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Off equilibrium dynamics and aging in unfrustrated systems

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Abstract

We analyse the Langevin dynamics of the random walk, the scalar field, the X-Y model and the spinoidal decomposition. We study the deviations from the equilibrium dynamics theorems (FDT and homogeneity), the asymptotic behaviour of the systems and the aging phenomena. We compare the results with the dynamical behaviour of (random) spin-glass mean-field models.

1 Introduction

Spin-glasses and other disordered systems have 'critical' dynamics throughout their low-temperature phase. Their most striking dynamical effect is that of aging: they do not reach thermal equilibrium after very long times and experiments are performed out of equilibrium showing a dependence on the history of the system [1, 2].

These phenomena have been studied with numerical simulations [3, 4, 5, 6], several phenomenological models have been proposed [7, 8], and analytical techniques have applied to mean-field systems [9, 10, 11].

It has been recently proposed [9] that mean-field spin-glasses do not to reach a situation of dynamical equilibrium (*i.e.* homogeinity in time and the fluctuation-dissipation theorem (FDT) are violated) even after infinitely long times.

In other systems an intermediate situation known as *interrupted aging* occurs in which non-equilibrium effects tend to dissapear, but very slowly as compared to the relaxation of ordinary non-critical systems (*e.g.* paramagnets, ferromagnets) [12, 13].

The persistence of out-of equilibrium effects after very long times, and in particular the violation of the equilibrium theorems is a feature not restricted to disordered systems such as spin-glasses. It is interesting in itself to study the deviations from the equilibrium theorems in simpler examples with hamiltonians that are deterministic (non random), and even not disordered or frustrated.

The aim of this note is to analyse the Langevin dynamics of some such simple examples: the random walk, the free scalar field and the X-Y model, and the spinoidal decomposition of a ferromagnetic Ising-like system. In each case we shall study the deviations from the equilibrium theorems and we shall analyse the long-time behaviour of the correlation and response functions, and the total response to a constant perturbation applied during a finite time-interval (the equivalent of the 'thermoremanent magnetization' in spin-glass experiments).

The organisation of the paper is the following. In section 2 we present some general remarks on the FDT and its possible generalization. In section 3 we analize the simplest non equilibrium model, *i.e.* the random walk. In section 4 we consider the case of the D-dimensional free scalar field theory. In section 5 we study the dynamics of the X-Y model at low temperature in two dimensions. Finally in section 6 we consider the dynamics of the spinoidal decomposition for the usual ferromagnetic Ising case. Our conclusions are presented in section 7.

2 The generalized fluctuation dissipation relation

Let us consider a system which has been quenched from high temperature at time $t = 0$. The auto-correlation function $C(t, t')$ among a local quantity O at two subsequent times t' and t is

$$C(t, t') = \langle\, O(t) O(t')\, \rangle \; . \tag{2.1}$$

Hereafter $\langle \cdot \rangle$ represents the mean over the thermal noise.

For large t and fixed $t - t'$, in an equilibrium dynamics situation the auto-correlation function behaves as

$$C(t,t') = C(t-t') \; , \tag{2.2}$$

i.e. it is homogeneous in time.

The response function to a pertubation is defined as the variation of the quantity $\langle\, O(t)\,\rangle$ with respect to a perturbation applied at time t'. More precisely, if we consider the perturbed Hamiltonian

$$H = H_o + \int dt \; h(t) O(t) \; , \tag{2.3}$$

the response function is defined as

$$R(t,t') = \frac{\delta\langle\, O(t)\,\rangle}{\delta h(t')} \tag{2.4}$$

and, because of causality, it equals zero if $t' > t$. The response function is not independent of the correlation if the system is in equilibrium. Indeed, it is related to the correlation function by the celebrated fluctuation-dissipation theorem (FDT):

$$R(t,t') = \beta\, \theta(t-t')\, \frac{\partial C(t,t')}{\partial t'} \; , \tag{2.5}$$

and it is also homogeneous in time $R(t,t') = R(t-t')$.

However, if the system is out of equilibrium neither homogeneity nor the FDT (2.5) hold. The generalised relation between response and correlation functions can be written as

$$R(t,t') = \beta\theta(t-t')\, X(t,t')\, \frac{\partial C(t,t')}{\partial t'} \tag{2.6}$$

with X a function of both times t' and t that characterizes the approach to equilibrium.

The FDT and its violation can be partially understood from the following considerations. Let us consider a system described a variable $y(t)$ which satisfies the Langevin equation

$$\frac{d}{dt} y(t) = -F[y](t) + \eta(t) \tag{2.7}$$

where η is a Gaussian random noise with zero mean and correlation

$$\langle\, \eta(t)\, \eta(t')\, \rangle = 2T\, \delta(t-t')\ , \tag{2.8}$$

T being the temperature.

Taking $t > t'$ for definiteness, the equation of motion (2.7) implies

$$(\frac{\partial}{\partial t'} - \frac{\partial}{\partial t})\, C(t,t') = 2T R(t,t') + A(t,t') \tag{2.9}$$

where we have used $\langle\, y(t)\, \eta(t')\, \rangle = 2T\, R(t,t')$ and

$$A(t,t') \equiv \langle\, F[y](t)\, y(t') - F[y](t')\, y(t)\, \rangle\ . \tag{2.10}$$

At equilibrium the correlation functions satisfy $\langle\, B(t)D(t')\, \rangle = \langle\, B(t')D(t)\, \rangle$, if $B(t)$ and $D(t')$ are any two functions of $y(t)$. This is a consequence of the time reversal symmetry. Hence the asymmetry A vanishes and the fluctuation-dissipation theorem may be recovered by using the invariance under translations in time of the correlation functions at equilibrium:

$$C(t,t') \;=\; C(t-t') \;\Rightarrow\; \left(\tfrac{\partial}{\partial t'} + \tfrac{\partial}{\partial t}\right) C(t,t') \;=\; 0 \tag{2.11}$$

and

$$R(t,t') = \beta\, \frac{\partial C(t,t')}{\partial t'}\ . \tag{2.12}$$

In the off-equilibrium situation the homogeneity in time (eq.(2.11)) is not valid and the asymmetry A may be present. Eq. (2.5) is not valid in general and the generalisation (2.6) must be considered.

In mean-field spin-glass models the auto-correlation and response functions are defined as $C(t,t') = (1/N)\sum_{i=1}^{N}\langle\, s_i(t)s_i(t')\,\rangle$ and $R(t,t') = (1/N)\sum_{i=1}^{N}\delta\langle\, s_i(t)\,\rangle/\delta h_i(t')$, respectively. In the analysis of the asymptotic dynamics [2] presented in Ref. [10] (see also [14]) it has been proposed that, for long enough times and small time differences, $t,t' \to \infty$ and $(t-t')/t << 1$, $X = 1$ and FDT is satisfied, while for long enough times and big time differences, $t,t' \to \infty$ and $(t-t')/t \sim O(1)$, the function X depends on the times only through [3] the correlation function $C(t,t')$, *i.e.*

$$R(t,t') = \beta\theta(t-t')\, X[C(t,t')]\, \frac{\partial C(t,t')}{\partial t'}\ . \tag{2.13}$$

A self-consistent asymptotic solution for the mean-field out of equilibrium dynamics within this assumption has been found both for the p-spin spherical and the Sherrington-Kirkpatrick models (Refs. [9],[10].

In the following sections we shall investigate the behaviour of the function X for various (non random) models and we shall compare the results with expression (2.13) at long times.

2.1 Scalings and aging phenomena

Another interesting problem is to study the scaling properties of the correlation, response and X functions, and the response of the system to a constant perturbation applied during a finite period $[0,t_w]$.

[2] Asymptotic means $t,t' \to \infty$ *after* $N \to \infty$

[3] This result is expected to hold for large times for systems with *finite* susceptibility.

For the p-spin spherical spin-glass model, if t and $t-t'$ are both large one analytically finds

$$C(t,t') \propto \frac{h(t')}{h(t)} \qquad (2.1.1)$$

within the assumptions described above. The numerical solution of the dynamical equations suggests that

$$C(t,t') = S(t'/t) \ , \qquad (2.1.2)$$

(*i.e.* h a power law). This is a new (non-homogeneous) scaling [4]. This is the simplest scaling that captures an essential feature of spin-glass phenomenology: the aging effects, *i.e.* the explicit dependence of the behaviour of the system on its history. The scaling (2.1.2) can be modified in many ways to describe in more detail the results of simulations of realistic models and experiments. Then, one sometimes assumes the slightly different form

$$C(t,t') = t^{-\delta}\, S(t'/t) \ , \qquad (2.1.3)$$

where δ is a small number, of the order of a few percent in spin-glass models [2, 8, 4]. The factor $t^{-\delta}$ implies an interruption of aging for large t (O(few years)).

The generalised FDT relation (2.6) can be written as

$$R(t,t') = \beta\,\Theta(t-t')\,X[t^{-\delta}, t'/t]\,\frac{\partial}{\partial t'}C(t,t'), \qquad (2.1.4)$$

If we substitute the scaling form (2.1.3) for the correlation function in eq.(2.9) and we assume that a similar form is valid for the asymmetry, we find that the response function scales as C/t' or $\partial C/\partial t'$.

[4]In the dynamics of other mean-field spin-glass models more complicated scalings can be present.

If instead we assume that the asymmetry A is zero, as will be the case below, we find

$$X(t,t') = X(\lambda) = \frac{1}{2}\left[1 + \lambda + \delta\,\frac{S(\lambda)}{S'(\lambda)}\right] \tag{2.1.5}$$

with $\lambda \equiv t'/t$.

In the typical aging experiments [1, 2] one measures the 'thermoremanent magnetization', *i.e.* the response of the system to a constant magnetic field h applied during the interval $[0, t_w]$, at constant temperature. t_w is interpreted as a 'waiting time'. In a general dynamical system described by the Langevin equation (2.7) the equivalent of the thermoremanent magnetization is

$$\chi_{t_w}(t) = \int_o^{t_w} dt' R(t,t')\ . \tag{2.1.6}$$

Aging experiments show that $\chi_{t_w}(t) = m_{t_w}(t)$ depends non-trivially on t and t_w [1, 2].

3 The Random Walk

The simplest example of a dynamical system that does not reach equilibrium is the random walk [16]. In the continuum limit the quantity $y(t)$ satisfies the very simple differential equation

$$\frac{d}{dt}y(t) = \eta(t), \tag{3.1}$$

with η a Gaussian noise with variance given by eq.(2.8).

It is easy to check that the correlation function $C(t,t') = \langle\, y(t)\, y(t')\, \rangle$ and the response function $R(t,t') = \delta \langle\, y(t)\, \rangle / \delta h(t') = (\beta/2) \,\langle\, y(t)\eta(t')\, \rangle$ are given by

$$
\begin{aligned}
C(t,t') &= 2T \; \min(t,t') \;, & (3.2)\\
R(t,t') &= \theta(t-t') \;. & (3.3)
\end{aligned}
$$

Hence, the relation (2.6) is satisfied with

$$
X(t,t') = 1/2 \;, \tag{3.4}
$$

$\forall t, t'$, a constant function but different from the usual FDT result, $X = 1$, the system never reaches equilibrium.

Indeed one finds that the scaling form (2.1.3) for the correlation is satisfied with $S(\lambda) = \lambda$ for $\lambda < 1$, but with a big value for δ, $\delta = -1$. Inserting this scaling in eq.(2.1.5) we also obtain $X = 1/2$, as expected since in the random walk problem the force F and the asymmetry A are zero. However, the scalings for the correlation and the total response are quite different from those observed in spin-glasses. In terms of the 'waiting time' t_w and $\tau \equiv t - t_w$, $C(\tau + t_w, t_w) = 2T\, t_w$ and $\chi_{t_w}(\tau + t_w) = t_w$ (*cfr.* eq. (2.1.6)). Both expressions are independent of τ but depend explicitly on t_w.

This example may seem trivial, but it captures the essence of the phenomenon that will be described in the rest of this note.

4 Free Gaussian Fields

In this section we study the behaviour of a simple free scalar field $\phi(\mathbf{x};t)$. The Hamiltonian is quadratic in the field and in dimensions D it reads

$$H = \frac{1}{2}\int d^D\mathbf{x_o}\ \left[(\nabla\phi)^2 + m_o^2\,\phi^2\right] \tag{4.1}$$

where m_o is the mass of the field (see *e.g.* Ref.[17]).

The relaxational dynamics is given by the Langevin equation

$$\frac{\partial}{\partial t_o}\phi(\mathbf{x}_o;t_o) = \Delta\phi(\mathbf{x}_o;t_o) - m_o^2\,\phi(\mathbf{x}_o;t_o) + \eta(\mathbf{x}_o;t_o)\ . \tag{4.2}$$

$\eta(\mathbf{x}_o;t_o)$ is a Gaussian noise ($\eta(\mathbf{k_o};t_o)$ its Fourier transform) with zero mean and correlations

$$\begin{aligned}
\langle\, \eta(\mathbf{x}_o;t_o)\, \eta(\mathbf{x}_o';t_o')\,\rangle &= 2T\ \exp\left(-\tfrac{x_o^2\Lambda^2}{4}\right)\ \delta(t_o-t_o') \\
\langle\, \eta(\mathbf{k_o};t_o)\, \eta(\mathbf{k_o}';t_o)\,\rangle &= 2T\ (2\pi)^D\ \exp\left(-\tfrac{k_o^2}{\Lambda^2}\right)\ \delta^D(\mathbf{k_o}+\mathbf{k_o'})\ \delta(t_o-t_o')
\end{aligned}$$

$x_o^2 \equiv |\mathbf{x}_o - \mathbf{x}_o'|^2$ and $k_o^2 \equiv |\mathbf{k_o}|^2$. We have introduced a spatial correlation over a typical lenght $1/\Lambda$ to simulate the lattice spacing. (This serves to cure some large k pathologies.)

Taking $\phi(\mathbf{x_o},0) = 0$ as the initial condition, the solution to the dynamical equation (4.2) for each noise realisation is

$$\phi(\mathbf{x}_o;t_o) = \int \frac{d^D\mathbf{k_o}}{(2\pi)^D}\ e^{i\mathbf{k_o}\mathbf{x_o}}\ \int_o^{t_o} d\tau\ e^{-(k_o^2+m_o^2)(t_o-\tau)}\ \eta(\mathbf{k_o},\tau)\ . \tag{4.3}$$

Since we are dealing with a field, the correlation, response and X functions depend on space-time coordinates. A standard calculation for the correlation function $C_o(\mathbf{x_o},\mathbf{x'_o};t_o,t'_o) = \langle\, \phi(\mathbf{x}_o;t_o)\, \phi(\mathbf{x}'_o;t'_o)\,\rangle$ gives

$$
\begin{aligned}
C_o(\mathbf{x_o},\mathbf{x'_o};t_o,t'_o) &= T\int \frac{d^D\mathbf{k_o}}{(2\pi)^D}\frac{1}{k_o^2+m_o^2} e^{i\mathbf{k_o}(\mathbf{x_o}-\mathbf{x'_o})} e^{-\frac{k_o^2}{\Lambda^2}} \\
&\qquad \left(e^{-(k_o^2+m_o^2)(t_o-t'_o)} - e^{-(k_o^2+m_o^2)(t_o+t'_o)}\right) \ . \qquad (4.4)
\end{aligned}
$$

The response function $R(\mathbf{x_o},\mathbf{x_o}';t_o,t'_o) = \partial\langle\, \phi(\mathbf{x}_o;t_o)\,\rangle/\partial h(\mathbf{x}'_o;t'_o)$ is given by

$$
\begin{aligned}
R_o(\mathbf{x_o},\mathbf{x_o}';t_o,t'_o) &= \int \frac{d^D\mathbf{k_o}}{(2\pi)^D}\, e^{i\mathbf{k_o}(\mathbf{x_o}-\mathbf{x'_o})} e^{-\frac{k_o^2}{\Lambda^2}}\, e^{-(k_o^2+m_o^2)(t_o-t'_o)} \\
&= \frac{1}{(4\pi)^{D/2}}\, \frac{e^{-m_o^2(t_o-t'_o)}}{(t_o-t'_o+\frac{1}{\Lambda^2})^{D/2}}\, e^{-\frac{x_o^2}{4(t_o-t'_o+\frac{1}{\Lambda^2})}} \ . \qquad (4.5)
\end{aligned}
$$

Here and in what follows we take unprimed times bigger than primed times and we omit the theta functions.

The preceeding formulæ suggest to measure space, time and mass in appropriate lattice units

$$
\begin{aligned}
t &\equiv \Lambda^2 t_o \ , && (4.6)\\
\mathbf{x} &\equiv \Lambda \mathbf{x}_o \ , && (4.7)\\
m &\equiv m_o/\Lambda && (4.8)
\end{aligned}
$$

and to rescale the correlation and response functions $C \equiv \Lambda^{(2-D)} C_o$, and $R \equiv (1/\Lambda^D) R_o$. Note that there is no rescaling of fields and correlations in $D=2$.

The function X that measures the departure from FDT reads, in terms of the rescaled coordinates:

$$
\begin{aligned}
X(\mathbf{x},\mathbf{x}';t,t') &\equiv T\frac{R(\mathbf{x},\mathbf{x}';t,t')}{\frac{\partial C(\mathbf{x},\mathbf{x}';t,t')}{\partial t'}} \\
&= \left[1+\left(\frac{t-t'+1}{t+t'+1}\right)^{D/2} \exp\left\{-2m^2t' + \frac{x^2 t'}{2[(t+1)^2-t'^2]}\right\}\right]^{-1}
\end{aligned}
\qquad (4.9)
$$

4.1 Large-times behaviour

Consider first the massive case. We have a time scale given by:

$$t_{eq} \sim m^{-1/2} \, . \tag{4.1.1}$$

For any x fixed and any two times $t, t' >> t_{eq}$ we have that $X = 1$. This identifies t_{eq} as an 'equilibration' time [5].

In the massless case $m = 0$ the equilibration time diverges and we have a more interesting situation. Let us concentrate in this case. For fixed x and large times t, t', we have

$$X(x; t, t') = X(x; \lambda) = \frac{1}{1 + \left(\frac{1-\lambda}{1+\lambda}\right)^{D/2}} \, , \tag{4.1.2}$$

with $\lambda = t'/t$. Hence X is non-trivial and FDT is violated, even for very long times.

If $\lambda \to 0$ then

$$X(x; \lambda) \to 1/2 \, . \tag{4.1.3}$$

If $\lambda \to 1$ and $D \neq 0$, then

$$X(x; \lambda) \to 1 \, , \tag{4.1.4}$$

and we recover FDT. $\lambda = 1$ corresponds to times t, t' satisfying $(t-t')/t << 1$, *i.e.* small time differences.

If we put $D = 0$ we recover $X = 1/2$ for all times, the result for the random walk.

[5] Note however that if x is of order $\sqrt{t'}$ or larger then X can be smaller than 1, even zero for small time differences. We shall not consider such diverging distances in the rest of the section.

4.2 Scalings

We now present the scalings. Since the massless scalar field turned out to be more interesting we shall concentrate in this model. If $m = 0$ the explicit computation of the integrals in eq. (4.4) gives

$$C(\mathbf{x}, \mathbf{x}'; t, t') = T \frac{1}{\pi^{D/2}} \frac{x^{2-D}}{4} \Gamma \left[\frac{D}{2} - 1; \frac{x^2}{4(t+t'+1)}, \frac{x^2}{4(t-t'+1)} \right] , \quad (4.2.5)$$

where $\Gamma[n; a, b]$ is the generalised incomplete Gamma function

$$\Gamma\left[n; a, b\right] \equiv \int_a^b dz \; z^{n-1} e^{-z} \; . \quad (4.2.6)$$

For equal space points $x = 0$ eq.(4.2.5) reduces to

$$C(0; t, t') = T \; \frac{1}{(4\pi)^{D/2}} \; \frac{1}{1 - \frac{D}{2}} \left[(t+t'+1)^{1-D/2} - (t-t'+1)^{1-D/2} \right] \; . \quad (4.2.7)$$

For long times and $\lambda < 1$, *i.e.* big time differences $(t - t')/t \sim O(1)$, this expression satisfies the scaling law (2.1.3) with $\delta = D/2 - 1$ and

$$S(\lambda) = T \; \frac{1}{(4\pi)^{D/2} (1 - \frac{D}{2})} (|1-\lambda|^{1-D/2} + |1+\lambda|^{1-D/2}) \; . \quad (4.2.8)$$

Hence, in this time scale the function X (eq. (4.1.2)) can be written as $X(0; t, \lambda) = X[t^\delta C]$ and in particular $X(0; t, \lambda) = X(C)$ for $D = 2$.

Considering again the general model, the total response (2.1.6) reads

$$\begin{aligned} \chi_{t_w}(t) &= \int d^D\mathbf{x} \int_o^{t_w} dt'' \frac{1}{(4\pi)^{D/2}} \frac{e^{-m^2(t-t'') - \frac{x^2}{4(t-t''+1)}}}{(t-t''+1)^{D/2}} \\ &= \frac{1}{m^2} e^{-m^2(t-t_w)} \left[1 - e^{-m^2 t_w} \right] \end{aligned} \quad (4.2.9)$$

and for large t_w, $t_w >> t_{eq}$ it reduces to

$$\chi_{t_w}(\tau + t_w) = \frac{1}{m^2} e^{-m^2 \tau} \,, \tag{4.2.10}$$

the typical relaxation in a system that has equilibrated; *i.e.* $\chi_{t_w}(\tau)$ depends only on $\tau = t - t_w$ and no aging is present.

Instead, in the massless limit

$$\chi_{t_w}(\tau + t_w) = t_w \tag{4.2.11}$$

which shows a dependence on the history for all t_w, although a rather unusual one.

The learned reader will notice that in the massless case the Hamiltonian is invariant under the transformation

$$\phi(x) \to \phi(x) + constant. \tag{4.2.12}$$

The correlation functions are not invariant under this transformation and therefore the symmetry is spontaneously broken. The slow approach to equilibrium is a reflection in the time domain of the Goldstone boson arising from the spontaneous breaking of the symmetry. In the next section we shall see a case where the symmetry group is implemented in a non linear way.

These results are in agreement with the general formulae discussed in Section 2 when the asymmetry is neglected (*cfr.* eq. (2.1.5)). Indeed the asymmetry is zero, because the force F is linear in the field ϕ.

5 The relaxational dynamics of the XY model

The Hamiltonian of the $O(2)$ non-linear σ model can be written in terms of the angular variable θ defined through $\mathbf{S}(\mathbf{x}_o) = (\cos\theta(\mathbf{x}_o), \sin\theta(\mathbf{x}_o))$. In two dimensions it reads

$$H = \frac{1}{2}\int d^2\mathbf{x}_o\, (\nabla\theta(\mathbf{x}_o;t_o))^2 \tag{5.1}$$

(see *e.g.* Ref. [17]).

The relaxational dynamics is given by the Langevin equation

$$\frac{\partial}{\partial t_o}\theta(\mathbf{x}_o;t_o) = -\frac{\delta H}{\delta\theta(\mathbf{x}_o;t_o)} + \eta(\mathbf{x}_o;t_o)\ , \tag{5.2}$$

with $\eta(\mathbf{x}_o;t_o)$ as in eq. (4.3).

We consider low temperatures such that vortices can be neglected and therefore we do not see the Kosterlitz-Thouless transition.

The solution to the dynamical equation (5.2) for each noise realisation is that of the massless scalar field problem in $D=2$, *cfr.* eq.(4.3).

The angle-angle correlation $\tilde{C}(\mathbf{x}_o,\mathbf{x}'_o;t_o,t'_o) \equiv \langle\, \theta(\mathbf{x}'_o;t'_o)\ \theta(\mathbf{x}_o;t_o)\,\rangle$ is given by

$$\tilde{C}(x_o;t_o,t'_o) = \frac{T}{4\pi}\,\Gamma\left[0; \frac{\Lambda^2 x_o^2}{4(1+\Lambda^2(t_o+t'_o))}, \frac{\Lambda^2 x_o^2}{4(1+\Lambda^2(t_o-t'_o))}\right]\ , \tag{5.3}$$

and in particular, the time correlation between the angles at the same space point ($x_o^2=0$) is

$$\tilde{C}(0;t_o,t'_o) = \frac{T}{4\pi}\log\left(\frac{1+\Lambda^2(t_o+t'_o)}{1+\Lambda^2(t_o-t'_o)}\right)\ . \tag{5.4}$$

The response to an external field $\tilde{h}(\mathbf{x}_o;t_o)$ acting like $\tilde{h}(\mathbf{x}_o;t_o)\ \theta(\mathbf{x}_o;t_o)$, $\tilde{R}_o(\mathbf{x}_o,\mathbf{x}'_o;t_o,t'_o) \equiv \delta\langle\,\theta(\mathbf{x}_o;t_o)\,\rangle/\delta\tilde{h}(\mathbf{x}'_o;t'_o) = 1/(2T)\,\langle\,\theta(\mathbf{x}_o;t_o)\ \ \eta(\mathbf{x}'_o;t'_o)\,\rangle$ is given by

$$\tilde{R}_o(x_o;t_o,t_o') = \frac{\Lambda^2}{4\pi}\,\frac{\exp\left(-\frac{\Lambda^2 x_o^2}{4(1+\Lambda^2(t_o-t_o'))}\right)}{1+\Lambda^2(t_o-t_o')}\ . \tag{5.5}$$

We now turn to calculating the physical quantities for which the angular character of θ is essential. We first calculate the 'composite' correlation

$$\begin{aligned} C(x_o;t_o,t_o') &\equiv \langle \sin\theta(\mathbf{x}_o;t_o)\ \sin\theta(\mathbf{x}_o';t_o')\,\rangle && (5.6)\\ &= \exp\left(-\frac{1}{2}\left(\tilde{C}(0;t_o,t_o)+\tilde{C}(0;t_o',t_o')\right)\right)\sinh\tilde{C}(x_o;t_o,t_o') && (5.7)\end{aligned}$$

and the associated response to a transverse field $h(\mathbf{x}_o;t_o)$ acting like $h(\mathbf{x}_o;t_o)\sin\theta(\mathbf{x}_o;t_o)$

$$R_o(x_o;t_o,t_o') \equiv \frac{\delta m(\mathbf{x}_o;t_o)}{\delta h(\mathbf{x}_o';t_o')} \tag{5.8}$$

where $m(\mathbf{x}_o;t_o)$ is the transverse magnetisation $m(\mathbf{x}_o;t_o) = \langle\sin\theta(\mathbf{x}_o;t_o)\,\rangle$. The 'composite' response can also be written in terms of the angle-angle correlation $\tilde{C}$ and its associated response function $\tilde{R}_o$:

$$\begin{aligned} R_o(x_o;t_o,t_o') &= \exp\left(-\frac{1}{2}\left(\tilde{C}(0;t_o,t_o)+\tilde{C}(0;t_o',t_o')\right)+\tilde{C}(x_o;t_o,t_o')\right)\\ &\quad\times\ \tilde{R}_o(x_o;t_o,t_o')\ . && (5.9)\end{aligned}$$

As in the previous section we now rescale space-time coordinates as

$$t \equiv \Lambda^2 t_o\ , \tag{5.10}$$

$$\mathbf{x} \equiv \Lambda\mathbf{x}_o\ , \tag{5.11}$$

and rescale the response function $R\equiv(1/\Lambda^2)R_o$, $\tilde{R}\equiv(1/\Lambda^2)\tilde{R}_o$ (but neither the correlations nor the angles).

In terms of the new coordinates we have

$$\tilde{C}(x;t,t') = \frac{T}{4\pi}\,\Gamma\left[0;\frac{x^2}{4(1+t+t')},\frac{x^2}{4(1+t-t')}\right]\ , \tag{5.12}$$

$$
\begin{aligned}
\tilde{R}(x;t,t') &= \frac{1}{4\pi}\frac{1}{1+t-t'}\exp\left(-\frac{x^2}{4(1+t-t')}\right) , &(5.13)\\
C(x;t,t') &= ((1+2t)(1+2t'))^{-\frac{T}{8\pi}}\sinh\tilde{C}(x;t,t') , &(5.14)\\
R(x;t,t') &= \frac{1}{2}((1+2t)(1+2t'))^{-\frac{T}{8\pi}}\tilde{R}(x;t,t')\exp\left(\tilde{C}(x;t,t')\right) . &(5.15)
\end{aligned}
$$

From eqs. (5.14) and (5.15), the function X reads

$$
X(x;t,t') = \frac{2\tilde{R}(x;t,t')}{A_-(x;t,t') + A_+(x;t,t')\exp(-2\tilde{C}(x;t,t'))} \qquad (5.16)
$$

with

$$
\begin{aligned}
A_\pm(x;t,t') &\equiv \frac{\partial\tilde{C}(x;t,t')}{\partial t'} \pm \frac{1}{2}\frac{\partial\tilde{C}(0;t',t')}{\partial t'}\\
&= \frac{T}{4\pi}\left[\frac{\exp\left(-\frac{x^2}{4(1+t-t')}\right)}{1+t-t'} + \frac{\exp\left(-\frac{x^2}{4(1+t+t')}\right)}{1+t+t'} \pm \frac{1}{1+2t'}\right] .
\end{aligned}
\qquad (5.17)
$$

5.1 Large-times behaviour

In this subsection we consider the large times limit, t and t' large ($t > t'$). In this limit the function X is

$$
\lim_{t\to\infty} X(x;t,t') = \lim_{t\to\infty}\frac{1}{1+\exp\left(-2\tilde{C}(x;t,t')\right)} \qquad (5.1.1)
$$

with $\tilde{C}(x;t,t')$ given by eq. (5.12). We shall analyse the function X and the correlation C in different regions determined by the space and time separations x and $t-t'$.

Equal Times

We first consider the correlation and response functions at equal times $t = t' \gg 1$. We consider separately the cases $x = 0$ (local values) and $x \gg 1$ (many 'lattice spacings').

a. $x = 0$

$$C(0; t, t) = \frac{1}{2} \tag{5.1.2}$$

which was to be expected, since the $O(2)$ symmetry is unbroken and $\langle \sin^2 \theta \rangle = 1/2$. We also have

$$X(0; t, t) = 1 \tag{5.1.3}$$

i.e. the system evolves locally with an equilibrium dynamics.

b. $x^2 \gg 1$

b.i $t \gg x^2 \gg 1$

$$\lim_{t \to \infty} X(x; t, t) = 1 \tag{5.1.4}$$

and

$$C(x; t, t) \to C_{static}(x) \simeq x^{-\frac{T}{2\pi}} \, . \tag{5.1.5}$$

b.ii $x^2 \gg t \gg 1$

$$\begin{aligned} \lim_{t \to \infty} X(x; t, t) &= \frac{1}{2} \, , \\ C(x; t, t) &\to 0 \, . \end{aligned} \tag{5.1.6}$$

We conclude that if we take a snapshot of the system at a large time t, within a range of length $\simeq t^{1/2}$ the system seems equilibrated in the sense that the correlations coincide with the static ones [17] and the response satisfies FDT. Well outside that range the angles are uncorrelated and $X = 1/2$, as in a random walk. In the following we shall see in more detail the nature of this 'equilibration'.

Different Times

We here consider different times, *i.e.* $t - t' \neq 0$ and we again analyse separately the cases $x = 0$ and $x \gg 1$.

a. $x = 0$

$$\lim_{t\to\infty} X(0; t, t') = \frac{1}{1 + \left(\frac{t+t'}{1+t-t'}\right)^{-\frac{T}{2\pi}}} \tag{5.1.7}$$

and the correlation $C(0; t, t')$ reads

$$C(0; t, t') \sim \frac{1}{2}(4tt')^{-\frac{T}{8\pi}} \left(\left(\frac{t+t'}{1+t-t'}\right)^{\frac{T}{4\pi}} - \left(\frac{t+t'}{1+t-t'}\right)^{-\frac{T}{4\pi}} \right) \tag{5.1.8}$$

b. Consider now two points widely separated $x \gg 1$ but well within the 'range of equilibration' for these times, *i.e.*

$$t > t' \gg x^2 \gg 1 \tag{5.1.9}$$

Two possibilities then arise:

b.i $\quad (t - t')/(t + t') \ll 1$

($\lambda \to 1$.) We reobtain the 'equilibrium' situation $X = 1$ and the correlation function goes, asymptotically in $t - t'$, as

$$C(x; t, t') \sim (t - t')^{-\frac{T}{4\pi}} . \tag{5.1.10}$$

$b.ii$ $\quad (t-t')/(t+t')>0$

$(\lambda<1.)$

$$\lim_{t\to\infty} X(x;t,t') = X(\lambda) = \frac{1}{1+(\frac{1+\lambda}{1-\lambda})^{-\frac{T}{2\pi}}} \ , \tag{5.1.11}$$

and the correlation function is

$$C(x;t,t') \simeq \frac{(2t)^{-\frac{T}{4\pi}}}{2}\lambda^{-\frac{T}{8\pi}}\left[\left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{T}{4\pi}} - \left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{T}{4\pi}}\right] \tag{5.1.12}$$

(*cfr.* eq. (5.1.8)), and $X = t^{-T/(4\pi)}\tilde{X}[C]$.

For relatively small time differences with respect to the total time and for any two points well within a 'domain' of equilibration these results are still those of a system evolving as in equilibrium, in other words $X=1$ and the correlation and response functions are homogeneous in time. However, even within a 'domain' when the time separation is large enough, the correlation and response give manifestly out of equilibrium results: X has a non-trivial time dependence and the correlation and response functions are not necessarily homogeneous in time. We conclude that one cannot picture these domains as regions in which a true (lasting) equilibrium has been established. This has to be contrasted with the behaviour of the massive scalar field, which after a certain t_{eq} and for fixed x evolves as in equilibrium.

5.2 Remanent magnetization

Let us now turn to studying the behaviour of the system in a 'thermoremanent magnetization' experiment.

The total response to a constant, uniform magnetic field $h(x;t,t')$ applied from $t'=0$ to $t'=t_w$ over the whole system is

$$\chi_{t_w}(t) = \frac{1}{4}(1+2t)^{-\frac{T}{8\pi}} \int_o^{t_w} dt'\ (1+2t')^{-\frac{T}{8\pi}}\ \ k\left(\frac{1+t-t'}{1+t+t'}, T\right) \tag{5.2.1}$$

with

$$k(w,T) \equiv \int_o^\infty du\ \exp\left(-\frac{u}{2}\right) \exp\left(-\frac{T}{4\pi}\Gamma\left[0; \frac{u}{2}, \frac{uw}{2}\right]\right)\ . \tag{5.2.2}$$

Defining a function (see Appendix A)

$$f(\lambda,T) = 2^{-(1+\frac{T}{4\pi})}\ \lambda^{\frac{T}{4\pi}-1} \int_{\frac{1-\lambda}{1+\lambda}}^{1} \frac{dw}{(1+w)^2} \left(\frac{1-w}{1+w}\right)^{-\frac{T}{8\pi}} k(w,T)\ , \tag{5.2.3}$$

we have that in the large t limit for every t_w and t

$$\chi_{t_w}(t) = t_w^{1-\frac{T}{4\pi}} f\left(\frac{t_w}{t}, T\right)\ , \tag{5.2.4}$$

and $f(\lambda_w,T)$ is an increasing function of λ_w, finite at $\lambda_w=1$. Asymptotically, for $t \gg t_w$

$$\chi_{t_w}(t) \propto t_w^{1-\frac{T}{8\pi}}\ t^{-\frac{T}{8\pi}} \tag{5.2.5}$$

Several considerations are in order about this behaviour. Throughout the low temperature phase $T<4\pi$, and the susceptibility diverges with t_w. This result was to be expected since the static magnetization grows as h^γ with $\gamma = T/(8\pi - T)$ [17].

For finite times and small fields the linear response theory holds, but becomes worse as an approximation for larger times and it fails completely at $t_w \to \infty$. This result is reminiscent of what seems to happen in spin-glasses with the reaction of the system when temperature is slightly changed: while experimentally (long times) this response is possibly non-symmetrical with respect to the sign of the temperature changes, it is still symmetrical in the relatively short times involved in most simulations.

The longer the waiting time during which the field has been applied, the slower the relaxation of the remanent magnetization. If we normalize the magnetization by its value at t_w, then the decay is a function of t/t_w. This is again reminiscent of what happens in spin glasses and other disordered systems, except for the fact that the susceptibility is finite in those cases.

6 Spinoidal Decomposition

We consider a normal ferromagnetic system (of Ising or Heisenberg type) and we supose that the dynamics is local, without local conservation of the magnetization. The Langevin equation described in the previous sections is a good example of such a dynamics. For definiteness we consider the Ising case.

We are interested in studying the evolution of such a system when we quench it from high temperature to a subcritical temperature. The problem is well studied in the literature [18]. The main result is the random formation of domains oriented in different directions which become larger and larger with increasing time. The size of the domains $\xi(t)$ grows as $t^{1/2}$. It is also well known that the equal time correlation function in the large time limit is well described by

$$\langle\, \phi(x,t)\phi(0,t)\,\rangle = F(x/\xi(t)), \tag{6.1}$$

for well separated space points; *i.e.* $x >> 1$. The function F is not very far from a Gaussian.

Throughout this section the brackets stand for average over initial conditions. We suppose that the field ϕ at time zero is Gaussian-distributed with a correlation function that goes to zero at large distances.

The most natural proposal for the correlation function at different times is

$$\langle\, \phi(x,t')\phi(0,t)\,\rangle = C(x/\xi(t), t/t') \tag{6.2}$$

and, if we consider the correlation at the same space point we would then have

$$\langle\, \phi(0,t')\phi(0,t)\,\rangle = C(t'/t)\;. \tag{6.3}$$

Intuitively we can understand this scaling as follows. The correlation function is proportional to the probability that both spins stay in the same cluster. At time t the spin stays in a cluster of size $\xi(t)$ which has a mean life proportional to $\xi(t)^2$, *i.e.* to t. Therefore it takes a time of order t to revert the magnetization.

Let us be more precise. We consider the following zero temperature Langevin equation:

$$\frac{\partial}{\partial t}\phi = \Delta\phi + \phi(1-\phi^2)\;. \tag{6.4}$$

One can treat this problem approximately as follows [19]: introduce a field ψ defined by

$$\phi = g(\psi) = \frac{\psi}{\sqrt{1+\psi^2}} \tag{6.5}$$

Eq. (6.4) becomes, in terms of ψ,

$$\frac{\partial \psi}{\partial t} = \Delta\psi + \psi - \left(\frac{\partial^2 g}{\partial \psi^2}\right)\left(\frac{\partial g}{\partial \psi}\right)^{-1} (\nabla\psi)^2 \ . \tag{6.6}$$

The approximation consists in neglecting the last term; then one assumes

$$\frac{\partial \psi}{\partial t} = \Delta\psi + \psi \ . \tag{6.7}$$

Note the sign of the mass term. We do not discuss here the range of validity of this approximation, which is widely done in the literature [18].

The strategy we follow is similar to the one used in the preceding section: we first solve a simple linear problem and then calculate the physical correlations as correlations of composite operators.

The ψ-correlation function in Fourier space is given by

$$\langle\, \psi(\mathbf{k},t)\, \psi(\mathbf{k}',t')\, \rangle \propto \delta^D(\mathbf{k}+\mathbf{k}')\, \exp((t+t')(1-k^2)) \tag{6.8}$$

which corresponds to

$$\langle\, \psi(\mathbf{x},t)\, \psi(\mathbf{x}',t')\, \rangle \propto \frac{1}{(t+t')^{D/2}} \exp(t+t' - \frac{x^2}{4(t+t')}) \tag{6.9}$$

in position space. For large times the absolute value of ψ becomes exponentially large, ϕ goes to ± 1 (*cfr.* eq. (6.5)) and one has

$$C(t,t') = \langle\, \mathrm{sgn}(\psi(0,t))\, \mathrm{sgn}(\psi(0,t'))\, \rangle \ . \tag{6.10}$$

The correlation of the random Gaussian variables $\psi_1 \equiv \psi(\mathbf{x}=0,t)$ and $\psi_2 \equiv \psi(\mathbf{x}=0,t')$, is

$$\langle\, \psi_i\, \psi_j\, \rangle \propto \frac{1}{(t_i+t_j)^{D/2}} e^{t_i+t_j} \qquad i,j = 1,2 \ , \tag{6.11}$$

their randomness comes from that of the initial conditions.

Eq. (6.10) becomes

$$C(t,t') = \frac{\int d\psi_1\, d\psi_2\ \mathrm{sgn}(\psi_1)\mathrm{sgn}(\psi_2)\ \exp\left(-\frac{\psi_1^2}{2\langle\, \psi_1^2\,\rangle} - \frac{\psi_1\psi_2}{\langle\, \psi_1\psi_2\,\rangle} - \frac{\psi_2^2}{2\langle\, \psi_2^2\,\rangle}\right)}{\int d\psi_1\, d\psi_2\ \exp\left(-\frac{\psi_1^2}{2\langle\, \psi_1^2\,\rangle} - \frac{\psi_1\psi_2}{\langle\, \psi_1\psi_2\,\rangle} - \frac{\psi_2^2}{2\langle\, \psi_2^2\,\rangle}\right)}\ . \quad (6.12)$$

Changing variables

$$\begin{aligned} \psi_1 &\rightarrow \frac{\exp(t')}{(2t')^{D/4}}\ \psi_1 \\ \psi_2 &\rightarrow \frac{\exp(t')}{(2t')^{D/4}}\ \psi_2 \end{aligned} \quad (6.13)$$

we obtain

$$C(t,t') = \frac{\int d\psi_1\, d\psi_2\ \mathrm{sgn}(\psi_1)\mathrm{sgn}(\psi_2)\ \exp\left(-A\,\psi_1^2 - 2B\,\psi_1\psi_2 - \psi_2^2\right)}{\int d\psi_1\, d\psi_2\ \exp\left(-A\,\psi_1^2 - 2B\,\psi_1\psi_2 - \psi_2^2\right)}\ , \quad (6.14)$$

with

$$A = \lambda^{D/2} \qquad B = \left(\frac{1+\lambda}{2}\right)^{D/2}\ , \quad (6.15)$$

$\lambda = t'/t$. Since we are only interested in the scaling we do not explicitly compute this integral, but notice that

$$C(t,t') = C(\lambda)\ . \quad (6.16)$$

In a similar way one can prove the other scaling laws. The important result is that this approximation gives expressions for the correlations which are in very nice agreement with the aging formulæ .

A similar analysis could have been done for the response function and for the correlation at finite temperature, but this would make this paper too long.

7 Conclusions

We have seen that in many systems in which equilibrium is slowly approached some form of aging phenomena are present. These systems are characterized by a correlation length that is infinite in the static limit but is finite for finite times: it diverges with a power law in time.

A remarkable feature is that in these systems the energy landscape is flat: no high barriers in energy are present. On the contrary the flatness of the potential in certain directions, *i.e.* the presence of zero modes, is at the origin of this very slow approach to equilibrium.

These systems are an evident proof that it is not possible to conclude for the existence of energy activated barrier-crossing only from the presence of aging. It would be rather interesting to see if there are some peculiar phenomena, which may distinguish the effects of barriers from those due to flat directions.

Appendix A

The total response to a constant magnetic field $h(\mathbf{x};t)$ applied during the interval $[0,t_w]$ over the whole system defined in eq. (2.1.6) is

$$\chi_{t_w}(t) = \frac{1}{4}(1+2t)^{-\frac{T}{8\pi}} \int_o^{t_w} dt' \, (1+2t')^{-\frac{T}{8\pi}} \; k\left(\frac{1+t-t'}{1+t+t'}, T\right) \quad \text{(A.1)}$$

with

$$k(w,T) \equiv \int_o^\infty du \; \exp\left(-\frac{u}{2}\right) \exp\left(-\frac{T}{4\pi}\Gamma\left[0; \frac{u}{2}, \frac{uw}{2}\right]\right) \; . \qquad \text{(A.2)}$$

Changing variables

$$\omega \;=\; \tfrac{1+t-t'}{1+t+t'} \qquad \Rightarrow \qquad \omega_w \;=\; \tfrac{1+t-t_w}{1+t+t_w} \tag{A.3}$$

the total response is

$$\chi_{t_w}(t) = \frac{1}{2}(1+2t)^{-\frac{T}{8\pi}}(1+t)\int_{\omega_w}^{1} \frac{d\omega}{(1+\omega)^2}\left[1+2(1+t)\frac{1-\omega}{1+\omega}\right]^{-\frac{T}{8\pi}} \quad k(\omega,T) \tag{A.4}$$

$\forall t, t'$.

Let us now consider the λ scale, *i.e.* $t \gg 1$ and $0 < \lambda_w \equiv t_w/t < 1$. In this case we can use $1+at \sim t$ and $1+t-t' \sim t-t'$ in the lower limit of the integral. Then,

$$\chi_{t_w}(t) = 2^{-(1+\frac{T}{4\pi})}\ \lambda_w^{-1+\frac{T}{4\pi}}\ t_w^{1-\frac{T}{4\pi}} \int_{\frac{1-\lambda_w}{1+\lambda_w}}^{1} \frac{d\omega}{(1+\omega)^2}\left(\frac{1-\omega}{1+\omega}\right)^{-\frac{T}{8\pi}} \quad k(\omega,T) \tag{A.5}$$

and finally

$$\chi_{t_w}(t) = t_w^{1-\frac{T}{4\pi}}\ f(\lambda_w, T) \tag{A.6}$$

with $f(\lambda_w, t)$ given by eq.(5.2.3). Note that the integrand in (A.5) diverges for $\omega = 1$ and gor $\omega = 0$, but the integral over ω is still convergent, and (A.5) is valid for all λ_w.

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