Interfaces and lower critical dimension in a spin glass model

S. Franz (1), G. Parisi (2), M.A. Virasoro (2)

(1) NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
(2) Università di Roma I "la Sapienza", P.le A. Moro 5, 00185 Rome Italy

December 6, 1994

Abstract

In this paper we try to estimate the lower critical dimension for replica symmetry breaking in spin glasses through the calculation of the additional free-energy required to create a domain wall between two different phases. This mechanism alone would say that replica symmetry would be restored at the lower critical dimension $D_c = 2.5$.

cond-mat/9405007
NORDITA preprint 94/21

The calculation of free energy increase due to an interfaces is a well known method to obtain information about the lower critical dimension for spontaneous symmetry breaking. We perform the first analytic computation of this free energy increase in spin glasses and we use it to suggest the value of the lower critical dimension.

Let us introduce the basic definitions. In the simplest case we can consider a system with two possible coexisting phases (A and B). We will study what happens in a finite system in dimensions D of size $M^d\ L$ with $d = D - 1$. Let us assume that in the d transverse direction we have periodic or free boundary conditions, while in the other direction (which we call t), we put the system in phase A at $t = 0$ and in phase B at $t = L$. The free energy of the interface is the increase in free energy due to this choice of boundary conditions with respect to choosing the same phase at $t = 0$ and $t = L$ (for simplicity we suppose that the two choices A or B lead to the same value for the free energy).

In many cases we have that the free energy increase $\delta F(L, T)$ behaves for large M and L as:

$$\delta F(L, M) = M^d / L^\omega \tag{1}$$

where ω is independent from the dimension. There is then a critical dimension at which the free energy of the interface is finite:

$$D_c = \omega + 1. \tag{2}$$

Heuristic arguments [1], which sometimes can be made rigourous , tell us that at this dimension (the lowest critical dimension) the two phases mix in such a way that symmetry is restored.

In most cases the value of ω from mean field theory is the exact one and therefore we can calculate in this way the value of the lower critical dimension. The simplest examples are the ferromagnetic Ising model $\omega = 0$ and the ferromagnetic Heisenberg model $\omega = 1$.

In this note we study the problem for spin glasses. For convenience we consider two replicas of the same system described by a total hamiltonian:

$$H = H[\sigma] + H[s] \tag{3}$$

where H is the Hamiltonian of a short range spin glass.

In this case the order parameter is the local overlap $q_i = \langle \sigma_i s_i \rangle$ between the two replicas. In mean field theory one finds that this overlap is constant in space, and all values in the interval $[q_{min}, q_{max}]$ are possible. A perturbative expansion around the solution with q constant in space can be done in sufficiently high dimensions, while infrared divergences appear in low dimension.

The aim of this note is to compute the free energy increase corresponding to imposing an expectation value of q equal to p_1 at $t = 0$ and p_2 at $t = L$. We consider here only the case where both p's and non zero and are in the interval $[q_{min}, q_{max}]$.

We find that

$$\delta F = V(|p_1 - p_2|/L)^{5/2}, \tag{4}$$

with $V \equiv M^d L$. As a consequence, the *naive* prediction of mean field theory for the lower critical dimension for spontaneous replica symmetry breaking is $D_c = 2.5$. If a similar value would have been obtained for the interface free energy in zero magnetic field and with p_1 and p_2 of opposite sign, one would argue that the EA parameter should vanish at $D = 2.5$.

We stress that these predictions are *naive*; corrections to the mean field theory are neglected. While in ferromagnets there is regime (low temperature) at which these corrections can be shown to be small, in the spin glass case the corrections to the mean field theory do not vanish even at zero temperature. We do not have here a regime in which we can show that the corrections to the free energy interface do not change qualitatively its L dependence.

A simple testable prediction of our computation is that on a L^D system with boundary condition $q = q_{EA}$, the expectation value of q in the centre of the box should go for large L as

$$q(L) = q_{EA} - const/L^{(2D/5-1)} \tag{5}$$

Our paper is organized as follows. In section 1 we briefly outline the results of the mean field theory for short range spin glasses and we discuss its extension to the case of two constrained real replicas. In section 2 we show that equation (4) can be derived using rather general scaling arguments under the technical assumption that the overlap between the two constrained replicas varies linearly in space. A confirmation for that behaviour is found in section 3 where we find a variational approximation to the free energy increment δF. In the final section we draw some conclusions.

1 The model

The model we consider is the standard D-dimensional Edwards-Anderson spin-glass [2] on a square lattice in a finite volume V, which for simplicity will be taken as a box of size $L >> 1$. This is defined by the Hamiltonian:

$$H[s_i] = - \sum_{<ij>} J_{ij} s_i s_j - h \sum_i s_i \tag{6}$$

where with standard notations we have denoted by $< i, j >$ the nearest neighbours on the lattice. The spins are Ising variables $s_i = \pm 1$, and the couplings J_{ij} are independent gaussian variables with zero average and variance $\bar{J_{ij}^2} = J^2 = 1$. h is the magnetic field.

In our discussion we will make extensive use of the results of the mean field theory (MFT) of the model near the critical temperature. Let us summarize here, without derivation, the main results. For a more complete exposition of the theory see e.g. [3, 4, 5, 6].

The study of the equilibrium properties of the model can be performed in the frame of replica method. The relevant order parameter is a space dependent overlap matrix $Q_i^{ab} = \langle s_i^a s_i^b \rangle$ which, analogously to the long range case, describes the statistics of overlaps between pure states. In MFT, where the system is treated in the saddle point approximation, one finds a (de Almeida-Thouless) line of second order phase transition to a glassy phase, which terminates for $h = 0$ in $T_c = 1$. In the vicinity of T_c, the free energy as a functional of the order parameter admits a Landau expansion in which the original lattice is coarse grained, and one considers the order parameter averaged in small regions of the space V_x centered in x, $Q_{ab}(x) = \frac{1}{|V_x|} \sum_{x \in V_x} Q_i^{ab}$. The free energy functional in terms of $Q_{ab}(x)$ is then written as:

$$
\begin{aligned}
-2nF = \int \mathrm{d}^D x [\tfrac{1}{2}\mathrm{Tr} Q(x) \Delta Q(x) + \tau \mathrm{Tr} Q^2(x) + (1/3) \mathrm{Tr} Q^3(x) \\
+ (y/4) \sum_{ab} Q_{ab}^4(x) + h^2 \sum_{ab} Q_{ab}(x)]
\end{aligned}
\tag{7}
$$

where Δ is the Laplacian operator, $\tau = T_c - T = 1 - T$, $y = 2/3$. The integration extends to the square box of size L, and 'Tr' denotes the trace in replica space. As usual, among all the quartic terms in Q which should be written in the expansion, we have only included the one responsible for the de Almeida-Thouless instability [7]. This leads to the phase transition into a replica symmetry breaking phase.

The main advantage of this reduced model is that it allows for a complete analysis of the r.s.b.: the saddle point equations can be solved exactly above and below the transition temperature. In the low temperature phase, the solution to the saddle point equations is found in the framework of Parisi Ansatz [8], which in the present context consists in parametrizing the space dependent matrices $Q_{ab}(x)$ as a space dipendent functions $q(x, u)$ with $0 \le u \le 1$. For free boundary conditions, the relevant saddle point is found to be constant in space, and the analysis become identical to that of the long range SK model [7]. At the saddle point one finds:

$$
q(x, u) = q(u) = \begin{cases} q_{min} & u \le u_0 \\ \frac{u}{3y} & u_0 \le u \le u_1 \\ q_{max} & u \ge u_1 \end{cases}
\tag{8}
$$

with $u_0 = 3yq_{min}$, $u_1 = 3yq_{max}$. q_{min} and q_{max} are specified by the relations

$$\begin{aligned} 2yq_{min}^3 &= h^2 \\ \tau &= q_{max}(1 - \tfrac{3y}{2} q_{max}) \end{aligned} \tag{9}$$

As in long range models, the appearence of r.s.b. imply the existence of many pure states with a non-trivial distribution of the mutual overlaps [9]. The theory predicts that these overlaps are constant in space.

Let us now enter into the core of our discussion. We want to introduce boundary conditions in the model to force spatial dishomogeneity of the order parameter. In analogy with what is done in ordered systems we would like to put the system in two different equilibrium states at the two boundaries along a given direction. We observe that this can not be done by imposing boundary values to the function $q(x,u)$. Any variation with respect to the form (8) would take us outside of the saddle point, where the free energy functional has no physical meaning. We shall follow instead a procedure introduced in a previous paper [10](referred as I in the following), to study long range spin glasses off-equilibrium.

Consider two identical (same J_{ij}) replicas of the system, which undergo different thermal histories (for equal temperature and magnetic field). We constrain the overlaps between these two real replicas (RR in the following) on the two boundaries along one direction and leave free boundary conditions on all others. In this case the Saddle Point overlap will be constant in all but the privileged direction, and we will have to solve an effective one dimensional problem. Denoting ∂_1 and ∂_2 the boundaries on which we impose the non trivial constraint, we can write the partition function for the doubled system as

$$\begin{aligned} Z_c &= \sum_{\{s_i,\sigma_i\}} \exp\{-\beta H[s] - \beta H[\sigma]\} \prod_{x\in\partial_1} \delta\left(\frac{1}{|V_x|}\sum_{x\in V_x} s_i\sigma_i - p_1\right) \\ &\cdot \prod_{x\in\partial_2} \delta\left(\frac{1}{|V_x|}\sum_{x\in V_x} s_i\sigma_i - p_2.\right) \end{aligned} \tag{10}$$

The object of interest will be the free energy difference δF between this situation and the unconstrained case where the delta functions are not present in the partition sum. We will get an estimate of δF in mean field theory. This will enable us to estimate the probability of fluctuations of the overlap of amplitude $|p_1 - p_2|$ over a scale L through $Prob(|p_1 - p_2|, L) \sim \exp(-\beta\delta F)$. If this probability remains finite for large L, this kind of fluctuations destroy replica symmetry breaking. In high enough dimension we will find a free energy difference divergent with L. The critical dimension D_c will be then identified as the dimension at which this free energy difference become finite.

In the replica treatement of the problem, one has to replicate both the s and the σ spins [10, 11]. Thus three space dependent $n \times n$ matrices will appear: $Q^{11}_{ab}(x)$ describing the overlaps among s spins, $Q^{22}_{ab}(x)$ describing the overlaps among σ spins and $Q^{12}_{ab}(x)$ describing the overlaps between s and σ spins. For symmetry reasons $Q^{12}_{ab}(x) = Q^{21}_{ba}(x)$; the diagonal elements Q_{aa} are as usual taken to be zero by convention. We will choose in the following saddle points for which $Q^{11}_{ab}(x) = Q^{22}_{ab}(x) \equiv Q_{ab}(x)$ and $Q^{12}_{ab}(x) = Q^{21}_{ab}(x) \equiv P_{ab}(x)$. The constraint introduced in the partition function reflects itself in the order parameter through the fixing of the values of the diagonal elements of Q^{12}_{ab} on the border:

$$Q^{12}_{aa}(x)|_{x\in\partial_1} = p_1; Q^{12}_{aa}(x)|_{x\in\partial_2} = p_2. \tag{11}$$

All other elements of the matrices are to be determined from the variational equations for the free energy. In dealing with the matrices Q^{rs}_{ab} $r,s = 1,2$ $a,b = 1,...,n$ it is useful to introduce new indices $\alpha \equiv (r,a)$, $\beta \equiv (s,b)$ and a $2n \times 2n$ matrix $\mathbf{Q}_{\alpha\beta} = Q^{rs}_{ab}$ that contains all the three matrices. In term of this new matrix, the free energy functional is formally identical to that for a single real replica in term of the usual Q_{ab}. The difference lies in the variational procedure, where one has to keep into account the constraint (11).

Near the critical temperature, the free energy admits a Landau like expansion (see 7),with the single replica matrix Q substituted by $\mathbf{Q}$. The saddle points equations in terms of the matrices Q and P are:

$$\begin{aligned} \Delta Q_{ab} &= 2\tau Q_{ab} + (Q^2)_{ab} + (P^2)_{ab} + yQ^3_{ab} + h^2 \\ \Delta P_{ab} &= 2\tau P_{ab} + 2(QP)_{ab} + yP^3_{ab} + h^2 \end{aligned} \tag{12}$$

As usual, to solve these equations we need an ansatz that will eventually allow us to do the analytical continuation to $n \to 0$. As in I we assume both matrices Q and P to be hierarchical matrices. Namely, we parametrize $Q_{ab}(x)$ by a function of $u \in [0,1]$ $q(x,u)$ and the matrix $P_{ab}(x)$ by a diagonal element $P_{aa}(x) = \tilde{p}(x)$ and a function $p(x,u)$. Physically $\tilde{p}(x)$ represents the space dependent overlap between two RR constrained on the boundary. A tentative discussion of the physical meaning of the functions $q(x,u)$ and $p(x,u)$ can be found in I.

Clearly, for the chosen boundary conditions, the various parameters will be constant in all but one direction. Labelling the coordinate along this direction by t we find that q p and $\tilde{p}$ will depend on space only through t. In this way, denoting the integrals of the kind $\int_0^1 \mathrm{d}u\, g(t,u)$ as $\langle g \rangle$, the saddle point equations become:

$$\frac{\partial^2}{\partial t^2} q(t,u) = 2(\tau - \langle q \rangle) q(t,u) + 2(\tilde{p} - \langle p \rangle) p(t,u)$$

$$
\begin{aligned}
&+ \quad \int_0^u \mathrm{d}v \; [q(t,u)-q(t,v)]^2 + \int_0^u \mathrm{d}v \; [p(t,u)-p(t,v)]^2 + yq^3(t,u) + h^2 \\
\frac{\partial^2}{\partial t^2} p(t,u) \quad &= \quad 2(\tau - \langle q \rangle) p(t,u) + 2(\tilde{p} - \langle p \rangle) q(t,u) \qquad (13) \\
&+ \quad 2\int_0^u \mathrm{d}v \; [q(t,u)-q(t,v)][p(t,u)-p(t,v)] + yp^3(t,u) + h^2 \\
\frac{\partial^2}{\partial t^2} \tilde{p}(t) \quad &= \quad 2\tau \tilde{p}(t) - 2\langle qp \rangle + y\tilde{p}(t)^3 + h^2
\end{aligned}
$$

If $p_1 = p_2$ and $q_{min} \le p_1 \le q_{max}$ one can find a solution without spatial dependence at all, with $\tilde{p} = p_1$. Then the problem reduces to the one discussed in I, There we showed, on very general grounds, that for any p_1 in this interval, there exist a solution to the saddle point equation which has the same free-energy density of the unconstrained system ($\delta F = 0$). The results of I in the present context are:

$$
q(x,u) = q(u) = \begin{cases} q_{min} & u \le u_0/2 \\ 2x/(3y) & u_0/2 < u \le u_p/2 \\ \tilde{p} & u_p/2 < u \le u_p \\ x/3y & u_p < u \le u_1 \\ q_{max} & u_1 < u \le 1. \end{cases} \qquad (14)
$$

$$
p(x,u) = p(u) = \begin{cases} q_{min} & u \le u_0/2 \\ 2x/(3y) & u_0/2 < u \le u_p/2 \\ \tilde{p} & u_p/2 < u \le 1 \end{cases} \qquad (15)
$$

$$
\tilde{p}(x) = \tilde{p} = p_1 \qquad (16)
$$

The parameters q_{min}, q_{max}, u_0 and u_1 are those of the unconstrained solution (8) and $u_p = 3yp_1$ is the point where the function (8) is equal to p_1. This solution reflects the fact that the multiplicity of states does not give an extensive contribution to the free energy. The space of equilibrium states of the two copies constrained at an overlap p_1 is simply a restriction of the cartesian product of the equilibrium states of two free system. The freedom in the choice of p_1, which is a zero mode of the free energy, is the spin glass analog to the Goldstone zero mode found in ordered models with a spontaneously broken continuous symmetry. It is now clear that if we impose $p_1 \ne p_2$, but both in the interval $[q_{min}, q_{max}]$, we are choosing at the boundaries two of the possible overlaps admitted by the *free* problem. The additional free energy will have to go to zero in the limit when the boundaries become very far from each other. Our aim is to know with what power it goes to zero.

2 A dimensional argument.

To study the situation $p_1 \neq p_2$ we could perturb around the solution with $p_1 = p_2$. Above the critical dimension the laplacian term in (7) can be treated as a small perturbation. The relevant parameter of the expansion turns out to be $|p_2 - p_1|/L$ which is always arbitrary small.

It is reasonable to think that the solution to the saddle point equations would be in this case *similar* in form to eq.(15), but with $\tilde{p} = \tilde{p}(t)$ a function interpolating between p_1 at $t = 0$ and p_2 at $t = L$. On physical grounds we expect that after averaging over the quenched disorder, $\tilde{p}(t)$ interpolates linearly between the boundary values, namely

$$\tilde{p}(t) = p_1(1 - t/L) + p_2 t/L. \tag{17}$$

This assumption will enable us to determine the lower critical dimension by mere dimensional analysis. Our hypothesis will be validated *a posteriori* in the next section, where we will derive the linearity in the context of an approximate maximization of the free energy.

To first order in the perturbative expansion in $p_1 - p_2$, the variation of the free-energy is $n\delta F = \int \mathrm{d}t\ \mathrm{Tr}\mathbf{Q}\Delta\mathbf{Q}$ computed at the unperturbed saddle point. It is easy to see that this variation is zero for the saddle point (15):

$$\begin{aligned} & \tfrac{1}{n} \int \mathrm{d}t \sum_{\alpha\beta} \mathbf{Q}_{\alpha\beta} \Delta \mathbf{Q}_{\alpha\beta} && (18) \\ = & \int \mathrm{d}t \left(\frac{\mathrm{d}\tilde{p}(t)}{\mathrm{d}x} \right)^2 [1 - \int \mathrm{d}u \theta(u - u_0/2)\theta(u_0 - u) - \int \mathrm{d}u\theta(u - u_0/2)] = 0. \end{aligned}$$

A non zero δF at this level would have implied, by simple dimensional analysis, $\delta F \sim L^{D-2}$, i.e. the same result found for ordered systems with a continuous symmetry. The vanishing of this term means that D_c is higher than 2. One could say that fluctuations due to the zero mode in spin glasses cost less and become therefore important earlier than the usual Goldstone modes in ordered systems.

Substituting the expresion (15) into the saddle points equations (13) and denoting $u_p = 3y\tilde{p}$, $\chi = 3y(\mathrm{d}\tilde{p}/\mathrm{d}t)^2$ we obtain

$$\begin{aligned} \frac{\partial^2 q(t,u)}{\partial t^2} &= \chi[\delta(u - u_p) - \frac{1}{2}\delta(u - u_p/2)] \\ \frac{\partial^2 p(t,u)}{\partial t^2} &= -\chi\delta(u - u_p/2). \end{aligned} \tag{19}$$

We find in this way that the expresion (15) satisfies the saddle point conditions for all u except $u_p/2$ and u_p. It is reasonable to suppose that the effect of the Laplacian term in the free-energy will result in a (small) smoothing of the functions q and p around $u_p/2$ and u_p. Thus we suppose that the functions will have variations of a given order ϵ in regions of order ϵ' around $u_p/2$ and u_p. ϵ and ϵ', as well as u_p will be in general function of t. Using the monotonicity of the functional relation between $\tilde{p}$ and t, we will consider all parameters as functions of $\tilde{p}$.

We now show that under this hypothesis the following remarkable facts happen:

- δF, defined as $F(p_2, p_1) - F(p_1, p_1)$ does not depend neither on the temperature τ nor on the magnetic field h.
- δF must behave as $L^D \chi^{(5/4)}$, in order that the linear interpolation (17) maximizes the free energy.

Let us write the variational matrix $\mathbf{Q}$ as $\mathbf{Q} = \mathbf{Q}^{(0)} + \delta\mathbf{Q}$ where $\mathbf{Q}^{(0)}$ solves the variational problem in the absence of the perturbation. δF can be written as

$$
\begin{aligned}
-2n\delta F &= \int \mathrm{d}x \left[\frac{1}{2} \,\mathrm{Tr}(2\delta\mathbf{Q}\Delta\mathbf{Q}^{(0)} + \delta\mathbf{Q}\Delta\delta\mathbf{Q}) \right. && (20)\\
&+ \tau \mathrm{Tr}\delta Q^2 + \mathrm{Tr}\mathbf{Q}^{(0)}\delta\mathbf{Q}^2 + \frac{1}{3}\mathrm{Tr}\delta\mathbf{Q}^3 && (21)\\
&+ \left. \frac{y}{4}\sum_{\alpha\beta}(3{\mathbf{Q}^{(0)}_{\alpha\beta}}^2\delta\mathbf{Q}^2_{\alpha\beta} + 4\mathbf{Q}^{(0)}_{\alpha\beta}\delta\mathbf{Q}^3_{\alpha\beta} + \delta\mathbf{Q}^4_{\alpha\beta}) \right] && (22)
\end{aligned}
$$

Because of the unperturbed SPE the only terms linear in $\delta\mathbf{Q}$ that can appear arise from the Laplacian term. For a given t (or equivalently $\tilde{p}$), one can evaluate the variation of the free energy *density* δf just dropping the integration over x in (22).

In this free-energy we want to study the dependence on τ and h *before* optimizing with respect to $\delta\mathbf{Q}$. As one can see from (15), the function p does not depend on τ and the function q depends on τ only in the region $u > u_{max}$ where $q(u) = q_{max}$. In the same way one observes that both functions depend on h only in the region $u < u_{min}$ where they take the value q_{min}. To study the dependence of δF on the temperature and the magnetic field we can use q_{max} and q_{min} as independent variables instead of using τ and h which give rise to simpler algebra. The structure of $\delta\mathbf{Q}$ we choose implies that the only terms which can depend on q_{max} are:

- $\tau\mathrm{Tr}\delta\mathbf{Q}^2$, trough its τ dependence ($\tau = q_{max}(1 - 3yq_{max}/2)$), and
- $\mathrm{Tr}\mathbf{Q}^{(0)}\delta\mathbf{Q}^2$

The other terms do not depend on q_{max} because for the α and β such that $\mathbf{Q}^{(0)}_{\alpha\beta} = q_{max}$ one has $\delta\mathbf{Q}_{\alpha\beta} = 0$. So we find that

$$
-2n\frac{\partial\delta f}{\partial q_{max}} = (1 - 3yq_{max})\mathrm{Tr}\delta\mathbf{Q}^2 + \sum_{\alpha\beta|\mathbf{Q}^{(0)}_{\alpha\beta}=q_{max}} \delta\mathbf{Q}^2_{\alpha\beta} \qquad (23)
$$

It is easy to show, using the algebra of ultrametric matrices, that the second addendum on the r.h.s. of (23) is exactly equal to $-(1-3yq_{max})\mathrm{Tr}\delta\mathbf{Q}^2$ and consequently δf is independent of τ. Along the same lines one shows that the only possible dependence on q_{min} is in the term $\mathrm{Tr}\mathbf{Q}^{(0)}\delta\mathbf{Q}^2$, but its derivative w.r.t. q_{min} is equal to zero. Therefore the free-energy density variation can only depend on $\tilde{p}$. Let us analize the dimensions of each term in δf in terms of $\tilde{p}$, ϵ, ϵ' and χ. We remind that while $\tilde{p}$ and χ are fixed parameter in the problem ϵ and ϵ' are to be determined by sadlle point equations. In the considerations which follow we can safely assume, that dimensionally $\epsilon' \sim \epsilon$. Keeping the two quantities different would only complicate the formulae, but not the scaling of the free energy.

We find for the dimensions of the different terms in (22):

$$\begin{aligned}
\mathrm{Tr}2\delta\mathbf{Q}\Delta\mathbf{Q}^{(0)} &\sim \epsilon\chi \\
\mathrm{Tr}\delta\mathbf{Q}\Delta\delta\mathbf{Q} &\sim \epsilon\chi\left(\frac{\partial\epsilon}{\partial\tilde{p}}\right) \\
\tau\mathrm{Tr}\delta\mathbf{Q}^2+\mathrm{Tr}\mathbf{Q}^{(0)}\delta\mathbf{Q}^2 &\sim \tilde{p}\epsilon^4 \\
\mathrm{Tr}\delta\mathbf{Q}^3 &\sim \epsilon^5 \\
\sum_{\alpha\beta}\mathbf{Q}^{(0)^2}_{\alpha\beta}\delta\mathbf{Q}^2_{\alpha\beta} &\sim \tilde{p}^2\epsilon^3 \\
\sum_{\alpha\beta}\mathbf{Q}^{(0)}_{\alpha\beta}\delta\mathbf{Q}^3_{\alpha\beta} &\sim \tilde{p}\epsilon^4 \\
\sum_{\alpha\beta}\delta\mathbf{Q}^4_{\alpha\beta} &\sim \epsilon^5
\end{aligned} \tag{24}$$

If we now rescale:

$$\epsilon \to \tilde{p}\eta \tag{25}$$

$$\chi \to \tilde{p}^4\phi \tag{26}$$

δf becomes an homogeneous function of order 5 in $\tilde{p}$. If the linear form $\tilde{p}(t)$ has to be a maximum of the free-energy, the free energy density must be independent of $\tilde{p}$. This can be seen from the fact if one optimizes for fixed $\tilde{p}(t)$ one finds in general $\delta f = \delta f(\tilde{p}, (\mathrm{d}\tilde{p}/dt))$. The further minimization with respect to $\tilde{p}(t)$ leads to Euler-Lagrange equations that give zero 'acceleration' $\mathrm{d}^2\tilde{p}/\mathrm{d}t^2$ only if δf is independent of $\tilde{p}$. So the $\tilde{p}^5$ dependence must be compensated at the saddle point by the dependence on $\phi \equiv \chi/\tilde{p}^4$. In this way, writing in all generality for the saddle point $\delta f = \tilde{p}^5 g(\phi)$ one must have $g(\phi) \sim \phi^{(5/4)} \equiv \chi^{(5/4)}/\tilde{p}^5$. Using $\chi = 3y(p_2-p_1)/L$ for the form (17), we find that the total free-energy variation scales as:

$$\delta F \sim L^D\chi^{(5/4)} \sim |p_2-p_1|^{(5/2)}L^{D-5/2}. \tag{27}$$

Equation (27) is the main result of this paper. It tells that in the context of mean field theory, the lower critical dimension at which δF become finite for finite $|p_2 - p_1|$ is $D_c = 2.5$. Moreover for $D > 2.5$ one can expect fluctuations in space of the order parameter to scale with the distance as $|q(x) - q(y)| \sim |x - y|^{1-2D/5}$. It is worth noticing that the value for the critical dimension we get is fully compatible with the one found by Bray and Moore for the glassy transition in absence of field [12] with totally different methods. The same and other authors [13, 14] claimed that in any finite dimension the spin glass transition is destroyed by the presence of a magnetic field. Our findings disagree with this claim, as we find δf to be independent on the magnetic field.

3 A variational approximation.

Let us now turn to an explicit computation of the free-energy density increment through a variational approach. Instead of solving the full SPE (13) we will here propose an explicit parametrization of the small variation to the form (15) in the neighbourhood of the points $u_p/2$ and u_p, and we will *maximize* the free energy with respect to the parameters of this variation. We expect that the numerical value so obtained for δF is a lower bound to the real value. Furthermore we will show that it scales as discussed in the previous section. Thus this section proves that the leading behaviour previously obtained does not accidentally cancel. Moreover here we will not need to assume the linear form (17) for $\tilde{p}(t)$: this will be found as the optimum of the free energy.

We choose to smooth the singularities around $u_p/2$ and u_p (cf. (19)) by interpolating with an arc of parabola the step-wise linear behaviour of the functions q and p in the surroundings of these points. Our variational functions will then be:

$$q(u) = \begin{cases} q_{min} & u < u_0 \\ 2u/(3y) & u_0 < u < u_1 \\ \tilde{p} - a(u - u_2)^2 & u_1 < u < u_2 \\ \tilde{p} & u_2 < u < u_3 \\ \tilde{p} + a'(u - u_3)^2 & u_3 < u < u_4 \\ u/(3y) & u_4 < u < u_5 \\ q_{max} & u > u_5 \end{cases} \tag{28}$$

$$p(u) = \begin{cases} q_{min} & u < u_0 \\ 2u/(3y) & u_0 < u < u_1 \\ \tilde{p} - a(u - u_2)^2 & u_1 < u < u_2 \\ \tilde{p} & u > u_2 \end{cases} \tag{29}$$

with

$$\begin{array}{lll} u_0 = 3yq_{min}/2 & u_1 = 3y\tilde{p}/2 - \delta/2 & u_2 = 3y\tilde{p}/2 + \delta/2 \\ u_3 = 3y\tilde{p} - \delta'/2 & u_4 = 3y\tilde{p} + \delta'/2 & u_5 = 3yq_{max} \\ a = 1/(3y\delta) & a' = 1/(6y\delta'). & \end{array} \tag{30}$$

The reader should not be confused by the notation at this point, although q_{min} and q_{max} are the same as in the presious sections and of (15), we changed here the notation for the points u_i ($i = 0, ..., 5$). The only variational parameters that appear in the the free energy functional are δ and δ'. Denoting as in the previous section $\chi = 3y\left(\frac{\mathrm{d}\tilde{p}}{\mathrm{d}t}\right)^2$, the free energy density increment as a function of δ and δ' takes the form:

$$\delta f = \frac{31\delta^5}{102060y^3} - \frac{\delta^4\tilde{p}}{324y^2} + \frac{\chi\delta}{9y} + \frac{\chi\delta'}{18y} - \frac{\delta^3\delta'^2}{9720y^3} + \frac{\delta^2\tilde{p}\delta'^2}{648y^2} - \frac{\tilde{p}\delta'^4}{5184y^2} - \frac{11\delta'^5}{3265920y^3} \tag{31}$$

which has to be maximized with respect to δ and δ'. Equation (31) is consistent with the scaling established in the previous section. The change of variables:

$$\begin{aligned} \delta &= y\tilde{p}\frac{a+b}{2} \\ \delta' &= y\tilde{p}\frac{b-a}{4} \\ \chi &= y^2\tilde{p}^4\phi \end{aligned} \tag{32}$$

gives us

$$\begin{aligned} \delta f &= y^2\tilde{p}^5\frac{b}{52254720}\Big(29\,a^4 - 10080\,a^2\,b - 252\,a^3\,b \\ &+ 142\,a^2\,b^2 - 84\,a\,b^3 - 11\,b^4 + 2903040\,\phi\Big)\,. \end{aligned} \tag{33}$$

It is apparent form the variational equations for a and b that a should be of order $\phi^{1/2}$ while $b \simeq \phi^{1/4}$. To lowest order in ϕ the solution is $a = -b^2/240$, $b = 12\sqrt{2}(7/11)^{(1/4)}\phi^{1/4}$ which gives for the free-energy density

$$\delta f = 0.673659\ y^2\chi^{5/4}. \tag{34}$$

This result confirms in a specific example the behaviour in $\chi^{5/4}$ of δf which is the only one compatible with $\tilde{p}(t)$ linear in t. The total free-energy of the interface, $\delta F = \delta f L^D$ is proportional to $L^{D-5/2}|p_1 - p_2|^{5/2}$, confirming the analysis of the previous section. Let us observe here the proportionality of δF to y^2, the coefficient of the quatic term in (7).

4 Conclusions

In this paper we have shown that it is possible to estimate the cost in free energy of a domain wall between two different phases in spin glasses. This information was used to indicate a possible value for the lower critical dimension $D_c = 2.5$ for replica symmetry breaking.

We are aware of the several criticisms that could be raised against this indication. In ordered system MFT gives a reliable estimate of D_c because the fluctuation of the order parameter are negligible at very low temperature. In the spin glass case there is not such evidence. Moreover, in ordinary systems, this kind of analysis is confirmed by the behaviour of the perturbation expansion. In $O(N)$ models, for example, the free propagator $G(k) \sim k^{-2}$ in the ordered phase and therefore the fluctuations of the order parameter diverge in $D = 2$. In the spin glass case, there is a whole family of propagators, the most divergent of which, in presence of magnetic field, has the behaviour $G(k) \sim k^{-3}$ [15]. If this behaviour is not modified by renormalization it would imply a lower critical dimension $D_c = 3$.

At present we do not know if any of this criticisms will be substantiated and the actual critical dimension in spin glasses is larger then 2.5. But if this happens, the replica symmetry restoration mechanism must be different from the simple 'instantonic' one that we have proposed in this paper.

The present state of the art in numerical simulations of spin glasses indicates that in dimension two there is no transition [16] while in dimension four there is full replica symmetry breaking [17]. Unfortunatly the situation is far from clear in dimension three. Recent simulations by one of us [18] obtained from large lattices were compatible with a finite temperature phase transition but also with an essential singularities at $T = 0$, which would indicate $D_c = 3$.

All this calls for further research. One can expect that simulations on larger lattices will eventually resolve the problem of the existence of the transition in three dimensions. On the more analytical side a crucial problem to attack is the renormalization of the k^{-3} behaviour of the most singular propagator in perturbation theory.

Acknowledgments

It is a pleasure to thank P.W.Anderson, C. de Dominicis, M.Mézard, D.Sherrington for interesting discussions.

References

[1] See e.g. G. Parisi, *Statistical Field Theory*, (Addison Wesley 1988)

[2] S.F. Edwards and P.W.Anderson *J.Phys. F: Metal Phys.* **5**965 (1975)

[3] M. Mézard, G. Parisi and M.A. Virasoro, *Spin glass theory and beyond* (World Scientific 1987).

[4] G. Parisi, *Field Theory, Disorder and Simulations*, (World Scientific 1992)

[5] K. Binder A.P. Young *Rev. Mod. Phys.* **58** 801 (1986)

[6] K.H. Fischer, J.A. Hertz, "Spin Glasses" Cambridge University Press, (1991).

[7] G.Parisi, *J.Phys.A* **13** 1887 (1980)

[8] G.Parisi, *Phys. Lett.* **73 A** 203 (1979)

[9] G.Parisi, *Phys. Rev. Lett.* **50** 1946 (1984)

[10] S.Franz, G.Parisi, M.A.Virasoro, *J.Phys. I (France)* **2** 1869 (1992)

[11] S.Franz, G.Parisi, M.A.Virasoro, *Europhys. Lett.* **22** (6) 405 (1993)

[12] A.Bray Moore *J.Phys.C* **17** L463 (1984)

[13] A.Bray Moore *J.Phys.C* **17** L613 (1984)

[14] D.S. Fisher D.A. Huse *Phys.Rev.Lett.* **56** 1601 (1986)

[15] C. de Dominicis, I.Kondor *J.Phys.A* **22** L743

[16] A.T.Ogielsky Phys.Rev.B 32 (1985) 7384

[17] G.Parisi, F.Ritort *J.Phys.A* **26** (1993) 6711, J.C.Ciria, G.Parisi, F.Ritort *J.Phys.A***26** (1993) 6731

[18] G.Parisi V.Marinari F.Ritort *J.Phys.A* **27** (1994) 2687