamics of the system that go to infinity. Instead it is the age t_w (the time elapsed since the quench from above the transition temperature) that regulates the slowing down of the system. For longer waiting times, the system has the opportunity of finding deeper 'traps' and narrower channels, it becomes less susceptible to changes and thus it ages.

3 Assumptions on the Correlation and Response Functions

The solution of the models is based on some assumptions that take into account the preceding scenario. We now describe these assumptions.

i. 'Weak-ergodicity breaking'

After any time t_w (that may be interpreted as a waiting time) the system continues to drift away and it reaches, asymptotically, the maximum distance compatible with the remanent magnetization. Thus, the correlation function $C(t+t_w,t_w) \equiv (1/N)\,\sum_{i=1}^{N} \overline{\langle s_i(t+t_w)s_i(t_w)\rangle}$ satisfies

$$\frac{\partial C(t+t_w,t_w)}{\partial t} \leq 0 \qquad \frac{\partial C(t,t')}{\partial t'} \geq 0 \tag{3}$$

$t>t'$, and in the absence of a magnetic field

$$\lim_{t\to\infty} C(t+t_w,t_w) = 0 \qquad \forall \text{ fixed } t_w \tag{4}$$

This hypothesis is supported by the numerical simulations of various mean-field [9, 17, 18] and realistic [19] models. The numerical solution of the mean-field dynamical equations for the p-spin spherical model (see Ref. [6]) also support this assumption: in Fig. 2.a. we plot the decay of the correlation function $C(t+t_w,t_w)$ *vs.* $t+t_w$. It is clear that the sign of the derivative is negative for the whole time-interval plotted. Moreover, Fig. 2.b. shows the decay of the correlation $C(t,t')$ as a funtion of t' and the derivative is positive in this case. ($T=.3$, $p=3$ in both cases.) [2]

ii. 'Weak long-term memory'

The response to a constant (small) magnetic field applied from $t'=0$ up to $t'=t_w$, $h_{t_w}(t') = h\theta(t_w-t')\theta(t')$, *i.e.* the thermoremanent magnetization, decays to zero for a long-enough time t

$$\lim_{t\to\infty} m_{t_w}(t) = \lim_{t\to\infty}\int_0^{t_w} dt' G(t,t') = 0 \qquad \forall \text{ fixed } t_w \tag{5}$$

[2]We present these figures to show the qualitative tendency of the correlation and response functions, a more precise numerical analysis is in order to obtain quantitative results.

The associated experimental curves [20, 21] and the simulations of the D-dimensional spin-glass cell [9] are compatible with this assumption.

It is this property that makes solutions to the long-time dynamics possible: Because the memory of the system of its history is strong when integrated over long times but weak at each given time, one does not need to solve all the short time (far off-equilibrium) details.

iii. Fast and slow relaxations.

After a (long) time t' there is a quick relaxation in a further time $\tau = t-t'$ (τ short compared to t') to some value q (the dynamical Edwards-Anderson parameter), followed by a slower drift away. Fast and slow relaxations correspond to relaxations within and away from a trap, respectively. Within a trap the system behaves as if it were in a local equilibrium, the equilibrium properties (Fluctuation-Dissipation Theorem (FDT) and homogeneity in time) are assumed to hold. The value q is the size of the traps or the width of the channels through which the system evolves.

$$\begin{aligned} C(t,t') &= C_{FDT}(t-t') + \mathcal{C}(t,t') \\ G(t,t') &= G_{FDT}(t-t') + \mathcal{G}(t,t') \end{aligned} \qquad G_{FDT}(t-t') = \frac{\partial C_{FDT}(t-t')}{\partial t'}$$

with

$$\begin{array}{rclrcl} C_{FDT}(0) &=& 1-q & \lim_{t-t'\to\infty} C_{FDT}(t-t') &=& 0 \\ \mathcal{C}(t,t) &=& q & \lim_{t\to\infty} \mathcal{C}(t,t') &=& 0\,. \end{array}$$

The relaxation away from a trap is slow:

$$\frac{\partial \mathcal{C}(t,t')}{\partial t} \sim 0 \qquad \frac{\partial \mathcal{C}(t,t')}{\partial t'} \sim 0 \quad \text{for large} \quad t,t' \tag{6}$$

In the simulations of the D-dimensional hypercubic cell of Ref. [9], the $\log(C(t+t_w,t_w))$ vs. $\log(t)$ curves show that for short times $t = (t+t_w)-t_w$ compared to $t+t_w$ all the curves for different waiting times t_w lie on each other. Hence, the correlation function is homogeneous in time for that range. The fast relaxation to a value q can be checked in the curves presented in Ref. [7].

In the numerical solution of the mean-field dynamical equations of the p-spin spherical model we also find support for this hypothesis. The Figs.

2.a and 2.b show a 'fast' decay from the value 1 at equal times to a certain value. This is more clear in Fig. 2.b. where $C(t,t')$ decays fastly from 1 to a value $q \sim .9$ This is justified since the total time involved is larger and the interval for the FDT decay must be short compared to the total time but it itself must be long.

Two further properties that are in part consequences of the assumptions already made are:

iv. For long times t, t' $\mathcal{C}$ and $\mathcal{G}$ are related by

$$\mathcal{G}(t,t') = X[\mathcal{C}(t,t')] \, \frac{\partial \mathcal{C}(t,t')}{\partial t'} \, \theta(t-t') \tag{7}$$

where X depends on the times *only through* $\mathcal{C}$.

The relation (7) with $X[t,t'] = \hat{X}[t,\mathcal{C}(t,t')]$ implies no assumption. One can show that in the large times limit the explicit dependence on t can be neglected if the energy, the susceptibility, and all the 'generalised susceptibilities' [6, 7] have a limit.

If we supplement the definition of $X[z]$ with $X[z] = 1$ for $q < z < 1$ then the relation (7) holds for all $C(t,t')$, t' large. $X[z]$ may be discontinuous in $z = q$, where it jumps from $X[q]$ to 1 ($X = 1 \Leftrightarrow$ FDT).

An assumption concerning the function $X(C)$, suggested by the simulations is:

v. $X[\mathcal{C}]$ is an increasing function of $\mathcal{C}$.

vi. 'Triangle relations' between the correlations at three large times $t_{min} \leq t_{int} \leq t_{max}$

The monotonicity of the correlation function with respect to both times described above, allows to write

$$C(t_{max}, t_{min}) = f[C(t_{max}, t_{int}), C(t_{int}, t_{min}), t_{min}] \tag{8}$$

We can consider the limit for large t_{min}, and for fixed $C(t_{max}, t_{min})$. The fact that such a limit exists implies that we can write, for large enough t_{min}:

$$C(t_{max}, t_{min}) = f[C(t_{max}, t_{int}), C(t_{int}, t_{min})] \tag{9}$$

where the explicit dependence on times have dissapeared, and the three correlations are related by 'triangle' relations. It is convenient to define the inverse relation:

$$C(t_{max}, t_{int}) = \overline{f}[C(t_{max}, t_{min}), C(t_{min}, t_{int})] \tag{10}$$

A simple computation involving four times shows that the function f is *associative* [7]

$$f(f(q_1, q_2), q_3) = f(q_1, f(q_2, q_3)) \tag{11}$$

The property of weak ergodicity breaking implies that

$$f(q_1, q_2) \leq \min(q_1, q_2) \tag{12}$$

One can now classify all the possible associative f that satisfy (12) as follows: consider the values q_i^* for which

$$f(q_i^*, q_i^*) = q_i^* \ . \tag{13}$$

We call them fixed points. Fixed points and intervals between consecutive fixed points are 'correlation scales'. The latter we call 'discrete scales' and, in particular, the interval $(q, 1)$ is one of them. A dense set of q_i^* corresponds to a dense set of scales.

It is shown in Ref. [7] that if q_1 and q_2 belong to different scales then

$$f(q_1, q_2) = \min(q_1, q_2) \tag{14}$$

If, instead, they belong to the same discrete scale:

$$f(q_1, q_2) = \jmath_k^{-1}[\jmath_k(q_1)\jmath_k(q_2)] \ , \tag{15}$$

where $\jmath_k$ are functions that may be different for each discrete scale.

In Fig. 3 we sketch the function $f(a, a)$ for a ferromagnet, a paramagnet, the p-spin spherical spin-glass and the SK spin-glass. In the next sections we present the analysis of the p-spin spherical model and we discuss the corresponding results for ths SK model [7].

4 A Simple Example

The analysis and assumptions in the preceding section are quite general and model-independent. The precise form of $X(C)$, the limits of the correlation scales, and the functions $\jmath_k$ associated with the discrete scales (if any) have to be extracted from the dynamical equations of each model.

The analysis of the SK model along these lines was done in Ref. [7], here we work out the dynamics of the p-spin spherical model [6] within this formalism. We shall afterwards discuss the differences between these two representative models.

The mean-field equations of motion for the p-spin spherical model read:

$$
\begin{aligned}
\frac{\partial C(t,t')}{\partial t} &= -\left(1-p\beta\,\mathcal{E}(t)\right)C(t,t') + 2\,G(t',t) \\
&\quad + \mu \int_0^{t'} dt''\, C^{p-1}(t,t'')\,G(t',t'') \\
&\quad + \mu\,(p-1)\int_0^{t} dt''\, G(t,t'')\,C^{p-2}(t,t'')\,C(t'',t')\,, \qquad (16)\\
\frac{\partial G(t,t')}{\partial t} &= -\left(1-p\beta\,\mathcal{E}(t)\right)G(t,t') + \delta(t-t') \\
&\quad + \mu\,(p-1)\int_{t'}^{t} dt''\, G(t,t'')\,C^{p-2}(t,t'')\,G(t'',t')\,, \qquad (17)
\end{aligned}
$$

where $\mu = p\beta^2/2$ and the energy $\mathcal{E}$:

$$
\mathcal{E}(t) = -\mu \int_0^t dt''\, C^{p-1}(t,t'')\,G(t,t'') \qquad (18)
$$

Considering assumption *iii* we can separate the small time difference (FDT) regime from the regime of widely separated times. We hence obtain:

$$
\left(\frac{\partial}{\partial\tau}+1\right)C_{\rm FDT}(\tau)+(\mu+p\beta\,\mathcal{E}_\infty)\,(1-C_{\rm FDT}(\tau)) = \mu\int_0^\tau d\tau''\,C_{\rm FDT}^{p-1}(\tau-\tau'')\,\frac{dC_{\rm FDT}}{d\tau''}(\tau'') \qquad (19)
$$

with the asymptotic energy $\mathcal{E}_\infty$ given by

$$
\mathcal{E}_\infty = -\frac{\beta}{2}\left[(1-q^p) + p\int_0^t dt''\,\mathcal{G}(t,t'')\,\mathcal{C}^{p-1}(t,t'')\right]\,. \qquad (20)
$$

The correlation decays to a value q determined by

$$
-\left(1-p\beta\,\mathcal{E}_\infty\right) + \mu\,(1-q^{p-1}) = -\frac{1}{1-q}\,. \qquad (21)
$$

Equation (19) corresponds to the dynamics à la Sompolinsky-Zippelius of the relaxation 'within a trap' [15].

For the long time differences we obtain:

$$\begin{aligned}
0 &= \mathcal{G}(t,t')\left[-(1-q)^{-1}+\mu\,(1-q)(p-1)\;\mathcal{C}^{p-2}(t,t')\right] \\
&\quad +\mu\,(p-1)\int_0^t dt''\,\mathcal{G}(t,t'')\,\mathcal{C}^{p-2}(t,t'')\,\mathcal{G}(t'',t')\;, \qquad (22)\\
0 &= \mathcal{C}(t,t')\left[-(1-q)^{-1}+\mu\,(1-q)\;\mathcal{C}^{p-2}(t,t')\right] \\
&\quad +\mu\int_0^{t'} dt''\,\mathcal{C}^{p-1}(t,t'')\,\mathcal{G}(t',t'') \\
&\quad +\mu\,(p-1)\int_0^t dt''\,\mathcal{G}(t,t'')\,\mathcal{C}^{p-2}(t,t'')\,\mathcal{C}(t'',t')\;. \qquad (23)
\end{aligned}$$

We have now two (coupled) sets of equations, corresponding to two time regimes. It is important to note that in order to claim that eqs. (22) and (23) are asymptotically valid *we need the property of weakness of the long-term memory*, which allows us to disregard in the integrals any finite-time interval. We can then neglect the fact that for the initial times the asymptotic equations do not hold.

Above the critical temperature $\mathcal{G}\sim 0$, the long-term memory is absent, and we are left with only equation (19) [15]. This is the same equation that arises in the mode-coupling theory of structural glasses [16].

In the glass phase, in order to determine any quantity, we have to solve the whole set of eqs. (19), (22) and (23).

We concentrate here on the solution of eqs. (22) and (23). We have neglected in them the time derivatives following *iii*. This brings as a consequence the fact that these equations (but not eq. (19)) have a continuous set of reparametrization invariances. Indeed, from a solution $\mathcal{C}$, $\mathcal{G}$ we obtain infinitely many others:

$$\hat{\mathcal{C}}(\hat{t},\hat{t}') = \mathcal{C}(h(t),h(t'))\;, \qquad \hat{\mathcal{G}}(\hat{t},\hat{t}') = h'(t')\,\mathcal{G}(h(t),h(t'))\;, \qquad (24)$$

with h any increasing function. This invariance is a consequence of having neglected the time derivatives in making the asymptotic limit. The full dynamical equations have no such invariances; because of causality their solution is *unique*. We face the *selection problem*, a rather common phenomenon in the asymptotic limit of solutions of differential equations. In this work we discuss solutions only modulo reparametrizations, the selection problem for this kind of dynamics remains to be solved.

Let us now use the assumptions *iv* and *vi* to write (22) and (23) in terms of X and $\bar{f}$. Defining

$$F[\mathcal{C}] \equiv -\int_{\mathcal{C}}^{q} d\mathcal{C}'\, X[\mathcal{C}'] \qquad H[\mathcal{C}] \equiv -\int_{\mathcal{C}}^{q} d\mathcal{C}'\, \mathcal{C}'^{p-2}\, X[\mathcal{C}'] \tag{25}$$

the dynamical equations (22) and (23) become

$$\begin{aligned} 0 &= -(1-q)^{-1}\, F[\mathcal{C}] + \mu\,(p-1)\,(1-q)\, H[\mathcal{C}] \\ &\quad +\mu\,(p-1) \int_{\mathcal{C}}^{q} d\mathcal{C}'\, X[\mathcal{C}']\, \mathcal{C}'^{p-2}\, F[\overline{f}(\mathcal{C}',\mathcal{C})]\,, \qquad (26) \\ 0 &= \left[-(1-q)^{-1} + \mu\,(1-q)\,\mathcal{C}^{p-2}\right] \mathcal{C} - \mu\,(p-1) \int_{0}^{\mathcal{C}} d\mathcal{C}'\, \mathcal{C}'^{p-2}\, F[\overline{f}(\mathcal{C},\mathcal{C}')] \\ &\quad +\mu\,(p-1) \int_{0}^{\mathcal{C}} d\mathcal{C}'\, X[\mathcal{C}']\, \mathcal{C}'^{p-2}\, \overline{f}(\mathcal{C},\mathcal{C}') \\ &\quad +\mu\,(p-1) \int_{\mathcal{C}}^{q} d\mathcal{C}'\, X[\mathcal{C}']\, \mathcal{C}'^{p-2}\, \overline{f}(\mathcal{C}',\mathcal{C})\,, \qquad (27) \\ \mathcal{E}(t) &= -\frac{\beta}{2}\left[(1-q^{p}) + p\int_{0}^{q} d\mathcal{C}'\, X[\mathcal{C}']\, \mathcal{C}'^{p-1}\right]. \qquad (28) \end{aligned}$$

In using the relation (7) and (9) we have eliminated the times. Hence we have *divided by the reparametrization group*. Indeed, (7) and (9) are reparametrization-invariant relations themselves.

The next step is to investigate the possibility of having 'discrete' correlation scales. The method is to propose that there is a scale, and then check which limits the equations allow the scale to have. Using the properties of the function $\overline{f}$ described in the previous section [7], these equations can be rewritten within a discrete scale $\mathcal{C} \in (a_2^*, a_1^*)$, $0 < a_1^* < q$,

$$\begin{aligned} 0 &= -(1-q)^{-1}\, F[\mathcal{C}] + \mu\,(p-1)\,(1-q)\, H[\mathcal{C}] - \mu\,(p-1)\, F[\mathcal{C}]\, H[a_1^*] \\ &\quad +\mu\,(p-1) \int_{\mathcal{C}}^{a_1^*} d\mathcal{C}'\, X[\mathcal{C}']\, \mathcal{C}'^{p-2}\, F[\overline{f}(\mathcal{C}',\mathcal{C})]\,, \qquad (29) \\ 0 &= \left[-(1-q)^{-1} + \mu\,(1-q)\,\mathcal{C}^{p-2} - \mu\,(p-1)\, H[a_1^*]\right] \mathcal{C} \\ &\quad +\mu\,(p-1) \int_{0}^{a_2^*} d\mathcal{C}'\, \mathcal{C}'^{p-2}\, \left(\mathcal{C}' X[\mathcal{C}'] - F[\mathcal{C}']\right) \\ &\quad +\mu\,(p-1) \int_{a_2^*}^{\mathcal{C}} d\mathcal{C}'\, \mathcal{C}'^{p-2}\, \left(X[\mathcal{C}']\overline{f}(\mathcal{C},\mathcal{C}') - F[\overline{f}(\mathcal{C},\mathcal{C}')]\right) \end{aligned}$$

$$+\mu\,(p-1)\int_{\mathcal{C}}^{a_1^*} d\mathcal{C}'\,X[\mathcal{C}']\,\mathcal{C}'^{p-2}\,\overline{f}(\mathcal{C}',\mathcal{C})\;. \tag{30}$$

Evaluating eq.(29) in $\mathcal{C}=a_1^*$ we have

$$-(1-q)^{-1}F[a_1^*]+\mu(p-1)\,((1-q)-F[a_1^*])\;H[a_1^*]=0\;, \tag{31}$$

and differentiating eq.(29) w.r.t. $\mathcal{C}$ and evaluating in $\mathcal{C}=a_1^*$ we get

$$(1-q)\left(a_1^{*p-2}-q^{p-2}\right)-a_1^{*p-2}F[a_1^*]-H[a_1^*]=0\;, \tag{32}$$

provided $X[a_1^*]\neq 0$. From eqs. (31) and (32) $F[a_1^*]$ is given by

$$F[a_1^*]=(1-q)\left(1-\sqrt{\frac{q^{p-2}}{a_1^{*p-2}}}\right)\;. \tag{33}$$

Differentiating again eq. (29) w.r.t. $\mathcal{C}$, evaluating the result in a_1^* and using the previous eqs. (32) and (33) we finally get

$$X[a_1^*]=\frac{(p-2)(1-q)}{a_1^*}\sqrt{\frac{q^{p-2}}{a_1^{*p-2}}}\;. \tag{34}$$

Since we assume that $X[\mathcal{C}]$ is an increasing function of $\mathcal{C}$ (assumption v) then

$$a_1^*=q \tag{35}$$

and

$$X[a_1^*]=X[q]=\frac{(p-2)(1-q)}{q}\;. \tag{36}$$

Once we have shown that the upper limit of the discrete scale must be q we need to obtain its lower limit. First, we shall show that $X[\mathcal{C}]$ must be constant inside the scale, *i.e.* $X[\mathcal{C}]=X=(p-2)(1-q)/q$, $\forall\mathcal{C}\in(a_2^*,q)$. Computing the derivatives of eqs. (29) and (30) w.r.t. $\mathcal{C}$, multiplying the former by $X[\mathcal{C}]$ and subtracting we have

$$\begin{aligned}0 \;=\;& \int_{a_2^*}^{\mathcal{C}} d\mathcal{C}'\,\mathcal{C}'^{p-2}\,X[\mathcal{C}]\,\frac{\overline{f}(\mathcal{C},\mathcal{C}')}{\partial\mathcal{C}}\,\left(X[\overline{f}(\mathcal{C},\mathcal{C}')]-X[\mathcal{C}']\right)\\ &+\int_{\mathcal{C}}^{q} d\mathcal{C}'\,\mathcal{C}'^{p-2}\,X[\mathcal{C}']\,\frac{\overline{f}(\mathcal{C}',\mathcal{C})}{\partial\mathcal{C}}\,\left(X[\overline{f}(\mathcal{C}',\mathcal{C})]-X[\mathcal{C}]\right)\;. \end{aligned} \tag{37}$$

This equation has $X[\mathcal{C}] = X$ as a solution. To show that it is the only admisible solution we derivate again w.r.t. $\mathcal{C}$ and evaluate in $\mathcal{C} = a_1^* = q$ to get

$$\frac{dX[\mathcal{C}']}{d\mathcal{C}'} = 0 \quad \Rightarrow \quad X[\mathcal{C}'] = X \ , \tag{38}$$

$\forall \mathcal{C} \in (a_2^*, q)$.

Finally, using the constancy of X we calculate $F[\mathcal{C}]$ and $H[\mathcal{C}]$

$$F[\mathcal{C}] = -X(q - \mathcal{C}) \qquad H[\mathcal{C}] = -\frac{X}{p-1}(q^{p-1} - \mathcal{C}^{p-1}) \ , \tag{39}$$

and from eq. (25) we get

$$a_2^* = 0 \quad \text{or} \quad a_2^* = q \ . \tag{40}$$

Hence the discrete scale is 'empty', there are no discrete scales at all but a continuos set of fixed points from 0 to q, or there only one discrete scale that spans the whole interval $(0, q)$ (apart from the FDT sector). We have checked that the former possibility is excluded by the dynamical equations, namely (27).

We can now turn to the explicit solution within the discrete scale spanning the interval $(a_2^* = 0, a_1^* = q)$. Eqs. (29) and (30) can now be written as

$$\begin{aligned} 0 &= q - \mathcal{C} - \mu\,(p-1)\,(1-q)^2\,(q^{p-1} - \mathcal{C}^{p-1}) \\ &\quad + \mu\,(p-1)\,(1-q)\,X \int_{\mathcal{C}}^{q} d\mathcal{C}'\,\mathcal{C}'^{p-2}\,\overline{f}(\mathcal{C}', \mathcal{C}) \ , \end{aligned} \tag{41}$$

$$\begin{aligned} 0 &= \mathcal{C}\left[-1 + \mu\,(p-1)\,(1-q)^2 \mathcal{C}^{p-2}\right] \\ &\quad + \mu\,(p-1)\,(1-q)\,X \int_{\mathcal{C}}^{q} d\mathcal{C}'\,\mathcal{C}'^{p-2}\,\overline{f}(\mathcal{C}', \mathcal{C}) \ . \end{aligned} \tag{42}$$

The value q can be immediately determined from these equations; they imply

$$\frac{1}{p-1} = \mu q^{p-2}(1-q)^2 \ . \tag{43}$$

From the analysis presented in Ref. [7] we know that the solution within a discrete scale has the form

$$\overline{f}(\mathcal{C}', \mathcal{C}) = \jmath^{-1}\left(\frac{\jmath(\mathcal{C})}{\jmath(\mathcal{C}')}\right) \ , \tag{44}$$

$\mathcal{C} < \mathcal{C}'$. We should now use the equations (41) and (42) to determine the function $\overline{f}(\mathcal{C}', \mathcal{C})$.

Making the change of variables:

$$\rho \equiv \ln \left(\frac{\mathcal{C}}{q} \right) \quad ; \quad k(\rho) \equiv \jmath^{-1}(q e^{\rho}) \tag{45}$$

eqs. (41) and (42) both become [3]

$$k(\rho)\,[1 - k^{p-2}(\rho)] \; + \; \frac{p-2}{p-1} \int_0^{\rho} d\rho' \, \frac{\partial k^{p-1}(\rho')}{\partial \rho'} \, k(\rho - \rho') = 0 \; . \tag{46}$$

The smooth solutions to this equation are:

$$k(\rho) = e^{\alpha \rho} \qquad \Rightarrow \qquad \overline{f}(\mathcal{C}', \mathcal{C}) = q^{-1} \, \frac{\mathcal{C}'}{\mathcal{C}} \tag{47}$$

Which in turn implies [7] that $\mathcal{C}$ is of the form:

$$\mathcal{C}(t, t') = q \, \frac{h(t')}{h(t)} \tag{48}$$

for any increasing h. *This is as far as we can go analytically without solving the selection problem.* The numerical solutions suggest [6] that $h(t) = t^{\gamma}$.

For large t_w one has, as $t - t_w \to \infty$

$$\mathcal{C}(t + t_w, t_w) \simeq q \left(\frac{h(t_w)}{h(t)} \right) \tag{49}$$

which explicitly shows aging.

5 Discussion

We have shown how to obtain some analytical results for spin-glass mean-field dynamics, on the basis of the assumptions made. In this solution the aging phenomena are explicit. The solution is not complete, some quantities we have only obtained *modulo time-reparametrizations*. However, the long-time

[3] Remarkably, this is the same equation that one obtains for long times in the Sompolinsky (time-homogeneous) dynamics [15] with ρ playing the role of time-differences.

limits of all quantities that depend on a single time (energy, magnetization, q, etc) are unaffected by reparametrizations, and hence fully determined.

We have found a solution for the p-spin spherical model with one discrete scale apart from the FDT scale. Let us now briefly describe what happens when one applies exactly the same procedure to the SK model [7]. It is found there that the dynamical equations do not admit any discrete scales (apart from the FDT $(q, 1)$ scale), but only a dense set of scales in the correlation interval $(0, q)$. This is very much like in the static treatment, discrete scales playing the same role as levels of replica symmetry breaking. There is however a difference: while for the SK model one finds that the values of the asymptotic energy, q, and transition temperature coincide (to $O(N)$) with those obtained in the static treatment, for the p-spin model they do not. Indeed, for this model one finds that the values for q and E correspond to those of the threshold level.

Another difference between the two models is in the behaviour of the thermoremanent magnetization. In the p-spin spherical model it decays with a rate that is inversely proportional to the waiting time the field has been on. In the SK model the decay of the magnetization after a long waiting time can be seen as taking place in steps, each of which is takes much longer than the preceding one. Hence, the SK model has a larger degree of 'freezing' of the magnetization.

To conclude, let us remark that the dynamical spin-glass phase can be viewed as a phase where a symmetry is spontaneously broken. The dynamical process has a supersymmetry (SUSY) group of invariances [22] with three generators

$$\mathbf{D}' \quad ; \quad \overline{\mathbf{D}'} \quad ; \quad [\overline{\mathbf{D}'}, \mathbf{D}']_+ = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \tag{50}$$

associated with probability conservation, FDT and time-homogeneity. The two-time (t, t') functions are defined in the time region whose two boundaries are the lines $t = 0$ and $t' = 0$, respectively.

The full SUSY group is broken down in certain time-regions down to the subgroup generated by only $\mathbf{D}'$. In the high temperature phase the effect of the initial (boundary) condition is to break the SUSY in a region of width $\simeq t_{eq}$ near the boundaries. In the low temperature phase the SUSY-breaking persists in an infinite region away from the boundaries. This is much like the effect of symmetry-breaking boundary conditions in ordinary sytems.

References

[1] M. Mézard, G. Parisi and M. A. Virasoro; *'Spin glass theory and beyond'*, World Scientific Lecture Notes in Physics Vol **9** (Singapore, 1987).

[2] D. S. Fisher and D. Huse; Phys. Rev. **B38**, 386 (1988).

[3] E. Marinari, G. Parisi and F. Ritort; *'On the 3D Ising Spin Glass'*, **cond-mat 9310041**, Univ. Roma I preprint 967-93.

[4] L. C. E. Struik; '*Physical aging in amorphous polymers and other materials*', Elsevier, Houston (1978).

[5] G. J. Koper and H. J. Hilhorst; J. Phys. France **49**, 429 (1988).
D. S. Fisher and D. Huse; Phys. Rev. **B38**, 373 (1988).
P. Sibani; Phys. Rev. **B35**, 8572 (1987).
K. H. Hoffmann and P. Sibani; Z. Phys. **B80**, 429 (1990).

[6] L. F. Cugliandolo and J. Kurchan; Phys. Rev. Lett. **71**, 1 (1993).

[7] L. F. Cugliandolo and J. Kurchan; *'On the out of equilibrium relaxation of the Sherrington - Kirkpatrick model'*, **cond-mat 9311016**, to be published in J. Phys. **A**.

[8] J.-P. Bouchaud; J. Phys. France **2** 1705, (1992).

[9] L. F. Cugliandolo, J. Kurchan and F. Ritort; Phys. Rev. **B49**, 6331 (1994).

[10] S. Franz and M. Mézard; *Off equilibrium glassy dynamics: a simple case*, LPTENS preprint, 1993, to be published in Europhys. Lett.

[11] J. Kurchan, G. Parisi and M. A. Virasoro; J. Phys. (France) **3** (1993), 1819.

[12] A. Crisanti, H-J Sommers; Z. Phys. **B87**, 341 (1992).

[13] A. J. Bray and M.A. Moore; J. Phys. **C12** L441 (1979).

[14] H. Sompolinsky; Phys. Rev. Lett. **47**, 935 (1981).

[15] A. Crisanti, H. Horner and H.-J. Sommers; Z. Phys. **B92**, 257 (1993).

[16] W. Gotze; in *'Liquids, Freezing and the Glass Transition'*, edited by J. P. Hansen, D. Levesque and J. Zinn-Justin (North Holland, Amsterdam, (1990).
W. Gotze and L. Sjorgen; Rep. Prog. Phys. **55**, 241 (1992).

[17] B. Strocka, H. Rieger and M. Schreckenberger; unpublished.

[18] A. Baldassarri, L. F. Cugliandolo, J. Kurchan and G. Parisi; unpublished.

[19] H. Rieger; J. Phys. **A26**, L615 (1993).

[20] L. Lundgren, P. Svedlindh, P. Nordblad and O. Beckman; Phys. Rev. Lett. **51** 911 (1983).

[21] E. Vincent, J. Hammann and M. Ocio; in *'Recent progress in Random Magnets'*, ed. D. H. Ryan, World Scientific, Singapore (1992), and references therein.

[22] J. Kurchan; J. Phys. France **2**, 1333 (1992).
S. Franz and J. Kurchan; Europhys. Lett. **20** (3), 197 (1992).