Fluctuations in a Spin Glass model with 1RSB

M. E. Ferrero, G. Parisi and P. Ranieri

Dipartimento di Fisica Università di Roma La Sapienza and INFN sezione di Roma I
Piazzale Aldo Moro, Roma 00185

Abstract. We discuss Gaussian fluctuations in a spin glass model with one replica symmetry breaking and we show how non-perturbative fluctuations of the break-point parameter can be included in the longitudinal propagator within linear response theory.

The aim of this letter is to discuss the fluctuations and more generally the corrections to the mean field theory of spin glass models where first order replica symmetry breaking occurs. We remind to the reader that replica symmetry can be broken in two different ways [1]:

- The function $q(x)$ is discontinuous and it takes only a finite number of values (in most cases two). Here the function $P(q)$ is the sum of a finite number of delta functions. For example in the case of only one step (1RSB) we have

$$q_m(x) = q_0 \quad \text{for} \quad x < m, \quad q_m(x) = q_1 \quad \text{for} \quad x > m. \tag{1}$$

  The corresponding function $P(q)$ is given by

$$P(q) = m\delta(q - q_0) + (1 - m)\delta(q - q_1). \tag{2}$$

- The function $q(x)$ is a continuous function and in this case also the function $P(q)$ has a continuous part.

Some models as the Sherrington Kirpatrick and the Edwards Anderson model belong to the second category, other models, the random energy model, Ising spins with p interactions and $p > 2$, the q state Potts model with $q > 4$, the ROME (random orthogonal matrix ensemble) belong to the first category.

The computation of the fluctuations and the corrections to the saddle point limit is rather difficult in the second case, where the form of the propagators is quite involved, and requires many powerful tools [2].

In the first case (1RSB) the situation was supposed to be much simpler, the propagator can be explicitly computed taking care of only the fluctuations of q_0 and q_1. The problems arise when the fluctuations in the variable m are considered.

Fluctuations changing m by a small amount are small in some sense and they have to be taken into account in the computations, but in some other sense they are large and the usual formalism (as we shall see) does not take them into account and must consequently be modified. Indeed it is true that when $m \to \tilde{m}$ $q_m(x) \to q_{\tilde{m}}(x)$ in some sense (for example in the L^p norm with finite p), but the quantity

$$\sup_x (q_m(x) - q_{\tilde{m}}(x)) \equiv |q_m - q_{\tilde{m}}|_\infty \tag{3}$$

does not go to zero in this limit.

One of the first results suggesting the necessity of taking care of fluctuations which correspond to variations of m is the following [3]. In the random energy model (REM) of Derrida the free energy can be written as

$$F(\beta) = -\frac{N}{\beta m}\log 2 + \frac{N\beta m}{2} \tag{4}$$

The correct result is obtained as a saddle point in m for large N. Corrections proportional to $1/N$ to the free energy density are clearly connected to fluctuations in m, while if we consider the formalism of [4] and we represent the REM as a Ising model with a p-spin interaction these corrections cannot come from fluctuations in the q parameters, which vanish in this limit.

More recently it has been shown (see [5]) that if one does not take into account the fluctuations of m one obtains the wrong result for the specific heat while the correct result could be obtained by taking into account the m- fluctuations using a simple (but at this stage arbitrary) prescription.

The aim of this note is to compute part of the fluctuations (the so called longitudinal propagator) by using the linear response theory, i.e. by evaluating the variation of the function $q(x)$ with respect to an external perturbation. This propagator contains singular terms, which are not found using the conventional approach. Correct results for the specific heat are obtained using this improved propagator.

We postpone the computation of the full propagator to a future investigation. Here we limit ourselves to the computation of the longitudinal propagator.

1. The Model

We use in our analysis a simple model that we consider representative of the class of models with a 1RSB saddle point. The model is simply obtained by adding an additional cubic term to the usual truncated free energy (W in the following), i.e.

$$W[Q] = -\lim_{n\to 0}\frac{1}{n}\left(\frac{\tau}{2}\mathrm{Tr}Q^2 + \frac{1}{6}\mathrm{Tr}Q^3 + \frac{\alpha}{6}\sum_{ab}(Q_{ab})^3 + \frac{\beta}{12}\sum_{ab}(Q_{ab})^4\right), \tag{5}$$

where $\tau = T_c - T$ and Tr stands for trace. We recall that in the SK model $\alpha = 0$ and $\beta = 1$ (while, for example, in the 3-state Potts model $\alpha = 1/2$ and β is negative [6]). In the framework of the Parisi ansatz the saddle point of Q is looked for in a subspace in which Q can be expressed in terms of a function $q(x)$ defined in the interval $[0,1]$. In this subspace the functional $W[Q]$ is given by

$$W[q] = \int_0^1 dx \left[\frac{\tau}{2} q^2(x) - \frac{1}{6} \left(x q^3(x) + 3 q^2(x) \int_x^1 q(y) dy \right) + \frac{\alpha}{6} q^3(x) + \frac{\beta}{12} q^4(x) \right]. \tag{6}$$

Below T_c stationarity with respect to the order parameter yields the 1RSB solution

$$q_m(x) = q_1 \theta(x - m) \tag{7}$$

where the parameters q_1 and m ($q_0 = 0$) are obtained by the saddle point conditions as a perturbative series in β,

$$\begin{aligned} q_1 &\simeq \frac{\tau}{1-\alpha} + \frac{5}{6} \frac{\beta \tau^2}{(1-\alpha)^3} + \frac{25}{18} \frac{\beta^2 \tau^3}{(1-\alpha)^5} \\ m &\simeq \alpha + \frac{\beta \tau}{1-\alpha} + \frac{5}{6} \frac{\beta^2 \tau^2}{(1-\alpha)^3}. \end{aligned} \tag{8}$$

The role played by the additional cubic term is to provide a breaking of replica symmetry which is located at $m \simeq \alpha$ while it is well known that in the SK model $m \sim \beta\tau$. To investigate the stability of this saddle point with respect to Q fluctuations we need the eigenvalues of the matrix

$$M_{ab,cd} = \frac{\partial^2 W}{\partial Q_{ab} \partial Q_{cd}}. \tag{9}$$

We find that the eigenvalues of this Hessian †, which should be positive in order to have a stable saddle point, are

$$\begin{array}{llll} \lambda_0 & = \lambda_{1,0} & = -\tau - q_1(m-1) & \to -\beta q_1^2/6 \\ \lambda_1 & = \lambda_{1,1} & = -\tau - \alpha q_1 - \beta q_1^2 - q_1(m-2) & \to -\beta q_1^2/6 + q_1(1-m) \\ & = \lambda_{2,0} & = -\tau - q_1(m/2-1) & \to -\beta q_1^2/6 + q_1 m/2 \\ & = \lambda_{2,1} & = -\tau - \alpha q_1 - \beta q_1^2 - q_1(m/2-2) & \to -\beta q_1^2/6 + q_1(1-m/2) \\ & = \lambda_{0,1,1} & = -\tau - q_1(m-1) & \to -\beta q_1^2/6 \\ & = \lambda_{1,2,2} & = -\tau - \alpha q_1 - \beta q_1^2 + q_1 & \to -\beta q_1^2/6 \\ \lambda_{0,1,2} & = \lambda_{0,2,1} & = -\tau - q_1(m/2-1) & \to -\beta q_1^2/6 + q_1 m/2 \\ & = \lambda_{0,2,2} & = -\tau + q_1 & \to -\beta q_1^2/6 + q_1 m. \end{array} \tag{10}$$

Using the saddle point values, we find that the minimum eigenvalue belongs to four degenerate sub-families ($\lambda_0 = \lambda_{1,0} = \lambda_{0,1,1} = \lambda_{1,2,2}$) and it is proportional to $-\beta\tau^2$. This shows that a coefficient $\alpha \neq 0$ allows the prescription of keeping a negative coefficient β in order to have a stable 1RSB ansatz (see also [8]), without a negative value for m and without a negative eigenvalue.

† For a complete and general analysis of the eigenvectors structure in the replica approach, see [7]. In what follows we use the notation presented in [7].

2. Fluctuations

Let us now consider this model in the Gaussian approximation. Our aim is to derive, within linear response theory, a longitudinal propagator which takes into account m fluctuations. In order to have a consistent check of our computation, at the end of this section we shall compare the specific heat obtained through this improved propagator with the usual expression obtained through the saddle point solution.

In the replica approach, the longitudinal propagator can be computed in the discrete formulation of replica symmetry breaking, i. e. by using the parameters q_0, q_1 and m and the global variations δq_0, δq_1 and δm (as done in [5]), or by considering the function $q(x)$ (see Eq. (6)) and the local variations $\delta q(x)$ (eventually followed by integration). Clearly, if we work in the local formulation, that is the first step toward the analysis in the full space, we need a method to deal with the m fluctuations. As previously mentioned, these fluctuations induce a non-perturbative (not small in the sense of Eq. (3)) variations on the function $q(x)$ and it is unclear how they can be taken into account in a perturbative computation.

In order to take into account these fluctuations let us introduce in this model an external field conjugate with the order parameter:

$$W[q] \to W[q] + \int_0^1 q(x)\epsilon(x). \tag{11}$$

The perturbation induced by this field on the saddle point solution can be parametrized as follows:

$$q_m^\epsilon(x) = q_1\theta(x-m-\delta m) + \delta q(x). \tag{12}$$

Within linear response theory, we define the longitudinal propagator by considering the response with respect to ϵ, i. e.

$$G(x,y) = \frac{\delta q_m^\epsilon(x)}{\delta\epsilon(y)} = -q_1\delta(x-m)\frac{\delta m}{\delta\epsilon(y)} + \frac{\delta q(x)}{\delta\epsilon(y)}. \tag{13}$$

The equations for the two components of the propagator follow from the equations of motion with the source $\epsilon \neq 0$ and from their expansion to first order in ϵ, i. e.

$$\left.\frac{\delta W[q]}{\delta q(x)}\right|_{q=q_\epsilon} = \epsilon(x) \tag{14}$$

$$\frac{\partial W[q_\epsilon]}{\partial m} = -\, q_1\epsilon(m). \tag{15}$$

Using this procedure we manage to consider in a perturbative approach a non-perturbative contribution. On the one hand the variations $\delta q(x)$ and δm defined in Eq. (12,13) play a different role. The introduction of $\delta(x-m)$ as a multiplicative factor of the component $\frac{\delta m}{\delta\epsilon(y)}$ is crucial in (13) because this delta-function separates without ambiguities the two contributions. On the other hand the two equations (14, 15) are qualitatively different: the first is a functional derivative of the free energy functional $W[q]$ while the latter is a derivative of a function $W(q_\epsilon(x)) = \tilde{W}_{q_1,m}(\delta q(x), \delta m)$ with respect to δm. There, while in the Eq. (15) the distribution functions are integrated and no ambiguity exists, in the Eq. (14) we have to deal with products of distribution functions (i.e. $\theta^2(x-m)$, $\theta(x-m)\delta(x-m)$ and $\theta^2(x-m)\delta(x-m)$). These products are, at this stage, ill defined and a regularization scheme is necessary. In what follows we choose a regularization such that

$$\theta^k(x-m) = \theta(x-m), \tag{16}$$

$$\theta^{k-1}(x-m)\delta(x-m) = \frac{1}{k}\delta(x-m), \tag{17}$$

where the function $\delta(x-m)$ that occurs in Eq. (13) is defined as the derivate of the function $\theta(x-m)$. Therefore relation (17) is the derivative of relation (16), that is the only arbitrary choice we make. One can also see that Eq. (17) involves the following prescription to evaluate the integral of the function $q^k(x)$ on a peaked measure:

$$\int_0^1 q(x)^k q_1 \delta(x-m) dx = \int_0^{q_1} q^k dq = \frac{q_1^{k+1}}{k+1}. \tag{18}$$

By expanding equations (14) and (15) to first order in ϵ and by using (16) and (17) we obtain following equations for the propagator components:

$$\begin{aligned} &-q_1\theta(x-m)\int_m^1 dy \frac{\delta q(y)}{\delta\epsilon(z)} + \frac{1}{6}\beta q_1^2 \frac{\delta q(x)}{\delta\epsilon(z)} + \frac{1}{2}q_1^2\theta(x-m)\frac{\delta m}{\epsilon(z)} = \delta(x-z) \\ &\frac{1}{2}q_1^2\int_m^1 dy \frac{\delta q(y)}{\delta\epsilon(z)} - \frac{1}{3}q_1^3\frac{\delta m}{\delta\epsilon(z)} = -q_1\delta(m-z). \end{aligned} \tag{19}$$

The corresponding result for the longitudinal propagator (13) is

$$\begin{aligned} G(x,y) = {}& G_0\,\delta(x-y) + G_1\,\theta(x-m)\theta(y-m) + \\ &- \Big(G_0^N\delta(x-m) + G_1^N\theta(x-m)\Big)\Big(G_0^N\delta(y-m) + G_1^N\theta(y-m)\Big) \end{aligned} \tag{20}$$

where

$$\begin{aligned} &G_0 = \frac{1}{q_1^2\beta/6}, \quad G_1 = \frac{1}{q_1^2\beta/6}\frac{q_1}{(q_1^2\beta/6-(1-m)\,q_1)}, \quad G_0^N = \sqrt{\frac{3(q_1^2\beta/6-(1-m)\,q_1)}{q_1\,(q_1^2\beta/6-(1-m)\,q_1/4)}}, \\ &G_1^N = \sqrt{\frac{3/4\;\; q_1}{(q_1^2\beta/6-(1-m)\,q_1/4)(q_1^2\beta/6-(1-m)\,q_1)}}. \end{aligned} \tag{21}$$

Two new terms, overlooked by the usual computation, appear in this longitudinal propagator, the term G_0^N and G_1^N. These terms, singular at $x \simeq m$, are the effect of the m fluctuations. Let us also note that the asymmetry between the regions $x > m$ and $x < m$ in this result is due to the assumption $q_0 = 0$ on the saddle point.

To conclude, let us verify the previous result and let us investigate its consequence on physical quantities, as the specific heat. It is well known that this quantity can be computed through the free energy evaluated at the saddle point or by computing the energy-energy fluctuations [9]. The computation of a specific heat in the mean field approximation through the 1RSB saddle point gives

$$C(\tau) = -\frac{d^2}{d\tau^2} W[q]_{SP} = -\frac{\tau}{1-\alpha} - \frac{\beta\tau^2}{(1-\alpha)^3} - \frac{35}{18}\frac{\beta^2\tau^3}{(1-\alpha)^5}, \tag{22}$$

where the dependence of the m parameter on the temperature implies a contribution to the specific heat also from the variation of m.

On the other hand, by considering the Gaussian fluctuations at zero-loop order and by using our prescriptions to deal with the distribution functions, we find also that

$$C(\tau) = \frac{1}{4}\langle \int dx\ dy\ q^2(x)\ q^2(y)\rangle_{conn} = \int_0^1 dx \int_0^1 dy\ q_{SP}(x) q_{SP}(y) G(x,y). \tag{23}$$

This shows that the new singular terms, which with our prescription (18) produce an effect in the computation of the physical quantity (23), are necessary to recover the correct result. Because of the nature of the x variable in the replica approach, the prescriptions for the singular measures are necessary to recover the correct result, while in the discrete formalism, where one has to deal with the parameters q_1 and m only, the regularization is not necessary and one naturally recovers the correct result.

In the case of a continuous breaking of the replica symmetry the longitudinal propagator computed using the linear response theory do coincide with the one obtained by the conventional approach [10]. Our result therefore suggests the following scenario

- If we break the replica symmetry in a continuous way by adding an appropriate external field, the longitudinal propagator is correctly given by the conventional techniques.
- If, by removing the external field, the function $q(x)$ becomes discontinuous, the longitudinal propagator computed via linear response theory goes to the correct one and therefore also the conventionally computed propagator will tend to the same value, which is different from the value obtained by applying directly the conventional techniques.
- We may only conjecture that a correct computation of all the components of the propagator (not only the longitudinal one) may be achieved by using the conventional approach after having introduced an external field which breaks the replica symmetry in a continuous way and then by sending the external field to zero.

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