On infrared divergences in spin glasses

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Abstract

By studying the structure of infrared divergences in a toy propagator in the replica approach to the Ising spin glass below T_c, we suggest a possible cancellation mechanism which could decrease the degree of singularity in the loop expansion.

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1 Introduction

The mean field approximation, when fluctuations are not taken into account, predicts a finite critical temperature T_c for the Ising spin glass [1]. In the replica approach [2], replica symmetry is spontaneously broken at T_c in a hierarchical and continuous way, yielding an ultrametric organization of equilibrium states (see [3, 4], [5] for a review).

Corrections to mean field theory up to third order in $\epsilon = 6 - D$ in the paramagnetic phase have been carried out in [6], while problems still exist in the computation of such corrections below T_c. The difficulties lie in the complexity of the replica approach, which leads to very complicated bare propagators, with severe infrared singularities (see [7, 8]).

A strong effort has been made in studying this difficult problem and many interesting results have been obtained. Among them, we recall the analysis of the modes with zero mass (see [9, 10]), the study of the explicit breaking of replica symmetry (see [11, 12]), the study of some strong renormalization effects (see [13, 14]) and, most recently, the derivation of a powerful method to compute the bare propagators (see [15, 16]), which provides a clearer derivation of the formulae first presented in [7, 8]. Unfortunately, the one loop contribution to the propagator has not been fully computed and even the lower critical dimension remains unknown.

Ignoring the possible renormalization effects, a naive prediction for the lower critical dimension would be 3. Indeed, disregarding the p^{-4} singularity, which appears only in the zero overlap propagator and is not present in a small magnetic field, the leading infrared singularity of the bare propagators is p^{-3}. However, it is not clear how much can really be inferred from these strong singularities in the framework of replicas. In particular, questions arise as to their origin and their consequences for measurable quantities. In fact, other approaches suggest that the lower critical dimension could be less that 3, or at least very near to 3. These follow from a computation of the free energy increase due to an interface among different phases and from numerical simulations [17, 18, 19].

The aim of this paper is to try to take a step towards a deeper understanding of the structure of these divergences. We focus on how the p^{-3} singularities are induced in the bare propagators and we suggest a possible mechanism for their cancellation.

The paper is organized as follows. After a brief summary of the replica approach, which can be skipped by the expert reader, we study a toy propagator defined using a field that explicitly breaks replica symmetry (already studied in [12] using a different method). For this propagator we are able to derive a differential equation which shows how the p^{-3} singularity due to continuous replica symmetry breaking cancels in the infrared limit and we illustrate how this cancellation results in a well-defined limit within the theory of distributions. Then we show how, in the framework of distributions, this propagator can be used to calculate the infrared behavior of a toy four-point function at tree level. We conclude our analysis by studying the structure of leading singularities in the full theory and we discuss the possible generalization of the cancellation mechanism.

2 The replica approach

The Ising spin glass, or E-A model [1], is defined by the following Hamiltonian,

$$H[s_i] = -\sum_{(i,j)} J_{ij} s_i s_j - h \sum_i s_i, \tag{1}$$

where $s_i = +1, -1$ are the spin variables, $i = 1, ..., N$, and () stands for nearest neighbors. The independent quenched parameters J_{ij} are chosen from a Gaussian distribution with zero average and variance $J^2 = 1/N$. In what follows, the magnetic field h is taken different from zero, the limit $h \to 0$ being performed after the limit $N \to \infty$.

The study of the equilibrium properties of the quenched problem can be performed in the replica approach [2]. In this approach an effective theory is obtained by averaging over the disorder, indicated by an overbar in the following. Rather than do this for the free energy, the replica approach averages over the disorder the partition function of n copies (replicas) of the original model, n being analytically continued to zero at the end. The effective theory is then symmetric with respect to permutations of the n replicas (replica symmetry), with a $n \times n$ matrix Q ($Q_{ab} = Q_{ba}$ and $Q_{aa} = 0$) for the order parameter.

At mean field level, a second order transition occurs at the de Almeida-Thouless line, which terminates at $T_c = 1$ for $h = 0$. On this line, roughly speaking, several states start to contribute to the Gibbs measure and ergodicity is lost. These results can be recovered, near T_c, through the expansion of the effective free energy density F in powers of the (small) order parameter

$$\beta F[Q] + \log 2 + \frac{\beta^2}{4} = W[Q] = -\lim_{n\to 0} \frac{1}{n}(\frac{\tau}{2}\mathrm{Tr}Q^2 + \frac{1}{6}\mathrm{Tr}Q^3 + \frac{1}{12}\sum_{ab}(Q_{ab})^4), \tag{2}$$

where $\tau = T_c - T$ and Tr stands for trace. In the framework of the Parisi ansatz [3], the saddle point of Q is looked for in a particular subspace using a hierarchical procedure. In this subspace, in which Q can be expressed in terms of a function $q(x)$ defined in the interval $[0, 1]$, the functional $W[Q]$ results in

$$W[q] = \int_0^1 (\frac{\tau}{2}q^2(x) - \frac{1}{6}(xq^3(x) + 3q^2(x)\int_x^1 q(y)dy) + \frac{1}{12}q^4(x))dx. \tag{3}$$

Below T_c stationarity with respect $q(x)$ yields the solution

$$\begin{aligned} q(x) &= x/2 \quad 0 < x \le x_1 \\ q(x) &= q(x_1) \ \ x_1 < x < 1, \end{aligned} \tag{4}$$

where x_1 is defined by $2\tau - x_1 + x_1^2/2 = 0$. Replica symmetry, which in this framework requires $q' = 0$, is spontaneously broken below T_c.

Replica symmetry breaking (RSB) is related to the probability of measuring a given value q for the overlap $\overline{< s_i >_a < s_i >_b}$ between two states, a and b, which differ by a finite amount in free energy. It has been shown that such a probability can be computed through the order parameter $q(x)$,

$$P(q) = \left(\frac{dq(x)}{dx} \right)^{-1} = 2\theta(q_1 - q) + (1 - 2q_1)\delta(q - q_1). \tag{5}$$

In the metric dictated by the overlap these states are organized ultrametrically: given three states at least two overlaps are equal, the third being greater than the other two. This organization allows the use of a tree to express the overlap between different states. By putting the states at the end of the branches of a tree, the overlap between the states can be represented by the distance between the top root and the level of the point where the branches coincide.

Before going beyond mean field, let us recall that the fluctuations of the order parameter Q around the RSB saddle point are usually divided into three families: longitudinal (L), anomalous (A) and replicon (R) (see [16] for the most recent and exhaustive analysis).

The longitudinal modes are by definition invariant under the action of the symmetry group which leaves invariant the ansatz of Q, and therefore correspond to fluctuations in $q(x)$. On the other hand, the anomalous and replicon modes break even this n replica permutation group and are parametrized in term of functions of two and three variables, as explained in detail in [16]. The conditions imposed by the breaking of this residual symmetry turn out to be strong enough to determine completely the R eigenvalues but not the L-A ones, where one has to explicitly solve the integral eigenvalue equations.

Zero modes of the fluctuations around the mean field saddle point appear in each family, the rest of the spectrum being positive. A remarkable result is that in the replicon sector, where one has a closed expression for the eigenvalues $\lambda(x, k, l)$, one finds that zero modes are present for a finite range of eigenvectors, i.e.

$$\lambda(x, x, x) = 0. \tag{6}$$

This can also be seen by explicit differentiation of the saddle point equation and using the fact that replica symmetry breaking is continuous (as first shown in [9]).

For future reference, let us also recall that an explicit RSB can be introduced in the theory (a deep analysis on the nature of the explicit RSB can be found in [11]) by adding to the effective free energy the term

$$\int_0^1 q(x)\epsilon(x)dx. \tag{7}$$

A finite conjugate field ϵ induces a shift in the order parameter $q(x)$ which provides a kind of infrared regulator because it induces a gap proportional to the slope of ϵ in the spectrum (as shown in [12]).

In the next section we consider such a field ϵ as an external source in order to perform a detailed analysis of the infrared limit in the replica approach.

3 The "projected" theory

For the time being, let us define a propagator in the subspace identified by $q(x)$. To define such propagator, we add to the functional $W[q]$ a kinetic term and consider how a small external conjugate field ϵ, explicitly breaking replica symmetry, perturbs the mean field solution. We have the following theory for a field $\delta q(x;p)$, in which x is a continuous internal degree of freedom,

$$W[q+\delta q] - \frac{p^2}{4}\int_0^1 (\delta q(x;p))^2 dx + \int_0^1 \delta q(x;p)\epsilon(x;p)dx, \tag{8}$$

and we introduce the bare propagator $G(x,y;p)$ through the relation

$$\delta q(x;p) = \int_0^1 G(x,y;p)\epsilon(y;p)dy. \tag{9}$$

For small ϵ the equation for $q(x)$ leads the following equation (see A),

$$p^2 G(x,y;p) + 2\int_0^x q(z)G(z,y;p)dz + 2q(x)\int_x^1 G(z,y;p)dz = 2\delta(x-y). \tag{10}$$

After repeated differentiation of the replica variable, we obtain

$$p^2\frac{\partial^2 G(x,y;p)}{\partial x^2} - 2q'(x)G(x,y;p) + 2q''(x)\int_x^1 G(z,y;p)dz = 2\delta''(x-y). \tag{11}$$

In what follows we are mainly interested in the diagonal sector $x = y < x_1$ when $p \to 0$, so we focus on this case. For $p = 0$ the previous equation leads to the solution

$$G(x,y;0) = -2\delta''(x-y), \tag{12}$$

while the solution to these equations for finite p, and $x, y < x_1$ is

$$G(x,y;p) = \frac{2\delta(x-y)}{p^2} - \frac{e^{-\frac{|x-y|}{p}}}{p^3} + g(x,y;p), \tag{13}$$

where the function $g(x,y;p)$ is not singular in the limit $p \to 0$ (A).

We observe that propagator already computed has been already studied in [12], using a different method, for the purpose of discussing the regularization induced by ϵ.

The results (12, 13) are interesting for two reasons. On the one hand, (12) shows how the small momentum behavior of this propagator can be cast into the form of a distribution. The integral kernel of the zero momentum equation for G is only $\min\{q(x), q(y)\}$ because the strictly diagonal contribution of the kernel vanishes on the mean field saddle point. This implies that eq. (12) is its inverse, which shows that a small $\epsilon(x) > 0$, independent of space, induces a shift of $q(x)$ for $x < x_1$ only through $-2\epsilon(x)'' > 0$. The propagator induced by ϵ for $x < x_1$ is massive.

On the other hand, (13) allows us to understand how continuous replica symmetry breaking gives rise before x_1 to the diagonal p^{-3} singularity for small p. In fact, when we add the kinetic term of order p^2 on the diagonal, the absence of a strictly diagonal contribution in the kernel implies a contribution p^{-2} on the diagonal of G. But now, to keep the off-diagonal elements of the product of the two matrices zero, a diverging off-diagonal contribution in G is also needed. Such a contribution can only be p^{-3} with an exponential prefactor because it has to be smooth and coalesces for $p = 0$.

We learn that care is needed in the limit $p \to 0$. This limit should be considered in the sense of distributions, paying attention to the cross-over between x and p. A finite momentum induces a regularization of the distribution which appears through two singular terms, which cancel in the limit of small p.

To proceed in this analysis let us now consider the propagator with a source $\epsilon \neq 0$. We know from the previous analysis (at $\epsilon = 0$) that for small p there is an off-diagonal contribution of width p and order p^{-3} that cancels the p^{-2} singularity and leads to a massive propagator. The analysis for $p = 0$ and small ϵ is similar with ϵ'/q' playing the role of p^2. At zero momentum the propagator does not diverge as ϵ'^{-1} but is instead given by

$$G(x, y; 0)_\epsilon = \frac{\delta q(x)_\epsilon}{\delta \epsilon(y)} = -\frac{\delta''(x-y)}{q'(x)_\epsilon}. \tag{14}$$

Let us use this result to compute the four point function at tree level. By taking two derivatives with respect to the conjugate field ϵ we obtain

$$G^{(4)}_{conn}(x, y, z, w; 0) = \frac{\delta^2 G(x, y; 0)_\epsilon}{\delta\epsilon(z)\delta\epsilon(w)}|_{\epsilon=0} = -2\frac{\delta''(x-y)\delta'''(x-w)\delta'''(x-z)}{q'(x)^5}. \tag{15}$$

This result, derived in detail in B, shows that in the four point function the infrared contributions from the two diagrams with four external legs, the one from the quartic vertex and the one from two cubic vertices with a propagator flowing between, are similar but do not cancel.

The absence of the complete cancellation is not too surprising for this propagator. In fact this propagator is massive and expresses the response of the order parameter to an explicit breaking of the replica symmetry. It is similar to the longitudinal propagator in an $O(N)$ model, which is regular in the infrared limit. However it is remarkable that in the long wavelength limit the propagator forces the two diagrams with four external legs, which are very different in structure, towards the same kind of contribution. This is exactly what happens in an $O(N)$ model, where the cancellation of a different four point function (transverse) is required by a Ward identity.

The possibility of using distributions to investigate the infrared limit of the complete propagators and to compute the complete four point function is appealing. Let us analyze the structure of the leading singularities of the full theory.

4 A preliminary analysis of the full theory

Our aim here is to investigate the possibility of extending the results obtained for the toy propagator, a two index object, to the complete propagators.

We have performed a preliminary study of these propagators using the results in [7, 8]. The propagators for the full theory are defined through the inverse of the mass matrix with a diagonal kinetic term

$$G_{\alpha\beta,\gamma\delta}(p) =< \delta Q_{\alpha\beta}(p)\delta Q_{\gamma\delta}(-p) >= \left(p^2 + \frac{\delta^2 W[Q]}{\delta Q \delta Q}\right)^{-1}_{\alpha\beta,\gamma\delta} . \tag{16}$$

Because the replicon eigenvalues are known in a closed form the Green functions are usually split (we follow the notation introduced in [8]) into two contributions,

$$^{R}G^{xx}_{z_1 z_2}(p) \quad \text{and} \quad ^{LA}G^{xy}_{z_1 z_2}(p), \tag{17}$$

where $\alpha \cap \beta = x$ if $q_{\alpha\beta} = q(x)$, $\alpha \cap \alpha = 1$, and $x = \alpha \cap \beta$, $y = \gamma \cap \delta$, $z_1 = \max\{\alpha \cap \gamma, \alpha \cap \delta\}$ and $z_2 = \max\{\beta \cap \gamma, \beta \cap \delta\}$.

We are interested in the divergences of order p^{-3} because we know from [7] that the p^{-4} singularity is confined to strictly zero overlap and disappears in a small magnetic field. From the full propagators given in [8] one obtains in the long wavelegth limit in the case of $0 < x, y < x_1$ (see C)

$$\begin{aligned}
z < x, y \qquad & ^{LA}G^{xy}_{zz}(p) \simeq \frac{e^{\frac{2z-u-v}{p}}}{up^3} \\
u < z \qquad & ^{LA}G^{xy}_{xz}(p) \simeq \frac{e^{\frac{u-v}{p}}}{up^3} \\
z_1, z_2 \geq x \qquad & ^{LA}G^{xx}_{z_1 z_2}(p) \simeq \frac{1}{xp^3} \\
z_1, z_2 > x \qquad & ^{R}G^{xx}_{z_1 z_2}(p) \simeq \frac{1}{x^2 p^2},
\end{aligned} \tag{18}$$

where the last two formulae have already been given in [7] and $u = min\{x, y\}$, $v = max\{x, y\}$. Let us analyze these results. The p^{-3} singularities are confined to the diagonal L-A contribution, where the ultrametric prefactor is u^{-1}. In the R propagator the singularities are of order p^{-2} and the ultrametric prefactor is x^{-2}.

We are tempted to understand all these singularities as being generated by the same mechanism as those of the toy propagator of the previous section. That is, the zero modes in the R fluctuations ($\lambda(x, x, x) = 0$) induce the p^{-3} singularity in the diagonal (upper indices) L-A propagator.

However, it seems that there is a difference with respect to the toy propagator. In fact in the toy propagator the p^{-3} singularity is canceled because of the diagonal p^{-2} singularity,

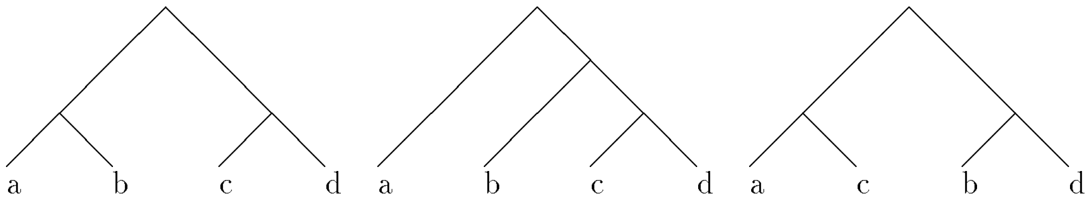

Figure 1: From the left to the right we show the ultrametric trees which correspond to the propagators G^{xy}_{zz}, G^{xy}_{xz} and $G^{xx}_{z_1 z_2}$.

opposite in sign. In the full theory the ultrametric prefactor of the diagonal R contribution is not the same as the L-A one, the sign also being the same.

To proceed in this analysis, let us observe that when one has four ultrametric indices, there are several different possibilities of arranging them on an ultrametric tree. In fig. 1 we show the trees corresponding to three of the possibilities. The three graphs correspond to the three possibilities of having two equal indices in the four index propagator. In the notation used for the propagators, where at least two indices must be equal, the graphs correspond, from left to right, to ${}^{LA}G^{x,y}_{zz}, {}^{LA}G^{x,y}_{xz}$ and ${}^{LA}G^{xx}_{z_1 z_2} +{}^{R} G^{xx}_{z_1 z_2}$. When the two equal indices are the upper ones ($x = y$), there can be both an R and an L contribution to the propagator (third graph in fig. 1), apart for the boundaries $z_1, z_2 = x$ where the R contribution is zero.

Because ${}^{LA}G^{xx}_{z_1 z_2} \sim p^{-3}$ does not depend on $z_1, z_2 \geq x$, we are interested in the *volume* of the corresponding L-A diagonal subspace, i.e. the sum

$$\sum_{abcd} {}^{LA}\theta_{ab,cd}, \tag{19}$$

where θ is equal to 1 only in the diagonal L-A region. This sum can be performed using the general result of Mézard in which a sum over replica indices can be transformed into a sum over all the possibilities of arranging the indices on an ultrametric tree [20].

A sum over several replica variables is equivalent to a sum of different contributions, each one corresponding to a possibility of arranging the variables on an ultrametric tree. Each tree has a weight that is given by the number of possible permutation of the tree multiplied by a factor which depends on the structure of the tree. If we specialize this result to the case of four indices, we obtain that for each node with three branches the factor is x while for each node with four branches the factor is $2x^2$. The previous prescription holds for all different indices. When the indices coincide pairwise there is an additional multiplicative factor $(-1)^m$, where m is the number of pairs.

Using these prescriptions to compute (19) we obtain

$$2x^2 + 2(1-x)^2 + 4x(1-x) - 4x - 4(1-x) + 2 = 0, \tag{20}$$

i.e., the volume of the L-A diagonal subspace vanishes. The volume of the R subspace $z_1, z_2 > x$ is given by

$$2(1-x)^2 - 4(1-x) + 2 = 2x^2, \tag{21}$$

while the volume of the off-diagonal subspace is

$$x + 4(1-x) - 4 + 2x = -x, \tag{22}$$

which is exactly the factor necessary for the L-A propagator to have the same x dependence found in the R one!

Let us conclude by considering the projection of the four index propagators in a two index subspace. In other words we want to define a two index object, through the four-index replica propagators, by integrating the lower indices. This operation is defined by the sum of all the propagators $G_{ab,cd}$ over all replicas a, b, c, d at fixed $x = a \cap b$ and $y = c \cap d$. Using distributions for small p we find that

$$\begin{aligned} \sum_{abcd} G_{ab,cd}(p) &\simeq -2x^2 \frac{\delta(x-y)}{x^2 p^2} + 2x \left(\frac{\delta(x-y)}{x p^2} - \frac{\delta''(x-y)}{x} \right) \\ &\simeq -2\delta''(x-y) \\ &= G(x, y; 0). \end{aligned} \tag{23}$$

5 Conclusions

In this paper we have analyzed a toy propagator that expresses how a small external field, explicitly breaking replica symmetry, induces a perturbation on the order parameter $q(x)$. This propagator, defined in the subspace identified by $q(x)$, turns out to be, as expected, the projection of the complete set of propagators in this subspace.

We have shown that for finite p this propagator is essentially given by two contributions, their singularities canceling for $p \simeq 0$. This cancellation results in a well-defined infrared limit within the theory of distributions that we have used to extract the infrared behavior of a toy four point function at tree level.

We have considered some aspects of the full theory. In particular we have analyzed the ultrametric structure of the subspace where the fluctuations are of order p^{-3}, which corresponds to the third graph in fig. 1. This subspace has a global volume equal to zero and includes, apart from the boundaries, the replicon subspace. Because the volume of the off-diagonal subspace is $-x$ one might conjecture that, by casting the infrared behavior of the propagators within the theory of distributions, these singularities cancel.

A careful analysis is necessary, and work is in progress in this direction.

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A Projected propagator

The equation of state at first order in ϵ is

$$\frac{\delta W[q]}{\delta q(x)}|_{\epsilon=0} + \int_0^1 \frac{\delta^2 W[q]}{\delta q(x)\delta q(y)}\delta q(y;p)dy - \frac{p^2 \delta q(x;p)}{2} + \epsilon(x;p) = 0 \tag{24}$$

which leads to following equation for $G(x,y)$

$$p^2 G(x,y;p) + 2\int_0^x q(z)G(z,y;p)dz + 2q(x)\int_x^1 G(z,y;p)dz = 2\delta(x-y). \tag{25}$$

The solution of eq. (25) for finite p can be achieved by solving the inhomogeneous equations, adding the most general solution of the homogeneous equations and searching for the coefficients which solve eq. (25). We obtain for $x, y < x_1$

$$G(x,y;p) = \frac{2\delta(x-y)}{p^2} - \frac{e^{-|x-y|/p}}{p^3} + g(x,y;p) \tag{26}$$

where the function $g(x,y)$ is

$$\begin{aligned}
g(x,y;p) = &\quad g^{++}e^{+(x+y)/p} + g^{--}e^{-(x+y)/p} + g^{+-}(e^{-|x-y|/p} + e^{+|x-y|/p}) \\
g^{++} = &\quad -\frac{p-(1-x_1)}{p^3}\frac{e^{-(x_1/p)}}{(p+(1-x_1))e^{+(x_1/p)} + (p-(1-x_1))e^{-(x_1/p)}} \\
g^{--} = &\quad +\frac{p+(1-x_1)}{p^3}\frac{e^{+(x_1/p)}}{(p+(1-x_1))e^{+(x_1/p)} + (p-(1-x_1))e^{-(x_1/p)}} \\
g^{+-} = &\quad +\frac{p-(1-x_1)}{p^3}\frac{e^{-(x_1/p)}}{(p+(1-x_1))e^{+(x_1/p)} + (p-(1-x_1))e^{-(x_1/p)}},
\end{aligned} \tag{27}$$

as obtained in [12] by using the longitudinal eigenvalues. As can be seen from eq. (25), when x or y goes beyond the breakpoint x_1 the solution, apart the delta function, does not depend any more on this variable. For x and $y > x_1$ we then obtain

$$G(x,y;p) = \frac{2\delta(x-y)}{p^2} - \frac{2}{p^2}\frac{e^{(x_1/p)} - e^{(-x_1/p)}}{(p+(1-x_1))e^{(x_1/p)} + (p-(1-x_1))e^{-(x_1/p)}}. \tag{28}$$

In the limit $p \to 0$ this gives

$$G(x,y;p) \simeq \frac{2\delta(x-y)}{p^2} - \frac{2}{p^2}\frac{1}{(1-x_1)}. \tag{29}$$

B Projected four point function

The aim of this appendix is to show how the result for four point function can be derived through the "projected" propagator. By a derivative with respect to ϵ we obtain

$$\begin{aligned}\frac{\partial G(x,y;0)}{\partial \epsilon(z)} &= \delta''(x-y)\frac{1}{q'(x)^2}\frac{d}{dx}\frac{\partial q(x)}{\partial \epsilon(z)} \\ &= \delta''(x-z)(-\delta'''(x-z)\frac{1}{q'(x)^3}+\delta''(x-z)\frac{q''(x)}{q'(x)^4}) \end{aligned} \tag{30}$$

and by an additional derivative we obtain

$$\begin{aligned}G^{(4)}_{conn}(x,y,z,w) &= \delta''(x-y)(3\delta'''(x-z)\frac{1}{q'(x)^4}\frac{d}{dx}\frac{\partial q(x)}{\partial \epsilon(w)} \\ &\quad -4\delta''(x-z)\frac{q''(x)}{q'(x)^5}\frac{d}{dx}\frac{\partial q(x)}{\partial \epsilon(w)}+\delta''(x-z)\frac{1}{q'(x)^2}\frac{d^2}{dx^2}\frac{\partial q(x)}{\partial \epsilon(w)}) \\ &= -\frac{\delta''(x-y)}{q'(x)^5}(3\delta'''(x-z)\delta'''(x-w)+\delta''(x-z)\delta''''(x-w)) \\ &= -\frac{\delta''(x-y)}{q'(x)^5}(2\delta'''(x-z)\delta'''(x-w)) \end{aligned} \tag{31}$$

In the last step the contributions of the two diagrams with four external legs, which do not cancel because a multiplicity factor 3, can be recognized.

This result can also be obtained using diagrams. The expansion of W around the mean field saddle point in the subspace identified by $q(x)$ gives a cubic and a quartic vertex

$$\begin{aligned}V^{(3)}(x,y,z) &= -\frac{\delta(x-y)\theta(z-y)+2\ \ permut.}{6} \\ V^{(4)}(x,y,z,t) &= +\frac{\delta(x-y)\delta(y-z)\delta(z-t)}{12}. \end{aligned} \tag{32}$$

Using these vertices, together with the projected propagator, to calculate the four point amputated function at the tree level and zero momenta we find

$$\begin{aligned}G^{(4)}_{amp}(x,y,z,t) &= -4!V^{(4)}(x,y,z,t)+3(3!)^2\int_0^1\int_0^1 dudv V^{(3)}(x,y,u)G(u,v;0)V^{(3)}(v,z,t) \\ &= \delta(x-y)\delta(y-z)\delta(z-t)(-2+6)\neq 0 \end{aligned} \tag{33}$$

which is exactly the same result obtained by differentiation on the propagator after cutting of the external legs.

C Infrared limit of the full propagators

In this section we use the results given in [8] for the full Gaussian propagators as the starting point to explicitly derive their behaviour in the infrared limit. Using the results and the notation presented in [8] the propagators are given as integrals of a kernel F_k^{uv} weighted with an ultrametric measure.

In the L-A sector we obtain that for small p the formulae for the leading contribution to the kernels in [8] simplify. We consider first the sector $k < u < v$. The leading contribution of order p^{-3} is independent of k and disappears in the integral by the derivative. However when k is within order p of u the kernel becomes of order p^{-2}. In fact we have

$$-\frac{1}{k}\frac{\partial}{\partial k}\,{}^{LA}F_k^{uv} \simeq -\frac{1}{k}\frac{e^{\frac{2k-u-v}{p}}}{p^3}(-\frac{2}{p}+\frac{8}{k}) \tag{34}$$

which, for the propagator, leads to

$$ {}^{LA}G_{zz}^{xy} \simeq \frac{e^{\frac{2z-u-v}{p}}}{up^3}. \tag{35}$$

For $u < k < v$ we find that the kernel for small p does not contribute to leading behavior of the second and third propagator, which are then given only by the kernel $k < u < v$

$$ {}^{LA}G_{xz}^{xy} \simeq \frac{e^{\frac{-|x-y|}{p}}}{up^3}. \tag{36}$$

The last L-A propagator (the diagonal) is then

$$ {}^{LA}G_{z_1 z_2}^{xx} \simeq \frac{1}{up^3}. \tag{37}$$

In the R sector the divergent contribution to the propagator for $0 < x < x_1$ is given by

$$\begin{aligned} {}^{R}G_{z_1 z_2}^{xx} &= \int_x^{z_1}\frac{dk_1}{k_1}\int_x^{z_2}\frac{dk_2}{k_2}\frac{\partial^2}{\partial k_1 \partial k_2}\frac{4}{4p^2+k_1^2+k_2^2-2x^2} \\ &\simeq \frac{8}{x^2}\int_0^{2x(z_1-x)} d\eta_1 \int_0^{2x(z_2-x)} d\eta_2 \frac{1}{(4p^2+\eta_1+\eta_2)^3} \\ &\simeq \frac{1}{x^2p^2} \end{aligned} \tag{38}$$

that is independent of z_1, z_2, if they are greater than x by a finite amount.

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