Theory of continuum percolation II. Mean field theory

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Abstract

I use a previously introduced mapping between the continuum percolation model and the Potts fluid to derive a mean field theory of continuum percolation systems. This is done by introducing a new variational principle, the basis of which has to be taken, for now, as heuristic. The critical exponents obtained are $\beta = 1$, $\gamma = 1$ and $\nu = 0.5$, which are identical with the mean field exponents of lattice percolation. The critical density in this approximation is $\rho_c = 1/V_e^{\dagger}$ where $V_e^{\dagger} = \int d\vec{x}\, p(\vec{x})\{\exp[-v(\vec{x})/kT] - 1\}$. $p(\vec{x})$ is the binding probability of two particles separated by $\vec{x}$ and $v(\vec{x})$ is their interaction potential.

1 Introduction

The first paper in this series [1], hereafter referred to as I, contained a general formalism of continuum percolation. Such a system consists of classical particles interacting through a pair potential $v(\vec{r}_i, \vec{r}_j)$, such that they can also bind (or connect) to each other with a probability $p(\vec{r}_i, \vec{r}_j)$. The function $q(\vec{r}_i, \vec{r}_j) = 1 - p(\vec{r}_i, \vec{r}_j)$ is the probability of disconnection .

The formalism is based on a quantitative mapping between the percolation model and an extension of the Potts model I have named the Potts fluid. For easy reference, I recall here the essential definitions and results.

The s-state Potts fluid is a system of N classical spins $\{\lambda_i\}_{i=1}^N$ interacting with each other through a spin-dependent pair potential $V(\vec{r}_i, \lambda_i; \vec{r}_j, \lambda_j)$, such that

$$V(\vec{r}_i, \lambda_i; \vec{r}_j, \lambda_j) \equiv V(i,j) = \begin{cases} U(\vec{r}_i, \vec{r}_j) & \text{if} \quad \lambda_i = \lambda_j \\ W(\vec{r}_i, \vec{r}_j) & \text{if} \quad \lambda_i \neq \lambda_j \end{cases} \tag{1.1}$$

The spins are coupled to an external field $h(\vec{r})$ through an interaction Hamiltonian

$$H_{int} = -\sum_{i=1}^{N} \psi(\lambda_i) h(\vec{r}_i) \tag{1.2}$$

where

$$\psi(\lambda) = \begin{cases} s-1 & \text{if} \quad \lambda = 1 \\ -1 & \text{if} \quad \lambda \neq 1 \end{cases} \tag{1.3}$$

Up to some unimportant constants, the Potts fluid partition function (more precisely, the configuration integral) is

$$Z = \frac{1}{N!} \sum_{\{\lambda_m\}} \int d\vec{r}_1 \cdots d\vec{r}_N \, \exp\left[-\beta \sum_{i>j} V(i,j) + \beta \sum_{i=1}^{N} h(i) \psi(\lambda_i) \right] \tag{1.4}$$

The magnetization of the Potts fluid is defined as

$$M = \frac{1}{\beta N (s-1)} \frac{\partial \ln Z}{\partial h} \tag{1.5}$$

where h is the now constant external field. The susceptibility is

$$\chi = \frac{\partial M}{\partial h} \tag{1.6}$$

The n-density functions of the Potts fluid, are defined as

$$\rho^{(n)}(\vec{r}_1, \lambda_1; \vec{r}_2, \lambda_2; \ldots, \vec{r}_n, \lambda_n) = \frac{1}{Z(N-n)!} \int d\vec{r}_{n+1} \cdots d\vec{r}_N \exp\left[-\beta \sum_{i>j} V(i,j) - \beta \sum_{i=1}^{N} h(i)\psi(\lambda_i)\right] \tag{1.7}$$

Of particular interest is the spin pair-correlation function, defined as

$$g_s^{(2)}(\vec{x}, \alpha; \vec{y}, \gamma) \equiv \frac{1}{\rho(\vec{x})\rho(\vec{y})} \rho^{(2)}(\vec{x}, \alpha; \vec{y}, \gamma) \tag{1.8}$$

which tends to 1 when $|\vec{x} - \vec{y}| \to \infty$. Here $\rho(\vec{x})$ is the numerical local density at the point $\vec{x}$. It is often useful to define a connected spin pair-correlation as

$$h_s^{(2)}(\vec{x}, \alpha; \vec{y}, \gamma) \equiv g_s^{(2)}(\vec{x}, \alpha; \vec{y}, \gamma) - 1 \tag{1.9}$$

This function tends to zero when $|\vec{x} - \vec{y}| \to \infty$.

Any continuum percolation model defined by $v(i,j)$ and $p(i,j)$ can be mapped onto an appropriate Potts fluid model with a pair-spin interaction defined by

$$\begin{aligned} U(i,j) &= v(i,j) \\ \exp\left[-\beta W(i,j)\right] &= q(i,j) \exp\left[-\beta v(i,j)\right] \end{aligned} \tag{1.10}$$

The relation between the Potts magnetization and the percolation probability, $P(\rho)$, is

$$\lim_{h\to 0} \lim_{N\to\infty} \lim_{s\to 1} M = P(\rho) \tag{1.11}$$

For densities lower than the critical density, the susceptibility is directly related to the mean cluster size, S.

$$\lim_{h\to 0} \lim_{N\to\infty} \lim_{s\to 1} \chi = \beta S \qquad\qquad (\rho < \rho_c) \tag{1.12}$$

An important quantity in the percolation model is the pair-connectedness function $g^{\dagger}(\vec{x}, \vec{y})$, the meaning of which is

$$\rho(\vec{x})\rho(\vec{y})\, g^{\dagger}(\vec{x}, \vec{y})\; d\vec{x}\, d\vec{y} \;=\; \text{Probability of finding two particles in regions } d\vec{x} \text{ and } d\vec{y} \text{ around the positions } \vec{x} \text{ and } \vec{y}\text{, such that they both belong to the same cluster.} \tag{1.13}$$

This function is related to the mean cluster size by

$$S = 1 + \rho \int d\vec{r}\, g^{\dagger}(\vec{r}) \tag{1.14}$$

where we assume, as we usually shall, that the system is translationally invariant, so that $g^{\dagger}(\vec{x},\vec{y}) = g^{\dagger}(\vec{x}-\vec{y})$ and $\rho(\vec{x}) = \rho(\vec{y}) = \rho$.

The pair-connectedness is related to the Potts pair-correlation functions by

$$\begin{aligned} g^{\dagger}(\vec{x},\vec{y}) &= \lim_{s\to 1}\left[g_s^{(2)}(\vec{x},\sigma;\vec{y},\sigma) - g_s^{(2)}(\vec{x},\sigma;\vec{y},\eta)\right] \\ &= \lim_{s\to 1}\left[h_s^{(2)}(\vec{x},\sigma;\vec{y},\sigma) - h_s^{(2)}(\vec{x},\sigma;\vec{y},\eta)\right] \end{aligned} \tag{1.15}$$

where $\sigma, \eta \neq 1$ and $\sigma \neq \eta$.

As usual, the first approach to any phase transition is to find the mean field theory. This is the aim of the present paper. The mean field theory for the magnetization (*i. e.*, the percolation probability) and for the susceptibility (*i. e.*, the mean cluster size) is developed in section 2 from a physical point of view. Then, in section 3, I show that the same results can be obtained from a variational principle, which however, remains at present a heuristic device. Section 4 uses the variational principle to obtain the pair-connectedness within the mean field approximation. Section 5 sums up the results.

2 Mean field theory for M and χ

As usual, the basic assumption of mean field theory is that every spin λ feels an average, homogeneous interaction $H(\lambda)/\beta$ due to all the other spins. Since the field h distinguishes spins in state 1 from all the others, we assume that $H(\lambda)$ takes two values, $H(\lambda = 1)$, and $H(\lambda \neq 1)$, this last being identical for all spin states other than 1. This is because the field preserves the symmetry between the non-1 spin states.

From Eq. (1.4), the configuration integral (or the partition function, up to nonimportant constants) is, in this approximation,

$$Z = \frac{1}{N!}\left\{\int d\vec{r} \sum_{\lambda} \exp\left[-H(\lambda) - \beta h(\vec{r})\psi(\lambda)\right]\right\}^{N} \tag{2.1}$$

Separating now the case $\lambda = 1$ from the $s-1$ cases $\lambda \neq 1$, and recalling the definition of $\psi(\lambda)$, Eq. (1.3), we have

$$Z = \frac{1}{N!}\left[\int d\vec{r}\ \Omega\right]^N \tag{2.2}$$

with

$$\begin{aligned}\Omega &\equiv \exp\left[-H(\lambda = 1) + \beta(s-1)h(\vec{r})\right] \\ &+ (s-1)\exp\left[-H(\lambda \neq 1) - \beta h(\vec{r})\right]\end{aligned} \tag{2.3}$$

Let us denote

$$\begin{aligned}B_1 &\equiv \exp\left[-H(\lambda = 1)\right] \\ B_r &\equiv \exp\left[-H(\lambda \neq 1)\right]\end{aligned} \tag{2.4}$$

For a constant field, Eq. (2.3) becomes

$$Z = \frac{V^N}{N!}\left[B_1 e^{\beta(s-1)h} + (s-1)B_r e^{-\beta h}\right]^N \tag{2.5}$$

The magnetization is obtained from Eq. (1.5),

$$M = \frac{1}{N(s-1)\beta}\frac{\partial \ln Z}{\partial h} = \frac{B_1 e^{\beta(s-1)h} - B_r e^{-\beta h}}{B_1 e^{\beta(s-1)h} + (s-1)B_r e^{-\beta h}} \tag{2.6}$$

In the percolation limit, $s \to 1$, we have

$$M = 1 - \lim_{s\to 1}\left(\frac{B_r}{B_1}\right)e^{-\beta s h} \tag{2.7}$$

or, for $h = 0$, when M becomes the percolation probability P,

$$P = 1 - \lim_{s\to 1}\left(\frac{B_r}{B_1}\right) \tag{2.8}$$

This equation is a self-consistency condition because (B_r/B_1) depends on M. The reason is as follows. A spin 1 interacts differently with other spins 1 than it does with spins in other states. As a result, B_1 must depend on the average number of spins in state 1 and of spins in all other states. The same reasoning holds for B_r. Now, let us denote

$$\begin{aligned} n_1 &\equiv \text{ number of spins in state 1} \\ n &\equiv \text{ number of spins in any state } \alpha \neq 1 \end{aligned} \tag{2.9}$$

I have shown in I [Eq. (4.9)] that

$$M = \frac{1}{N}(n_1 - n) \tag{2.10}$$

Therefore B_1 and B_r depend on M, (or P) through n_1 and n.

Finding the form of (B_r/B_1) requires several approximate arguments, which will be clearer if we start with a simple case, *i. e.*, $v(i,j) = 0$. In the percolation picture, this means no interactions . In the Potts fluid picture, this means that the only interactions take place between non-parallel spins, and these interactions are given by $\exp[-\beta W(i,j)] = q(i,j)$ [compare Eq. ()]. In this case, a spin 1 interacts only with the $(s-1)n$ spins which are in non-1 states. Similarly, a non-1 spin interacts only with the $n_1 + (s-2)n$ spins which are non-parallel to it.

The first step is to notice that here we can take already the limit $s \to 1$, since the term $(s-1)$ in the denominator in the definition of M, Eq. (1.5), has vanished in Eq. (2.7). In this limit, a spin 1 interacts with zero [the limit of $(s-1)n$] other spins, while any non-1 spin interacts with $n_1 - n = NM$ other spins.

The main argument is now that it is easier to find B_r and B_1 in the limit $M \to 1$, *i. e.*, when practically all the spins are in the state 1.

Consider first a state where all the spins are in the state 1. Such a state can be thought of, for example, as the limit of an infinitely strong field. Then there are no inter-spin interaction, and the partition function is

$$Z = \frac{1}{N!}\int d1 \cdots dN \exp[\beta N h(s-1)] = \frac{V^N}{N!}\exp[\beta N h(s-1)] \tag{2.11}$$

However, by definition, we also have

$$Z = \frac{V^N}{N!}B_1^N \exp[\beta N h(s-1)] \tag{2.12}$$

Comparing the equations, we have that $B_1 = 1$. In principle, this holds in the limit $M \to 1$ only; however, as mentioned above, in the $s \to 1$ limit, a spin

1 interacts with no other non-1 spins. On the other hand, in our case, the interaction with other spins 1 is $v(i,j) = 0$. Therefore, *all* the interactions vanish and the above result for B_1 can be extended to all values of M. Hence, $B_1 = 1$ in general.

Next consider a state where all spins but one are in the state 1 (in the thermodynamic limit, this is still $M = 1$). Again, the partition function can be calculated easily. For definiteness, assume that the non-1 spin is always numbered N. Then, from Eq. (1.4), we have

$$\begin{aligned} Z &= \frac{1}{(N-1)!}\int d1\cdots dN \, \exp\left\{-\beta\sum_{i=1}^{N-1} W(i,N) + \beta h[(N-1)s - N]\right\} \\ &= \frac{1}{(N-1)!}\left[\int d\vec{r}_i \, q(\vec{r}_N - \vec{r}_i)\right]^{N-1} \exp\left\{\beta h[(N-1)s - N]\right\} \end{aligned} \quad (2.13)$$

And we must also have

$$Z = \frac{V^N}{(N-1)!} B_r B_1^{N-1} \exp\left\{\beta h[(N-1)(s-1) - 1]\right\} \quad (2.14)$$

Hence,

$$B_r = \left[\frac{1}{V}\int d\vec{r} \, q(\vec{r})\right]^{N-1} \quad (2.15)$$

Using $p(\vec{r}) \equiv 1 - q(\vec{r})$ and denoting

$$V_e \equiv \int d\vec{r} \, p(\vec{r}) \,, \quad (2.16)$$

we have

$$B_r = \left[1 - \frac{V_e}{V}\right]^{N-1} \quad (2.17)$$

Now the power $N-1$ represents the number of spins which interact with the non-1 spin. In other words, it is $n_1 - n$ for this particular configuration. In the thermodynamic limit it can therefore be replaced by MN. In terms of the density $\rho = N/V$ we can write

$$B_r = \left[1 - \frac{\rho V_e}{N}\right]^{MN} \quad (2.18)$$

Now although this equation was derived for the case $M = 1$, it is actually written for an arbitrary M. Consequently, we'll *assume* now that we can use it for all values of M. This is part of the mean field approximation itself, *i. e.*, that for the case of $v(i,j)=0$, we suppose that in the thermodynamic limit,

$$\frac{B_r}{B_1} = \lim_{N\to\infty}\left[1-\frac{\rho V_e}{N}\right]^{MN} = \exp\left(-M\rho V_e\right) \tag{2.19}$$

Substituting this into Eq. (2.7) yields

$$M = 1-\exp\left(-M\rho V_e - \beta s h\right) \tag{2.20}$$

or, for $h=0$,

$$M = 1-\exp\left(-M\rho V_e\right) \tag{2.21}$$

This equation is very similar to the one obtained in the mean field theory of the Ising model. For low densities, its only solution is $M=0$. A non-zero solution appears first when the slopes of the two sides of the equation are equal at $M=0$. This condition yields for the critical density, ρ_c, the result

$$\rho_c = \frac{1}{V_e} \tag{2.22}$$

Typically [2], the function $p(\vec{r})$ is chosen to be

$$p(\vec{r}) = \begin{cases} 1 & \vec{r} \quad \text{belongs to some excluded volume}, V_{exc}, \\ & \quad\ \text{around the center of the particle} \\ 0 & \quad\ \text{else} \end{cases} \tag{2.23}$$

Then, $V_e = V_{exc}$, and Eq. (2.22) becomes

$$\rho_c = \frac{1}{V_{exc}} \tag{2.24}$$

Computer simulations [3] show this result to be correct in the limit of infinite spatial dimension. This is precisely where mean field theory is expected to be correct.

Expanding now Eq. (2.20) in the vicinity of the critical point, we have

$$M = M\frac{\rho}{\rho_c} - \frac{1}{2}M^2\left(\frac{\rho}{\rho_c}\right)^2 + \dots \tag{2.25}$$

Hence, the percolation probability, which is equal to M in this limit, is

$$P(\rho) = M \approx \frac{2\rho_c}{\rho^2}(\rho_c - \rho) \sim (\rho_c - \rho)^\beta \qquad (\rho \to \rho_c) \tag{2.26}$$

where in this equation, β is the critical exponent of $P(\rho)$. Therefore, we find

$$\beta = 1 \qquad \text{(Mean Field)} \tag{2.27}$$

We can now calculate the susceptibility. From Eq. (2.20),

$$\chi = \left.\frac{\partial M}{\partial h}\right|_{h=0} = \left[\beta + \rho V_e \left.\frac{\partial M}{\partial h}\right|_{h=0}\right] \exp\left(-M\rho V_e\right) \tag{2.28}$$

or

$$\chi = \frac{\beta e^{-M\rho V_e}}{1 - \rho V_e} \tag{2.29}$$

For $\rho < \rho_c$, we have that $\chi \to \beta S$. In this case, $M = 0$, so that the mean cluster size is

$$\frac{\chi}{\beta} \to S = \frac{1}{1 - \rho V_e} \sim (\rho_c - \rho)^{-\gamma} \tag{2.30}$$

Hence, the critical exponent γ is

$$\gamma = 1 \qquad \text{(Mean Field)} \tag{2.31}$$

We now wish to extend this argument to cases where $v(i,j) \neq 0$. The essentials remain unchanged. Again we look at the limit $M \to 1$. This time, when all the spins are in the state 1, the equivalents of Eqs. (2.11)and (2.12) yield

$$B_1^N = \frac{1}{V^N}\int d1 \cdots dN \exp\left[-\beta \sum_{i=1}^{N}\sum_{i>j} v(i,j)\right] \tag{2.32}$$

Similarly, if only one spin is in a state other than 1, we have

$$B_1^{N-1}B_r = \frac{1}{V^N}\int d1 \cdots dN \qquad \exp\left[-\beta \sum_{i=1}^{N-1}\sum_{i>j} v(i,j)\right] \times \prod_{i=1}^{N-1} q(i,N)\exp[-\beta v(i,N)] \tag{2.33}$$

Dividing the two equations, we find that in this limit (*i. e.*, $M \to 1$)

$$\frac{B_r}{B_1} = \frac{\int d1 \cdots dN \prod_{i=1}^{N-1} q(i,N) \exp\left[-\beta \sum_{i>j=1}^{N-1} v(i,j) - \beta v(i,N)\right]}{\int d1 \cdots dN \, \exp\left[-\beta \sum_{i>j=1}^{N} v(i,j)\right]} \tag{2.34}$$

In general, one cannot compute exactly this ratio. However, we don't need B_r/B_1 exactly, but only in the region $M \approx 0$, where the critical behavior occurs. Now, from the previous arguments, B_r/B_1 can depend on M only through the excess spin-1 density $M\rho = \rho_1 - \rho_\alpha$ (where ρ_1 is the density of spins 1 and ρ_α is the density of spins in any state $\alpha \neq 1$). Hence, we are interested in the region of small $M\rho$. At first sight, this seems unhelpful, because Eq. (2.34) corresponds to the limit $M \to 1$. However, $M\rho$ may be made small by decreasing the density rather than M. In other words, the region of interest $M\rho \to 0$ may be obtained by looking at the limit $\rho \to 0$ even if $M \to 1$.

We therefore take the small density limit of Eq. (2.34) and extract the leading behavior. Let us introduce the functions

$$\begin{aligned}
f(\vec{r}) &= \exp[-\beta v(\vec{r})] - 1 \\
f^*(\vec{r}) &= q(\vec{r}) \exp[-\beta v(\vec{r})] - 1 \\
f^\dagger(\vec{r}) &= f(\vec{r}) - f^*(\vec{r}) = p(\vec{r}) \exp[-\beta v(\vec{r})]
\end{aligned} \tag{2.35}$$

The function $f(\vec{r})$ is just the Mayer f-function [4]. Its usefulness stems from its being typically short ranged (*i. e.*, $f(\vec{r}) \to 0$ at $|\vec{r}| \to \infty$). Therefore, integrals over f (and over f^* and $f^\dagger$ as well) remain finite in the thermodynamic limit.

Eq. (2.34) now becomes

$$\frac{B_r}{B_1} = \frac{\int d1 \cdots dN \prod_{i=1}^{N-1} \left\{ \prod_{i>j} [1 + f(i,j)] \, [1 + f^*(i,N)] \right\}}{\int d1 \cdots dN \prod_{i=1}^{N-1} \prod_{i>j} [1 + f(i,j)]} \tag{2.36}$$

Expanding this expression, we find

$$\begin{aligned}
\frac{B_r}{B_1} &= \frac{\int d1\cdots dN\left\{1+\sum\limits_{i=1}^{N-1}\left[\sum\limits_{i>j} f(i,j)+f^*(i,N)\right]+\cdots\right\}}{\int d1\cdots dN\left\{1+\sum\limits_{i=1}^{N-1}\sum\limits_{i>j} f(i,j)+\cdots\right\}} \\
&= \frac{1+[(N-1)(N-2)/2V]\int d\vec{r}\, f(\vec{r})+(N-1/V)\int d\vec{r}\, f^*(\vec{r})+\cdots}{1+[(N-1)(N-2)/2V]\int d\vec{r}\, f(\vec{r})+(N-1/V)\int d\vec{r}\, f(\vec{r})+\cdots}
\end{aligned} \tag{2.37}$$

Keeping now only the leading terms in the density, we obtain

$$\frac{B_r}{B_1} \approx \frac{1+(N-1/V)\int d\vec{r}\, f^*(\vec{r})+\cdots}{1+(N-1/V)\int d\vec{r}\, f(\vec{r})+\cdots} = 1-\frac{N-1}{V}\int d\vec{r}\, f^\dagger(\vec{r})+\cdots \tag{2.38}$$

where we have used the definition $f^\dagger(\vec{r}) = f(\vec{r}) - f^*(\vec{r})$. Again, as in Eq. (2.18), we recognize that the term $N-1$ represents just $n_1 - n$ in this configuration. Hence once again, we can replace it with MN and write

$$\frac{B_r}{B_1} \approx 1-\frac{MN}{V}\int d\vec{r}\, f^\dagger(\vec{r})+\cdots = 1-M\rho V_e^\dagger \tag{2.39}$$

where

$$V_e^\dagger \equiv \int d\vec{r}\, f^\dagger(\vec{r}) \tag{2.40}$$

generalizes Eq. (2.40), to which it properly reduces if $v(\vec{r}) = 0$.

To the same order in ρ, we may also express Eq. (2.39) as

$$\frac{B_r}{B_1} \approx \exp\left[-M\rho V_e^\dagger\right] \tag{2.41}$$

which completes the analogy with Eq. (2.19) for the case $v = 0$. Because of this, all the results for the case $v = 0$ apply to the more general case as well. Hence, for arbitrary potentials and binding criteria, we now find

$$\begin{aligned}
P &= 1-\exp\left(-\rho P V_e^\dagger\right) \\
S &= \frac{1}{1-\rho V_e^\dagger} \qquad (\rho < \rho_c) \\
\rho_c &= \frac{1}{V_e^\dagger} = \frac{1}{\int d\vec{r}\, f^\dagger(\vec{r})} \\
\beta_{MF} &= 1 \\
\gamma_{MF} &= 1
\end{aligned} \tag{2.42}$$

3 A heuristic variational formulation

We would like now to calculate the connectedness function in the mean field approximation. The previous arguments are inadequate for this task. One knows, however, that in the Ising model, the correlation function may be calculated within mean field theory through a variational formulation based on the Bogolyubov-Feynman inequality [5], *i. e.*,

$$\frac{Z}{Z_0} \geq \exp\left(-\beta \langle \mathcal{H}_1 \rangle_0\right) \tag{3.1}$$

where the subscript 0 refers to some reference system and $\mathcal{H}_1 = \mathcal{H} - \mathcal{H}_0$, where $\mathcal{H}$ is the true Hamiltonian of the system and $\mathcal{H}_0$ is the Hamiltonian of the reference system.

Unfortunately, this inequality is inadequate to deal with our system, because the potential $v(i,j)$ will typically have a strongly repulsive part ("hard core") at short distances, or else the function $q(i,j)$ will typically vanish in some range (where the binding is certain), both of which make the r.h.s of the inequality undetermined. One would therefore like a variational formulation of the mean field which allows for strong effective interactions. I will show that the mean field developed in the previous section can be derived from such a principle, though I have been unable to formally prove the principle itself. Its status must therefore be taken at present to be essentially heuristic.

The motivation is as follows. Let us assume, as usual, some reference system in which the spins interact only with an external field B. The reference Hamiltonian, $\mathcal{H}_0$, is

$$\mathcal{H}_0 = -B \sum_{i=1}^{N} \psi(\lambda_i) \tag{3.2}$$

Then the partition function can be written as

$$\begin{aligned} Z &= \frac{1}{N!} \sum_{\{\lambda_m\}} \int d\vec{r}_1 \cdots d\vec{r}_N \left\{ \exp\left[-\beta \sum_{i>j} V(i,j) + \beta(h-B) \sum_{i=1}^{N} \psi(\lambda_i) \right] \right\} \\ &\qquad \times \exp\left[\beta B \sum_{i=1}^{N} \psi(\lambda_i) \right] \\ &= Z_0 \left\langle \exp\left[-\beta \sum_{i>j} V(i,j) + \beta(h-B) \sum_{i=1}^{N} \psi(\lambda_i) \right] \right\rangle_0 \end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
Z_0 &= \frac{1}{N!} \sum_{\{\lambda_m\}} \int d\vec{r}_1 \cdots d\vec{r}_N \, \exp\left[\beta B \sum_{i=1}^{N} \psi(\lambda_i)\right] \\
&= \frac{V^N}{N!} \left[e^{\beta B(s-1)} + (s-1)e^{-\beta B}\right]^N
\end{aligned}
\tag{3.4}
$$

and $\langle \quad \rangle_0$ means an average performed in the reference system with the Hamiltonian $\mathcal{H}_0$. The convexity of the exponential function then gives us the Bogolyubov-Feynman inequality,

$$
\frac{Z}{Z_0} \geq \exp\left[\left\langle -\beta \sum_{i>j} V(i,j) + \beta(h-B) \sum_{i=1}^{N} \psi(\lambda_i) \right\rangle_0\right] \tag{3.5}
$$

The usual mean field theory follows from extremization with respect to the parameter B. The resulting approximation to Z is expected to be good as long as βV is small. Now, if this is the case, we could as well write

$$
-\beta V(i,j) \approx \exp\left[-\beta V(i,j)\right] - 1 \tag{3.6}
$$

so that to this order

$$
\frac{Z}{Z_0} \sim \exp\left[\sum_{i>j} \left\langle e^{-\beta V(i,j)} - 1 \right\rangle_0 + \beta(h-B) \sum_{i=1}^{N} \langle \psi(\lambda_i) \rangle_0 \right] \tag{3.7}
$$

However, at this stage it is not clear what has become of the inequality in Eq. (3.5), because the replacement Eq. (3.6) is increasing the exponent, thereby countering the Bogolyubov-Feynman inequality.

Nevertheless, for small βV, we expect that $(1/N)\ln Z$ can be approximated by the following function,

$$
F_t(B) \equiv \frac{1}{N} \left\{ \ln Z_0 + \sum_{i>j} \langle \exp\left[-\beta V(i,j)\right] - 1 \rangle_0 + \sum_{i=1}^{N} \beta(h-B) \langle \psi(\lambda_i) \rangle_0 \right\} \tag{3.8}
$$

Typically, of course, $V(i,j)$ is not uniformly small. The surprising result, however, is that if we use for $(1/N)\ln Z$ the *extremum* value of $F_t(B)$ with respect to B, we shall recover the mean field theory of the previous section.

The proof is straightforward. First,

$$\langle\psi(\lambda_i)\rangle_0 = \frac{\sum_{\lambda_i}\psi(\lambda_i)\exp\left[\beta B\psi(\lambda_i)\right]}{\sum_{\lambda_i}\exp\left[\beta B\psi(\lambda_i)\right]} = \frac{(s-1)\left[e^{\beta B(s-1)} - e^{-\beta B}\right]}{e^{\beta B(s-1)} + (s-1)e^{-\beta B}} \qquad (3.9)$$

Next, we rewrite

$$\begin{aligned} \exp\left[-\beta V(i,\lambda_i; j,\lambda_j)\right] - 1 &= q(i,j)\exp[-\beta v(i,j)] - 1 \\ &+ p(i,j)\exp[-\beta v(i,j)]\delta_{\lambda_i,\lambda_j} \end{aligned} \qquad (3.10)$$

Taking the average with respect to the reference system of the two terms on the r.h.s, we find first

$$\langle q(i,j)\exp[-\beta v(i,j)] - 1\rangle_0 = \frac{1}{V^2}\int di\, dj\ \{q(i,j)\exp[-\beta v(i,j)] - 1\} \quad (3.11)$$

which is independent of B.

The second term is

$$\begin{aligned} &\left\langle p(i,j)\exp[-\beta v(i,j)]\delta_{\lambda_i,\lambda_j}\right\rangle_0 \\ &= \frac{1}{Z_0 N!}\int d1\cdots dN \sum_{\{\lambda_m\}} p(i,j)\exp\left[-\beta v(i,j) + \beta B\sum_{k=1}^{N}\psi(\lambda_k)\right]\delta_{\lambda_i,\lambda_j} \\ &= \left[\frac{1}{V^2}\int di\, dj\ p(i,j)e^{-\beta v(i,j)}\right]\frac{\sum_{\lambda_i,\lambda_j}\delta_{\lambda_i,\lambda_j}\exp\left\{\beta B[\psi(\lambda_i) + \psi(\lambda_j)]\right\}}{\left[e^{\beta B(s-1)} + (s-1)e^{-\beta B}\right]^2} \\ &= \left[\frac{1}{V^2}\int di\, dj\ p(i,j)e^{-\beta v(i,j)}\right]\frac{e^{2\beta B(s-1)} + (s-1)e^{-2\beta B}}{\left[e^{\beta B(s-1)} + (s-1)e^{-\beta B}\right]^2} \end{aligned} \qquad (3.12)$$

Note that from the definition of $V_e^\dagger$, Eq. (2.16), we have

$$\frac{1}{V^2}\int di\, dj\ p(i,j)\exp[-\beta v(i,j)] = \frac{V_e^\dagger}{V} \qquad (3.13)$$

Substituting Eqs. (3.4), (3.9) and () into the definition of $F_t(B)$, Eq. (3.8), we have

$$\begin{aligned} F_t(B) &= C + \ln\left[e^{\beta B(s-1)} + (s-1)e^{-\beta B}\right] \\ &+ \frac{N-1}{V}\left(\frac{V_e^\dagger}{V}\right)\frac{e^{2\beta B(s-1)} + (s-1)e^{-2\beta B}}{\left[e^{\beta B(s-1)} + (s-1)e^{-\beta B}\right]^2} \\ &+ \beta(h-B)(s-1)\frac{e^{\beta B(s-1)} - e^{-\beta B}}{e^{\beta B(s-1)} + (s-1)e^{-\beta B}} \end{aligned} \qquad (3.14)$$

where

$$C = \langle q(i,j)\exp[-\beta v(i,j)] - 1\rangle_0 + \ln V - \frac{1}{N}\ln(N!) \tag{3.15}$$

is independent of B.

We now have immediately that

$$\frac{\partial F_t}{\partial B} = \frac{\beta^2 s^2 (s-1)(h-B)e^{-\beta B s}}{[1+(s-1)e^{-\beta B s}]^2} + \left[\frac{(N-1)V_e^\dagger}{V}\right]\frac{\beta s(s-1)e^{-\beta B s}\left(1-e^{-\beta B s}\right)}{[1+(s-1)e^{-\beta B s}]^3} \tag{3.16}$$

The extremum condition $\partial F_t/\partial B = 0$ yields the equation

$$B - h = \left(\frac{\rho V_e^\dagger}{\beta s}\right)\frac{1-e^{-\beta B s}}{1+(s-1)e^{-\beta B s}} \tag{3.17}$$

In this approximation, the magnetization is

$$M = \frac{1}{\beta(s-1)}\frac{\partial F_t}{\partial h} = \frac{1}{\beta(s-1)}\left[\frac{\partial F_t}{\partial B}\frac{\partial B}{\partial h} + \left(\frac{\partial F_t}{\partial h}\right)_B\right] = \frac{1}{\beta(s-1)}\left(\frac{\partial F_t}{\partial h}\right)_B \tag{3.18}$$

where we have used the extremum condition $\partial F_t/\partial B = 0$.

From Eqs. () and (3.17), we have

$$M = \frac{e^{\beta B(s-1)} - e^{-\beta B}}{e^{\beta B(s-1)} + (s-1)e^{-\beta B}} = \frac{\beta s}{\rho V_e^\dagger}(B-h) \tag{3.19}$$

Hence we can rewrite Eq. (3.17) as

$$\beta s B = \beta s h + M\rho V_e^\dagger \tag{3.20}$$

Substituting this result back into Eq. (3.19) gives us

$$M = \frac{1 - e^{-M\rho V_e^\dagger}e^{-\beta s h}}{1+(s-1)e^{-M\rho V_e^\dagger}e^{-\beta s h}} \tag{3.21}$$

which is exactly the mean field equation, Eq. (2.6) with B_r/B_1 given by Eq. (2.41). Hence, indeed, all the mean field results may be obtained from the variational principle of F_t.

Finally, note that from Eq. (3.16), the extremum of F_t is actually a maximum. This might suggest the existence of an inequality $\ln Z \geq NF_t$, but I have been unable to prove it one way or another.

4 Correlation functions

The variational principle now allows us to find the correlation function in mean field theory. To this end, we need to relate the Potts n-density functions defined in Eq. () to the partition function. By analogy with a similar formalism in the theory of liquids [4], let us define a generalized functional differentiation operator, in the following way. Let $\mathcal{F}[t(\vec{r},\lambda)]$ be a functional of the function $t(\vec{r},\lambda)$, which depends on a position variable as well as on an associated discrete spin variable. Then the generalized functional derivative $\tilde{\delta}\mathcal{F}/\delta t$ will be defined through the relation

$$\delta\mathcal{F} = \int d\vec{r} \sum_{\lambda} \frac{\tilde{\delta}\mathcal{F}}{\delta t(\vec{r},\lambda)} \delta t(\vec{r},\lambda) \tag{4.1}$$

where $\delta\mathcal{F}$ is the change in $\mathcal{F}$ associated with a variation δt in $t(\vec{r},\lambda)$. The only difference with the usual functional derivative is in the added summation over the spin variable. This does not change any of the basic properties of the operator.

I shall show later that the Potts correlation functions are functional derivatives of the partition function, for fields h which are inhomogeneous. In order to extend the mean field theory to cover this case, we now need to adopt the following heuristic claim:

Heuristic Assumption: The F_t variational principle yields the proper mean field theory of continuum percolation even for the case of inhomogeneous fields $h(\vec{r})$ and $B(\vec{r})$. The extremum condition must then be generalized to be

$$\frac{\tilde{\delta}F_t}{\delta\left[\beta B(\vec{r})\psi(\lambda)\right]} = 0 \tag{4.2}$$

which determines the extremizing field $B(\vec{r})$.

The application of this extremum condition yields a complicated integral equation. However, for our purpose, we only need $F_t[B]$ in the region $h(\vec{r}) \to 0$ and in the vicinity of the critical region $M(\vec{r}) \to 0$. Looking at the results for the homogeneous case, Eq. (3.20), we can expect that the extremizing function $B(\vec{r})$ will be also very small in this case (the solution shows this assumption to be self-consistent). Therefore, one only needs to look at the extremum condition for very small $B(\vec{r})$. The details of this calculation are in the Appendix. The final result is the condition (to first order in B)

$$B(\vec{x}) = h(\vec{x}) + \frac{\rho}{s}\int d\vec{y}\ p(\vec{x}-\vec{y})e^{-\beta v(\vec{x}-\vec{y})}B(\vec{y}) \tag{4.3}$$

where we have assumed that $p(\vec{x},\vec{y}) = p(\vec{x}-\vec{y})$. Note that if B and h are uniform, we recover the first order limit of Eq. (3.17), as we should. This equation is easily solved in Fourier space where

$$\hat{B}(\vec{k}) = \frac{\hat{h}(\vec{k})}{1-(\rho/s)\hat{f}^{\dagger}(\vec{k})} \tag{4.4}$$

with

$$f^{\dagger}(\vec{r}) = p(\vec{r})e^{-\beta v(\vec{r})} \tag{4.5}$$

and functions with hats are Fourier transforms of the corresponding functions without hats.

Let us now calculate the Potts correlation functions. We take

$$t(\vec{r},\lambda) = \exp[\beta h(\vec{r})\psi(\lambda)] \quad . \tag{4.6}$$

Then we have for the partition function from Eq. (1.4)

$$Z = \frac{1}{N!}\sum_{\{\lambda_m\}}\int d\vec{r}_1\cdots d\vec{r}_N \prod_{i=1}^{N} t(\vec{r}_i,\lambda_i)e^{-\beta\sum_{i>j}V(i,j)} \tag{4.7}$$

It is now simple to see that

$$\frac{\tilde{\delta}Z}{\delta t(\vec{r},\lambda)} = \frac{N}{N!}\sum_{\{\lambda_2,\ldots,\lambda_N\}}\int d\vec{r}_2\cdots d\vec{r}_N \prod_{i=2}^{N} t(\vec{r}_i,\lambda_i)\left[e^{-\beta\sum_{i>j}V(i,j)}\right]_{\substack{\vec{r}_1=\vec{r}\\ \lambda_1=\lambda}} \tag{4.8}$$

Comparing this to Eq. () we see that

$$\rho^{(1)}(\vec{r},\lambda) = \frac{t(\vec{r},\lambda)}{Z}\frac{\tilde{\delta}Z}{\delta t(\vec{r},\lambda)} = \frac{\tilde{\delta}\ln Z}{\delta \ln t(\vec{r},\lambda)} \tag{4.9}$$

Similarly, we have that

$$\begin{aligned}&\frac{\tilde{\delta}^{(2)}Z}{\delta t(\vec{x},\alpha)\delta t(\vec{y},\gamma)}\\ &= \frac{1}{(N-2)!}\sum_{\{\lambda_3,\ldots,\lambda_N\}}\int d\vec{r}_3\cdots d\vec{r}_N \prod_{i=3}^{N} t(\vec{r}_i,\lambda_i)\left[e^{-\beta\sum_{i>j}V(i,j)}\right]_{\substack{\vec{r}_1=\vec{x};\ \lambda_1=\alpha\\ \vec{r}_2=\vec{y};\ \lambda_2=\gamma}}\end{aligned} \tag{4.10}$$

And a comparison with Eq. (1.7) shows that

$$\rho^{(2)}(\vec{x},\alpha;\vec{y},\gamma)-\rho^{(1)}(\vec{x},\alpha)\rho^{(1)}(\vec{y},\gamma)=t(\vec{x},\alpha)t(\vec{y},\gamma)\frac{\tilde{\delta}^2\ln Z}{\delta t(\vec{x},\alpha)\delta t(\vec{y},\gamma)} \tag{4.11}$$

Within the mean field theory we replace $\ln Z$ with $NF_t[B]$. Making use of the extremum condition, we now have from Eq. (4.9) and the definition of $t(\vec{r},\lambda)$, Eq. (4.6), that

$$\rho^{(1)}(\vec{x},\alpha)=\left[\frac{\tilde{\delta}NF_t}{\delta[\beta h(\vec{x})\psi(\alpha)]}\right]_B=\frac{N}{\Gamma}\exp\left[\beta B(\vec{x})\psi(\alpha)\right] \tag{4.12}$$

where

$$\Gamma\equiv\sum_\lambda\int d\vec{z}\,\exp\left[\beta B(\vec{z})\psi(\lambda)\right] \tag{4.13}$$

The easiest way to calculate the pair-correlation function is now to use the identity

$$\begin{aligned}\frac{\tilde{\delta}\rho^{(1)}(\vec{x},\alpha)}{\delta[\beta h(\vec{y})\psi(\gamma)]} &= t(\vec{y},\gamma)\frac{\tilde{\delta}}{\delta t(\vec{y},\gamma)}\left[t(\vec{x},\alpha)\frac{\tilde{\delta}NF_t}{\delta t(\vec{x},\alpha)}\right]\\ &= t(\vec{y},\gamma)\frac{\tilde{\delta}NF_t}{\delta t(\vec{x},\alpha)}\delta(\vec{x}-\vec{y})\,\delta_{\alpha,\gamma}+t(\vec{x},\alpha)t(\vec{y},\gamma)\frac{\tilde{\delta}^2NF_t}{\delta t(\vec{x},\alpha)\delta t(\vec{y},\gamma)}\\ &= \rho^{(1)}(\vec{x},\alpha)\delta(\vec{x}-\vec{y})\,\delta_{\alpha,\gamma}+\rho^{(1)}(\vec{x},\alpha)\rho^{(1)}(\vec{y},\gamma)h^{(2)}(\vec{x},\alpha;\vec{y},\gamma)\end{aligned} \tag{4.14}$$

On the other hand, we have from Eq. (4.13) that

$$\frac{\tilde{\delta}\rho^{(1)}(\vec{x},\alpha)}{\delta[\beta h(\vec{y})\psi(\gamma)]}=\left[1-\frac{1}{\Gamma}e^{\beta B(\vec{y})\psi(\gamma)}\right]\frac{N}{\Gamma}e^{\beta B(\vec{x})\psi(\alpha)}\frac{\tilde{\delta}[B(\vec{x})\psi(\alpha)]}{\delta[h(\vec{y})\psi(\gamma)]} \tag{4.15}$$

We are interested in this function in the range $\rho<\rho_c$ and $h=0$, which is the one relevant for percolation. In this case, both h and B vanish, so that we can evaluate all quantities at zero fields. Thus, using the fact that $\Gamma=sV$ in this limit (see the Appendix), we have that

$$\rho^{(1)}(\vec{x},\alpha)\Big|_{B=0}=\frac{N}{\Gamma}\Big|_{B=0}=\frac{\rho}{s} \tag{4.16}$$

where $\rho = N/V$ as usual. In the same limit, we also have from Eq. (4.15)

$$\frac{\tilde{\delta}\rho^{(1)}(\vec{x},\alpha)}{\delta[\beta h(\vec{y})\psi(\gamma)]}\bigg|_{h=0,B=0} = \frac{\rho}{s}\left[1-\frac{1}{sV}\right]\frac{\tilde{\delta}[B(\vec{x})\psi(\alpha)]}{\delta[h(\vec{y})\psi(\gamma)]}\bigg|_{h=0,B=0} \tag{4.17}$$

In the thermodynamic limit, the second term inside the brackets on the r.h.s vanishes (its presence is typical of the use of the canonical rather than the grand canonical ensemble). Combining Eqs. (4.17), (4.16) and (), we obtain

$$\frac{\rho^2}{s^2}h^{(2)}(\vec{x},\alpha;\vec{y},\gamma) = \frac{\rho}{s}\frac{\tilde{\delta}[B(\vec{x})\psi(\alpha)]}{\delta[h(\vec{y})\psi(\gamma)]}\bigg|_{h=0,B=0} - \frac{\rho}{s}\delta(\vec{x}-\vec{y})\,\delta_{\alpha,\gamma} \tag{4.18}$$

Since the system is now homogeneous, $h^{(2)}$ depends only on the difference $\vec{x}-\vec{y}$. Taking the Fourier transform of Eq. (4.18), we have that

$$\frac{\rho^2}{s^2}\hat{h^{(2)}}(\vec{k};\alpha,\gamma) = -\frac{\rho}{s}\delta_{\alpha,\gamma} + \frac{\rho}{s}\int d\vec{k}\; e^{i\vec{k}(\vec{x}-\vec{y})}\,\frac{\tilde{\delta}[B(\vec{x})\psi(\alpha)]}{\delta[h(\vec{y})\psi(\gamma)]}\bigg|_{h=0,B=0} \tag{4.19}$$

Finally, we note that from the definition of the generalized functional derivative,

$$\frac{\tilde{\delta}[B(\vec{x})\psi(\alpha)]}{\delta[h(\vec{y})\psi(\gamma)]} = \frac{\delta B(\vec{x})}{\delta h(\vec{y})}\delta_{\alpha,\gamma} \tag{4.20}$$

where the expression $\delta B/\delta h$ on the r.h.s is now a usual functional derivative.The properties of the usual functional derivative imply that

$$\int d\vec{k}\, e^{i\vec{k}(\vec{x}-\vec{y})}\frac{\delta B(\vec{x})}{\delta h(\vec{y})}\bigg|_{h=0,B=0} = \frac{\delta\hat{B}(\vec{k})}{\delta\hat{h}(\vec{k})}\bigg|_{h=0,B=0} = \frac{1}{1-(\rho/s)\hat{f^{\dagger}}(\vec{k})} \tag{4.21}$$

where in the last equality we have used Eq. (4.4). Substituting this result in Eq. (4.19) finally yields

$$\hat{h^{(2)}}(\vec{k};\alpha,\gamma) = \frac{\hat{f^{\dagger}}(\vec{k})}{1-(\rho/s)\hat{f^{\dagger}}(\vec{k})}\delta_{\alpha,\gamma} \tag{4.22}$$

Hence, substituting this into Eq. () finally yields the pair connectedness within the mean field approximation,

$$\hat{g^{\dagger}}(\vec{k}) = \frac{\hat{f^{\dagger}}(\vec{k})}{1-\rho\hat{f^{\dagger}}(\vec{k})} \tag{4.23}$$

This expression is consistent with Eq. () for the mean cluster size through the relation between S and $g^\dagger$ given by Eq. (1.14).

Since $f^\dagger(\vec{r}) = f^\dagger(-\vec{r})$, then for small $\vec{k}$, we have that

$$\hat{f}^\dagger(\vec{k}) = V_e^\dagger - V_1^\dagger k^2 + O(k^4) \tag{4.24}$$

where

$$\begin{aligned} V_e^\dagger &= \int d\vec{r} f^\dagger(r) \\ V_1^\dagger &= \int d\vec{r}\, \frac{r^2}{6} f^\dagger(r) \end{aligned} \tag{4.25}$$

Therefore, for small wavevectors, the pair-connectedness is

$$\hat{g}^\dagger(\vec{k}) \approx \left(\frac{\hat{f}^\dagger(\vec{k})}{1 - \rho V_e^\dagger} \right) \frac{1}{1 + [V_1^\dagger/(1 - \rho V_e^\dagger)]k^2} \tag{4.26}$$

We expect by analogy with other critical phenomena that

$$\hat{g}^\dagger(\vec{k}) \sim \frac{1}{1 + \xi^2 k^2} \tag{4.27}$$

where ξ is the *connectedness length*, which plays in percolation theory a role analogous the the one the correlation length plays in other critical phenomena [6]. Hence, we have obtained an expression for the connectedness length within mean field theory,

$$\xi^2 = \frac{V_1^\dagger}{1 - \rho V_e^\dagger} = \frac{V_1^\dagger \rho_c}{\rho_c - \rho} \sim (\rho_c - \rho)^{-\nu} \tag{4.28}$$

Thus, we find that the mean field value of the exponent ν is

$$\nu = \frac{1}{2} \qquad \text{(Mean Field)} \tag{4.29}$$

5 Discussion

The results of the mean field theory are summarized below.

$$P \;=\; 1 - \exp\left(-\rho P V_e^\dagger\right)$$

$$
\begin{aligned}
S &= \frac{1}{1-\rho V_e^\dagger} \qquad\qquad (\rho < \rho_c) \\
\xi^2 &= \frac{V_1^\dagger}{1-\rho V_e^\dagger} \\
\rho_c &= \frac{1}{V_e^\dagger} = \frac{1}{\int d\vec{r} f^\dagger(\vec{r})} \\
\beta_{MF} &= 1 \\
\gamma_{MF} &= 1 \\
\nu_{MF} &= \frac{1}{2}
\end{aligned}
\tag{5.1}
$$

The critical exponents obtained are identical to those obtained in percolation on a Bethe lattice, usually considered as the mean field equivalent for the lattice percolation model. This therefore addresses one of the questions presented in the introduction of I, namely the extent and nature of the continuum percolation universality class. As far as mean field theory is concerned, we have obtained theoretical proof that this universality class encompasses all interactions $v(i,j)$ and all binding criteria $p(i,j)$ and that it is identical to the universality class of lattice percolation. This last, incidentally, can be considered as a particular type of continuum percolation with interactions fixing the particles on the sites of a lattice. Therefore, the question of the independence of the critical exponents on the interaction is connected essentially to the relation between continuum and lattice percolation. We know, however, that mean field theory always grossly overestimates universality, as demonstrated by its being insensitive even to spatial dimensionality. Thus one cannot give too much weight to these conclusions. Nevertheless, the mean field result is an encouraging first step. Certainly had we found any difference between continuum and lattice percolation in the mean field approximation, or any dependence on the interaction or the binding criterion, it would have been highly improbable that a more elaborate calculation would have restored universality. Thus at least, our results are encouraging, if far from definitive.

As to the critical density, we do not expect it to be quantitatively adequate, as mean field results never are. However, as the dimensionality of the system increases, these results become better and better. Thus, Alon et al. [3] have shown that indeed, for a system of permeable hyper-cubes, $\rho_c V_{exc} \to 1$ for high dimensions.

As another example, for a D-dimensional system of hard-cores and soft-shells [7, 8], we have

$$v(i,j) = \begin{cases} \infty & \text{if} \quad |\vec{r}_i - \vec{r}_j| < a \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

$$p(i,j) = \begin{cases} \infty & \text{if} \quad |\vec{r}_i - \vec{r}_j| < d \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

where a and d are two length parameters. Then we obtain in the mean field approximation

$$\rho_c = \frac{1}{C_D\,(d^D - a^D)} = \frac{1}{C_D d^D\,(1-\eta^D)} \tag{5.4}$$

where $\eta \equiv d$ is the aspect ratio, and C_D id the D-dimensional volume of a unit sphere. Hence, in mean field theory, ρ_c is a monotonic function of the aspect ratio η. This contradicts the result of simulations which show ρ_c to have a pronounced minimum in 3 dimensions. However, some computer simulations [9] show that as the dimensionality increases, the minimum becomes weaker, and at high dimensions, it appears that ρ_c does increase ever more monotonically. Hence the mean field result corresponds again to the limit $D \to \infty$.

The importance of mean field theory is thus not so much in the numerical results is yields, but rather in the proof of the usefulness of the theoretical approach presented here. The mean field equations have been derived wholly within the Potts fluid picture, where heavy use is made of the available Hamiltonian formulation. The $s \to 1$ limit then yields the percolation quantities, which do indeed have the expected mean field properties. It is hard to see how such results could have been obtained directly in the percolation picture. This indicates that other methods which can be applied to the Potts fluid should yield interesting results for the continuum percolation problem. This will be the subject of future inquiries.

Appendix

The functional $F_t[B]$ is defined as in Eq. (3.8)

$$F_t(B) \equiv \frac{1}{N}\ln Z_0 + \frac{(N-1)}{2}\left\langle q(i,j)\exp\left[-\beta v(i,j)\right] - 1\right\rangle_0$$

$$
\begin{aligned}
&+ \quad \frac{(N-1)}{2} \left\langle p(i,j) \exp\left[-\beta v(i,j)\right] \delta_{\lambda_i,\lambda_j} \right\rangle_0 \\
&+ \quad \left\langle \beta \left[h(j) - B(j)\right] \psi(\lambda_j) \right\rangle_0
\end{aligned} \tag{A.1}
$$

We now have that

$$
\ln Z_0 = N \ln(\Gamma) - \ln(N!) \tag{A.2}
$$

where

$$
\Gamma \equiv \sum_{\alpha} \int d\vec{r} \exp\left[\beta B(\vec{r}) \psi(\alpha)\right] \tag{A.3}
$$

Similarly, we have immediately that

$$
\left\langle \beta \left[h(j) - B(j)\right] \psi(\lambda_j) \right\rangle_0 = \frac{1}{\Gamma} \sum_{\lambda_j} \int dj \, e^{\beta B(j) \psi(\lambda_j)} \left[h(j) - B(j)\right] \psi(\lambda_j) \tag{A.4}
$$

And

$$
\begin{aligned}
&\left\langle q(i,j) \exp\left[-\beta v(i,j)\right] \right\rangle_0 \\
&\quad = \quad \frac{1}{\Gamma^2} \sum_{\lambda_i,\lambda_j} \int di \, dj \;\; q(i,j) e^{-\beta v(i,j)} e^{\beta\left[B(i)\psi(\lambda_i) + B(j)\psi(\lambda_j)\right]}
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
&\left\langle p(i,j) \exp\left[-\beta v(i,j)\right] \delta_{\lambda_i,\lambda_j} \right\rangle_0 \\
&\quad = \quad \frac{1}{\Gamma^2} \sum_{\lambda_i,\lambda_j} \int di \, dj \;\; p(i,j) e^{-\beta v(i,j)} e^{\beta\left[B(i)\psi(\lambda_i) + B(j)\psi(\lambda_j)\right]} \delta_{\lambda_i,\lambda_j}
\end{aligned} \tag{A.6}
$$

Taking now the functional derivatives of these terms, we find

$$
\frac{\tilde{\delta} \ln Z_0}{\delta\left[N \beta B(\vec{x}) \psi(\lambda)\right]} = \frac{1}{\Gamma} \exp\left[\beta B(\vec{x}) \psi(\lambda)\right] \tag{A.7}
$$

$$
\begin{aligned}
&\frac{\tilde{\delta} \left\langle \beta \left[h(j) - B(j)\right] \psi(\lambda_j) \right\rangle_0}{\delta\left[\beta B(\vec{x}) \psi(\lambda)\right]} \\
&\quad = \quad e^{\beta B(\vec{x}) \psi(\lambda)} \frac{1}{\Gamma} \left\{-1 + \left[h(\vec{x}) - B(\vec{x})\right] \psi(\lambda)\right\} \\
&\quad - \quad e^{\beta B(\vec{x}) \psi(\lambda)} \frac{1}{\Gamma^2} \sum_{\lambda_j} \int dj \, e^{\beta B(j) \psi(\lambda_j)} \left[h(j) - B(j)\right] \psi(\lambda_j)
\end{aligned} \tag{A.8}
$$

$$
\begin{aligned}
&\frac{\tilde{\delta}\left\langle q(i,j)\exp\left[-\beta v(i,j)\right]-1\right\rangle_0}{\delta[\beta B(\vec{x})\psi(\lambda)]} \\
&= e^{\beta B(\vec{x})\psi(\lambda)}\frac{2}{\Gamma^2}\sum_{\lambda_i}\int d\vec{y}\, q(\vec{x}-\vec{y})e^{-\beta v(\vec{x}-\vec{y})}e^{\beta[B(\vec{y})\psi(\lambda_i)} \\
&- e^{\beta B(\vec{x})\psi(\lambda)}\frac{2}{\Gamma^3}\sum_{\lambda_i,\lambda_j}\int di\, dj\;\, q(i,j)e^{-\beta v(i,j)}\, e^{\beta[B(i)\psi(\lambda_i)+B(j)\psi(\lambda_j)]} \qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\tilde{\delta}\left\langle p(i,j)\exp\left[-\beta v(i,j)\right]\delta_{\lambda_i,\lambda_j}\right\rangle_0}{\delta[\beta B(\vec{x})\psi(\lambda)]} \\
&= e^{\beta B(\vec{x})\psi(\lambda)}\frac{2}{\Gamma^2}\int d\vec{y}\, p(\vec{x}-\vec{y})e^{-\beta v(\vec{x}-\vec{y})}e^{\beta[B(\vec{y})\psi(\lambda_i)} \\
&- e^{\beta B(\vec{x})\psi(\lambda)}\frac{2}{\Gamma^3}\sum_{\lambda_i,\lambda_j}\int di\, dj\;\, p(i,j)e^{-\beta v(i,j)}e^{\beta[B(i)\psi(\lambda_i)+B(j)\psi(\lambda_j)]}\delta_{\lambda_i,\lambda_j}
\end{aligned}
\qquad \text{(A.10)}
$$

We now expand these expressions around $B\to 0$. We have that

$$
\Gamma=\sum_\alpha\int d\vec{r}\,\exp\left[\beta B(\vec{r})\psi(\alpha)\right]\approx\sum_\alpha\int d\vec{r}\,[1+\beta B(\vec{r})\psi(\alpha)] \qquad \text{(A.11)}
$$

We note that

$$
\sum_\alpha\psi(\alpha)=(s-1)+(s-1)(-1)=0 \qquad \text{(A.12)}
$$

Therefore, to first order,

$$
\Gamma=sV+O(B^2) \qquad \text{(A.13)}
$$

The identity Eq. (A.12) simplifies greatly all the functional derivatives. To first order,

$$
\begin{aligned}
\frac{\tilde{\delta}\ln Z_0}{\delta[N\beta B(\vec{x})\psi(\lambda)]} &= \frac{1}{sV}\exp\left[\beta B(\vec{x})\psi(\lambda)\right]+O(B^2) \qquad \text{(A.14)} \\
\frac{\tilde{\delta}\left\langle q(i,j)\exp\left[-\beta v(i,j)\right]-1\right\rangle_0}{\delta[\beta B(\vec{x})\psi(\lambda)} &= O(B^2) \qquad \text{(A.15)}
\end{aligned}
$$

And

$$
\frac{\tilde{\delta}\left\langle\beta\left[h(j)-B(j)\right]\psi(\lambda_j)\right\rangle_0}{\delta[\beta B(\vec{x})\psi(\lambda)]}
$$

$$= \quad e^{\beta B(\vec{x})\psi(\lambda)} \frac{1}{sV} \{-1 + [h(\vec{x}) - B(\vec{x})]\, \psi(\lambda)\} + O(B^2) \tag{A.16}$$

$$\frac{\tilde{\delta} \left\langle p(i,j) \exp\left[-\beta v(i,j)\right] \delta_{\lambda_i,\lambda_j} \right\rangle_0}{\delta[\beta B(\vec{x})\psi(\lambda)]}$$

$$= \quad e^{\beta B(\vec{x})\psi(\lambda)} \frac{2}{s^2 V^2} \int d\vec{y}\, p(\vec{x}-\vec{y}) e^{-\beta v(\vec{x}-\vec{y})} B(\vec{y}) + O(B^2) \tag{A.17}$$

Summing up all these expressions and equating $\tilde{\delta} F_t / \delta[\beta B \psi] = 0$, we obtain the condition

$$B(\vec{x}) = h(\vec{x}) + \frac{\rho}{s} \int d\vec{y}\, p(\vec{x}-\vec{y}) e^{-\beta v(\vec{x}-\vec{y})} B(\vec{y}) + O(B^2) \tag{A.18}$$

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