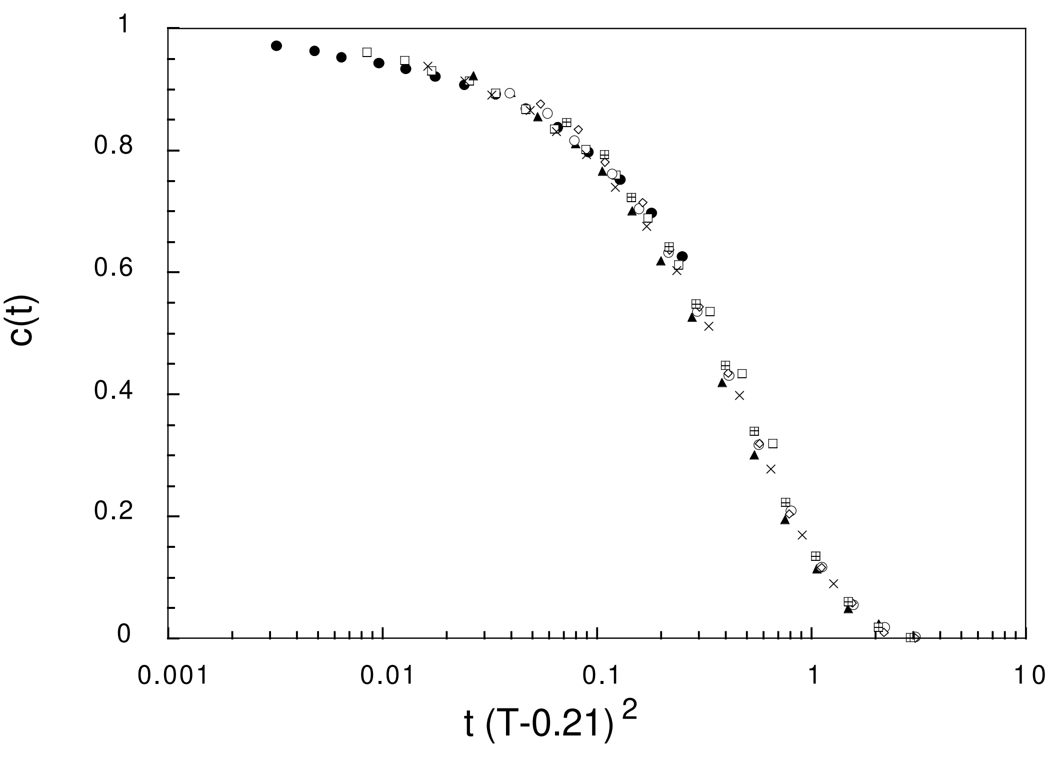

Figure 14: Finite-time scaling eq.(23) for the open case. Good scaling is obtained with $T_G \sim 0.21 \pm 0.02$ and $\gamma \sim 2$.

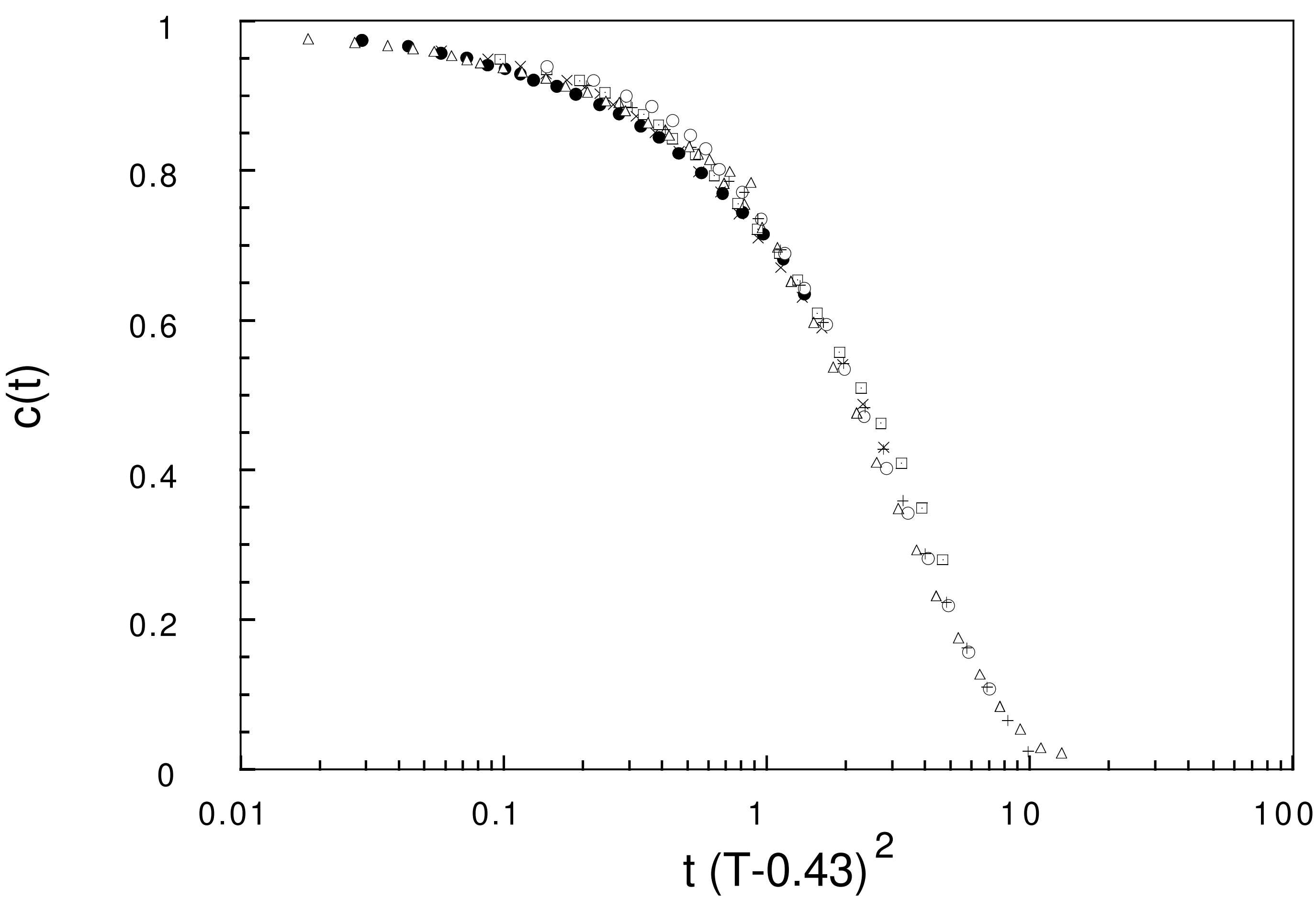

Figure 13: Finite-time scaling eq.(23) for the periodic case. Good scaling is obtained with $T_G \sim 0.43 \pm 0.02$ and $\gamma \sim 2$.

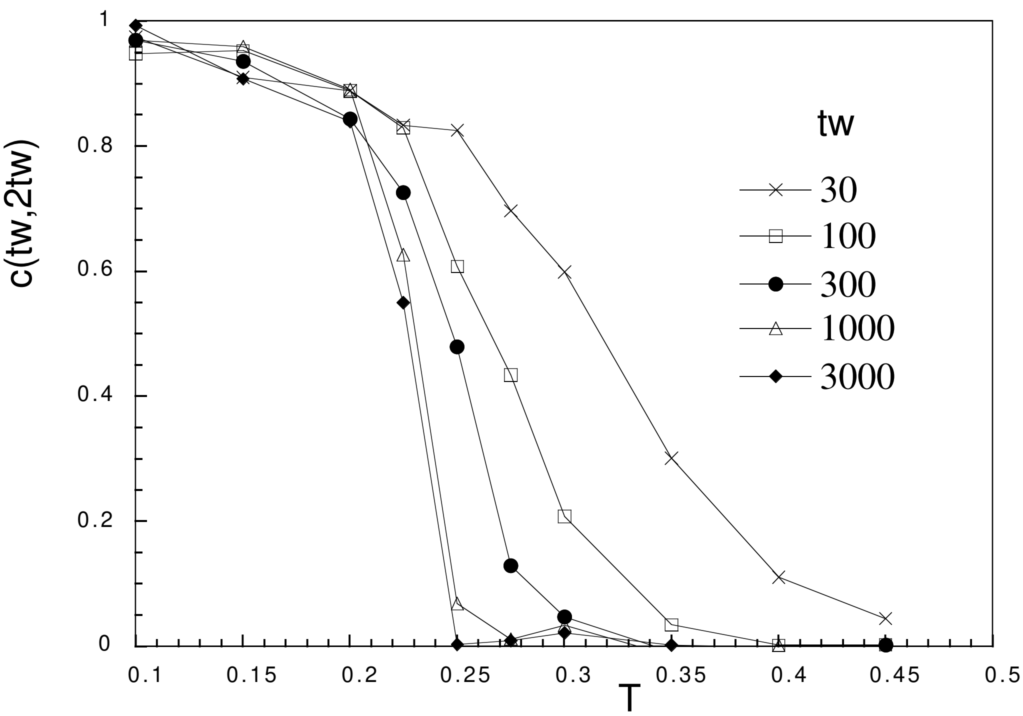

Figure 12: $C(t_w, 2t_w)$ for the open model for different values of $t_w = 30, 100, 300, 1000, 3000$ as a function of the temperature.

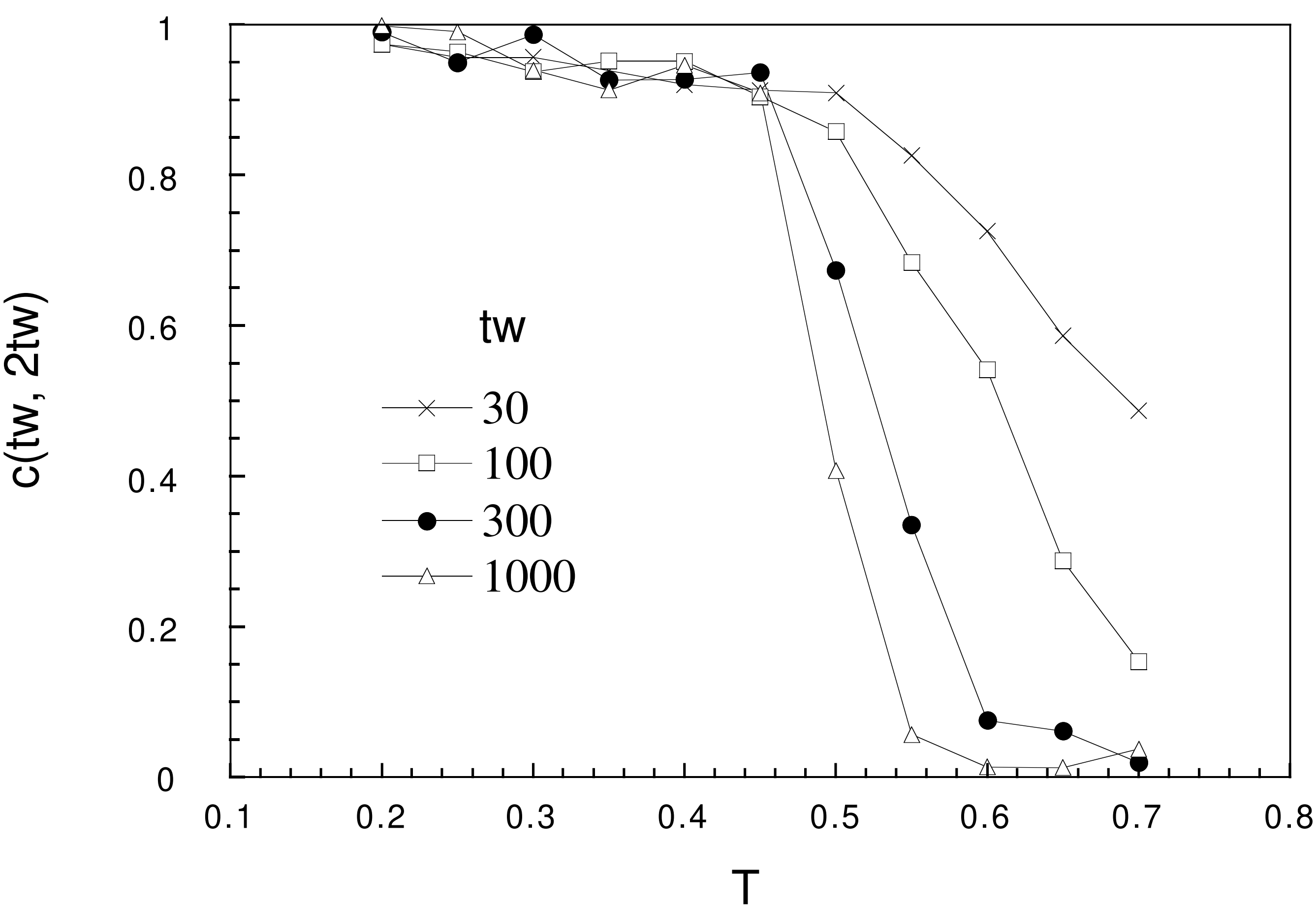

Figure 11: $C(t_w, 2t_w)$ for the periodic model for different values of $t_w = 30, 100, 300, 1000$ as a function of the temperature.

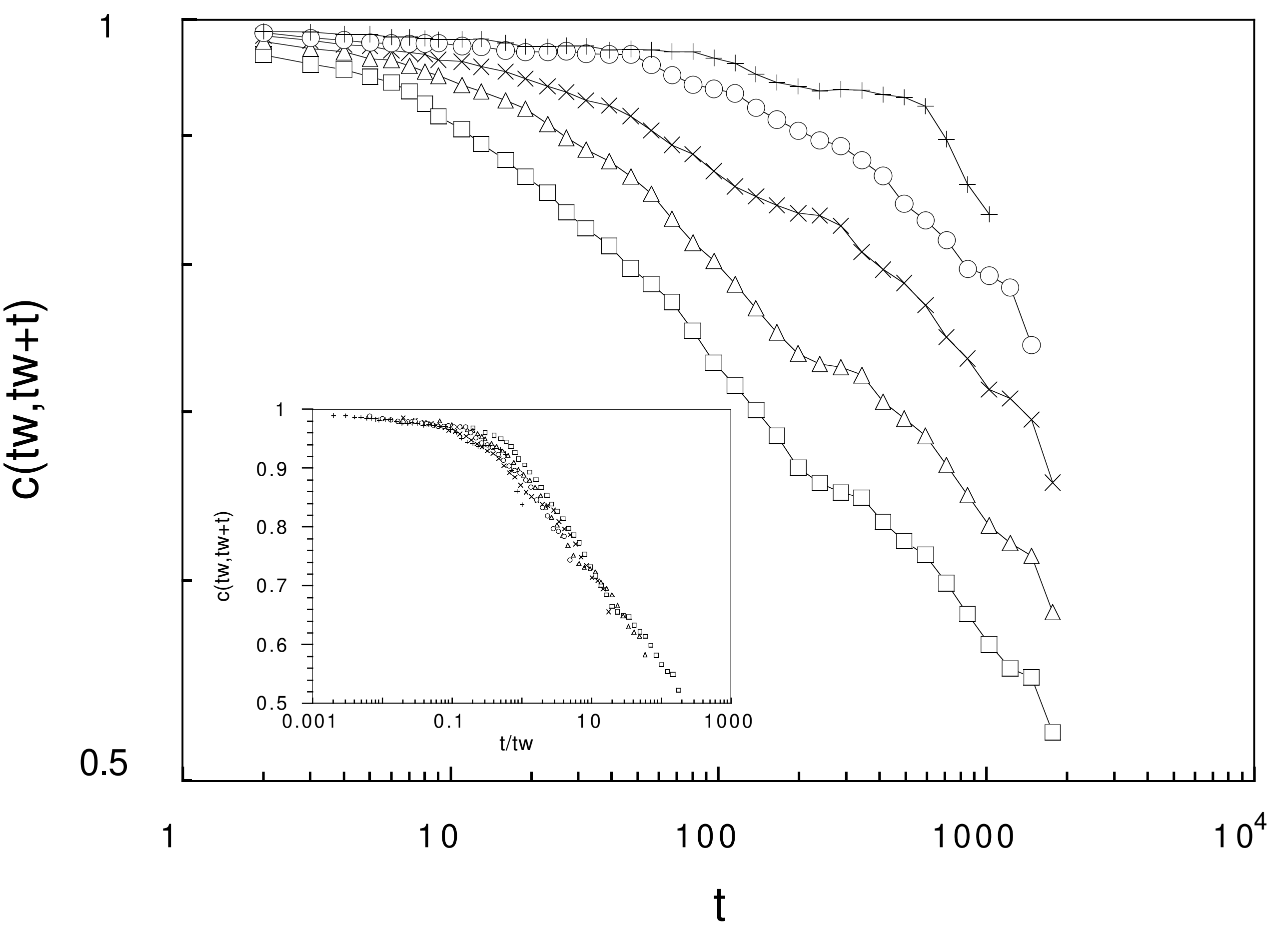

Figure 10: $C(t_w, t+t_w)$ for the open model for different values of t_w at the glass transition ($T_G \sim 0.19$). The inset shows the scaling law eq.(21). The size is $N = 10000$.

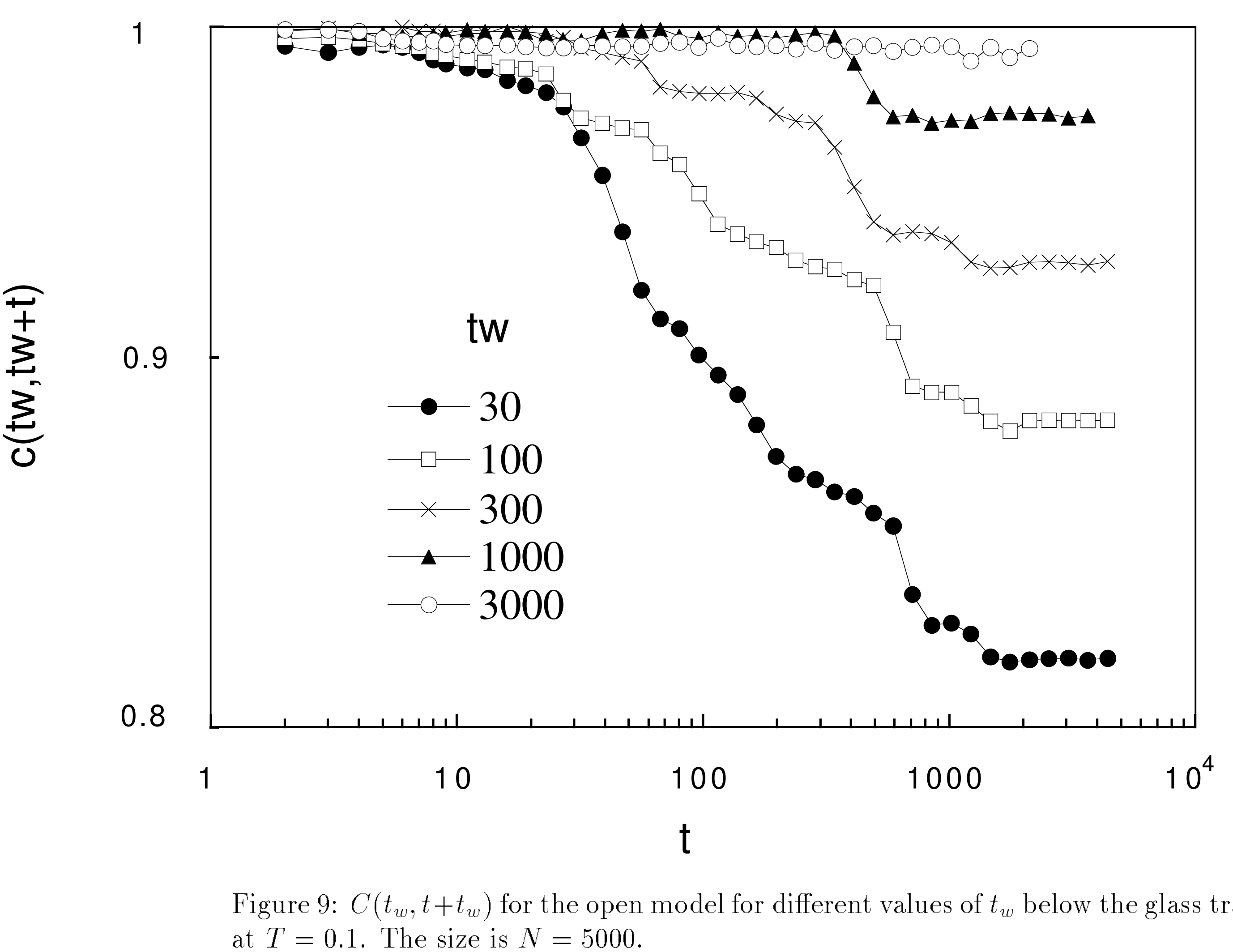

Figure 9: $C(t_w, t+t_w)$ for the open model for different values of t_w below the glass transition at $T = 0.1$. The size is $N = 5000$.

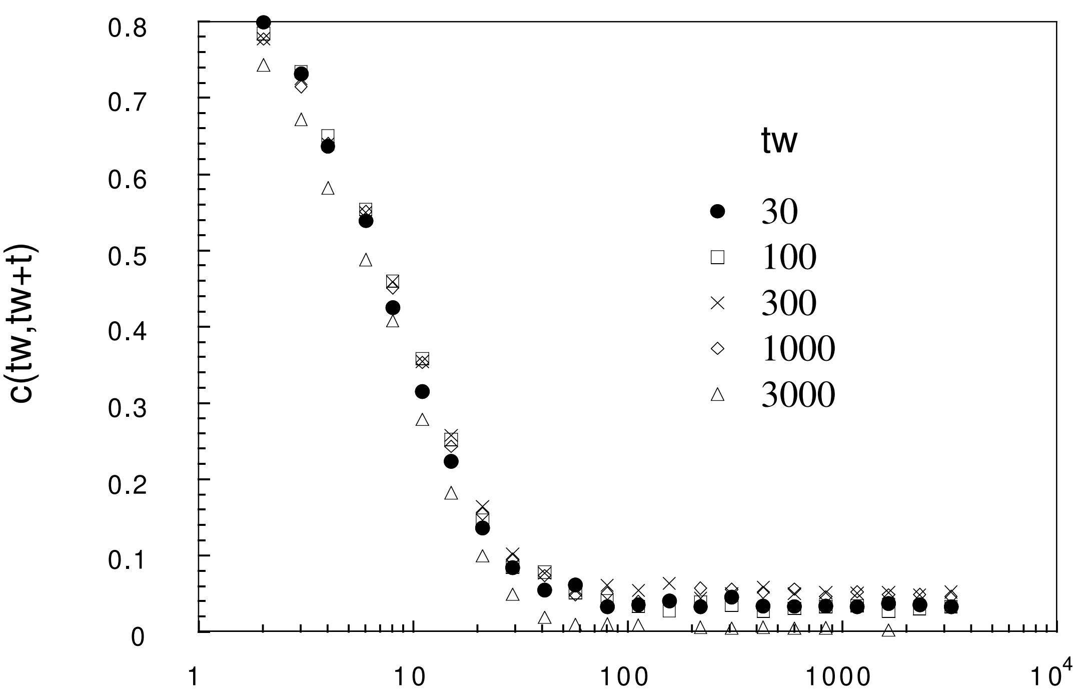

Figure 8: $C(t_w, t+t_w)$ for the open model for different values of t_w above the glass transition at $T = 0.45$. The size is $N = 5000$.

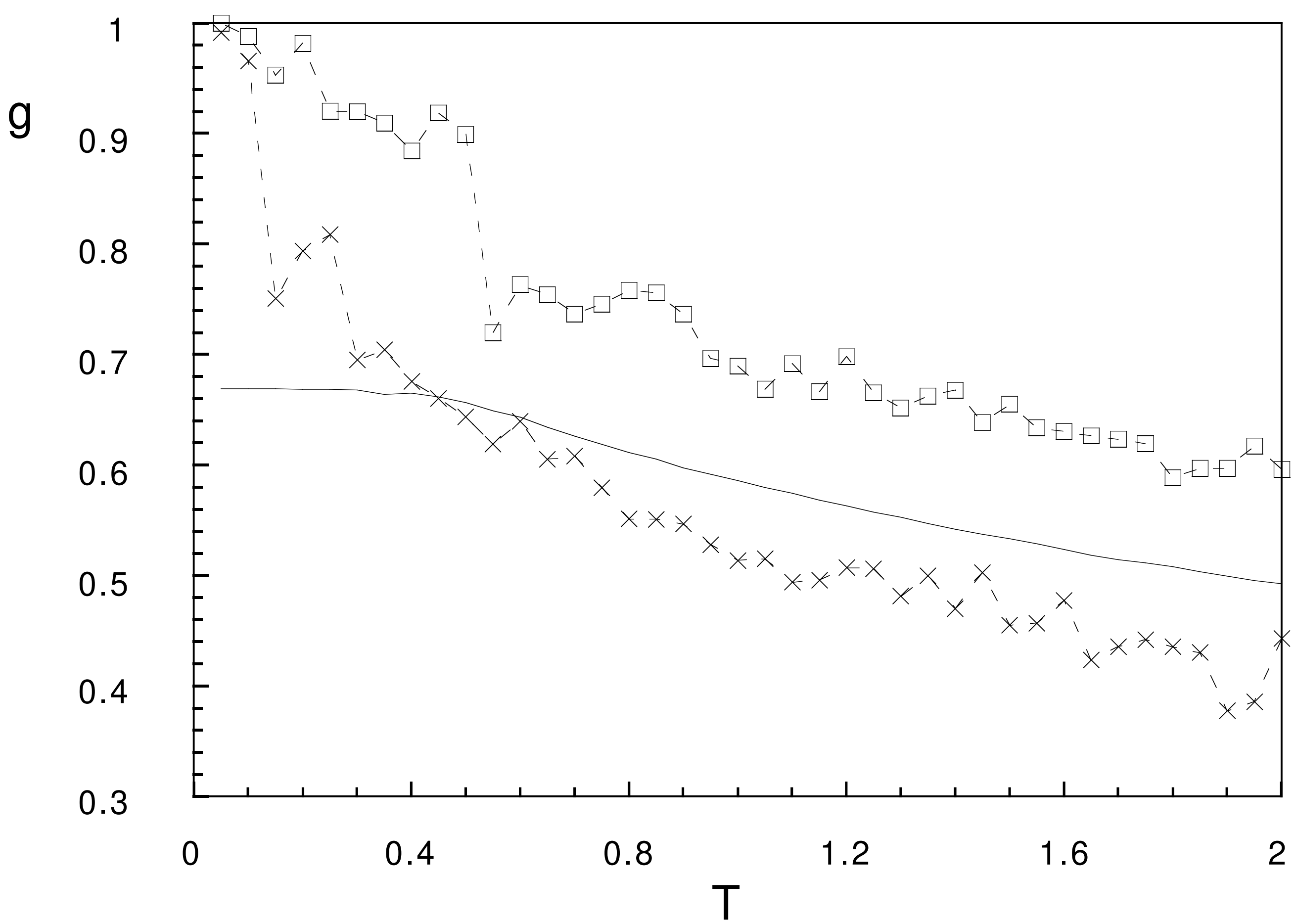

Figure 7: Binder parameter of the periodic and open models versus temperature. The continuous line is the approximate high-temperature result to the periodic case eq.(15). Data is shown for $N = 100$ in the periodic model and $N = 500$ in the open case.

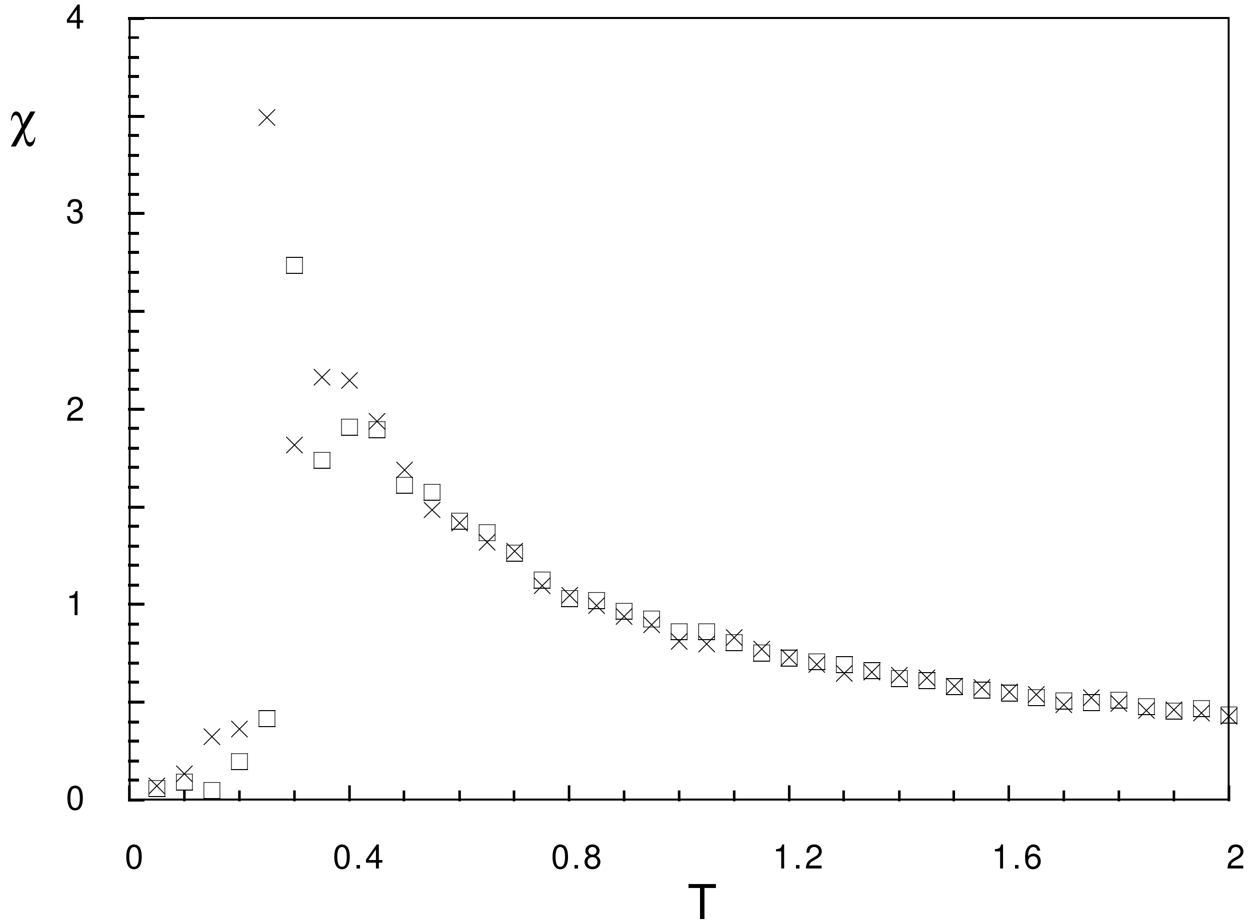

Figure 6: Magnetic susceptibility of the open model versus temperature. Simulation results are for $N = 500$ (squares) and $N = 1000$ (crosses).

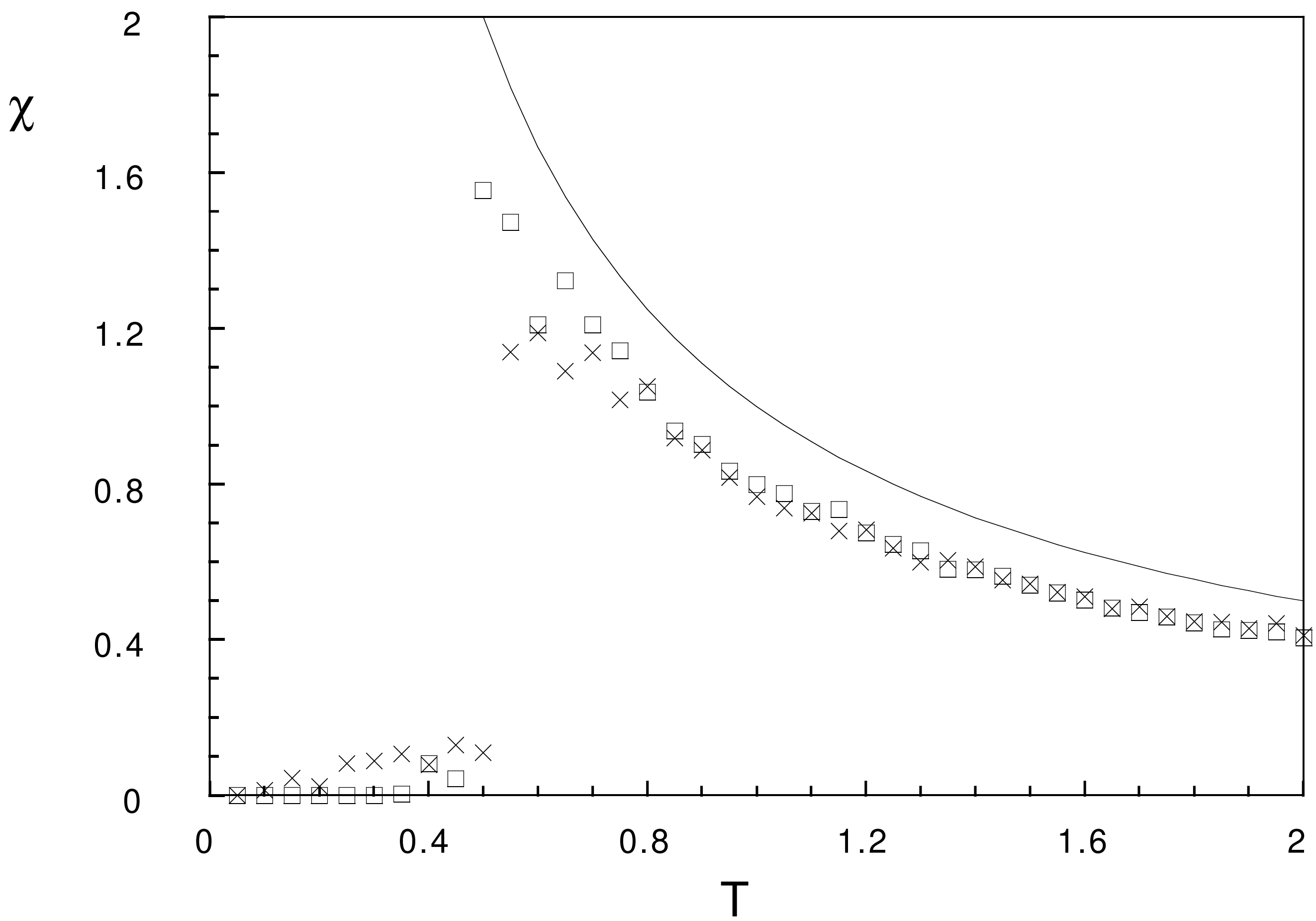

Figure 5: Magnetic susceptibility of the periodic model versus temperature. The continuous line is the approximation eq.(14). Simulation results are for $N = 100$ (squares) and $N = 500$ (crosses).

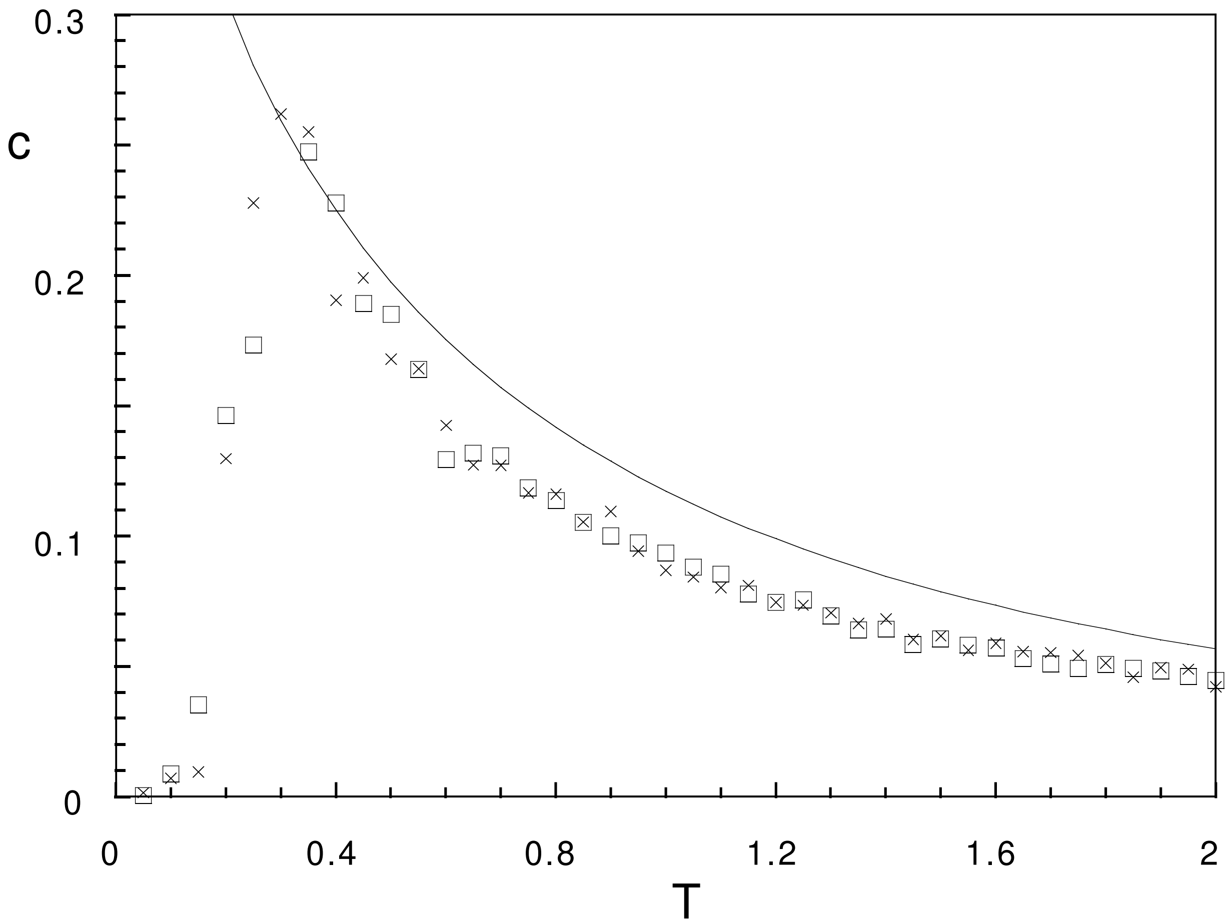

Figure 4: Specific heat of the open model versus temperature. The dashed line is the GB approximation. Simulation results are for $N = 100$ (squares) and $N = 500$ (crosses).

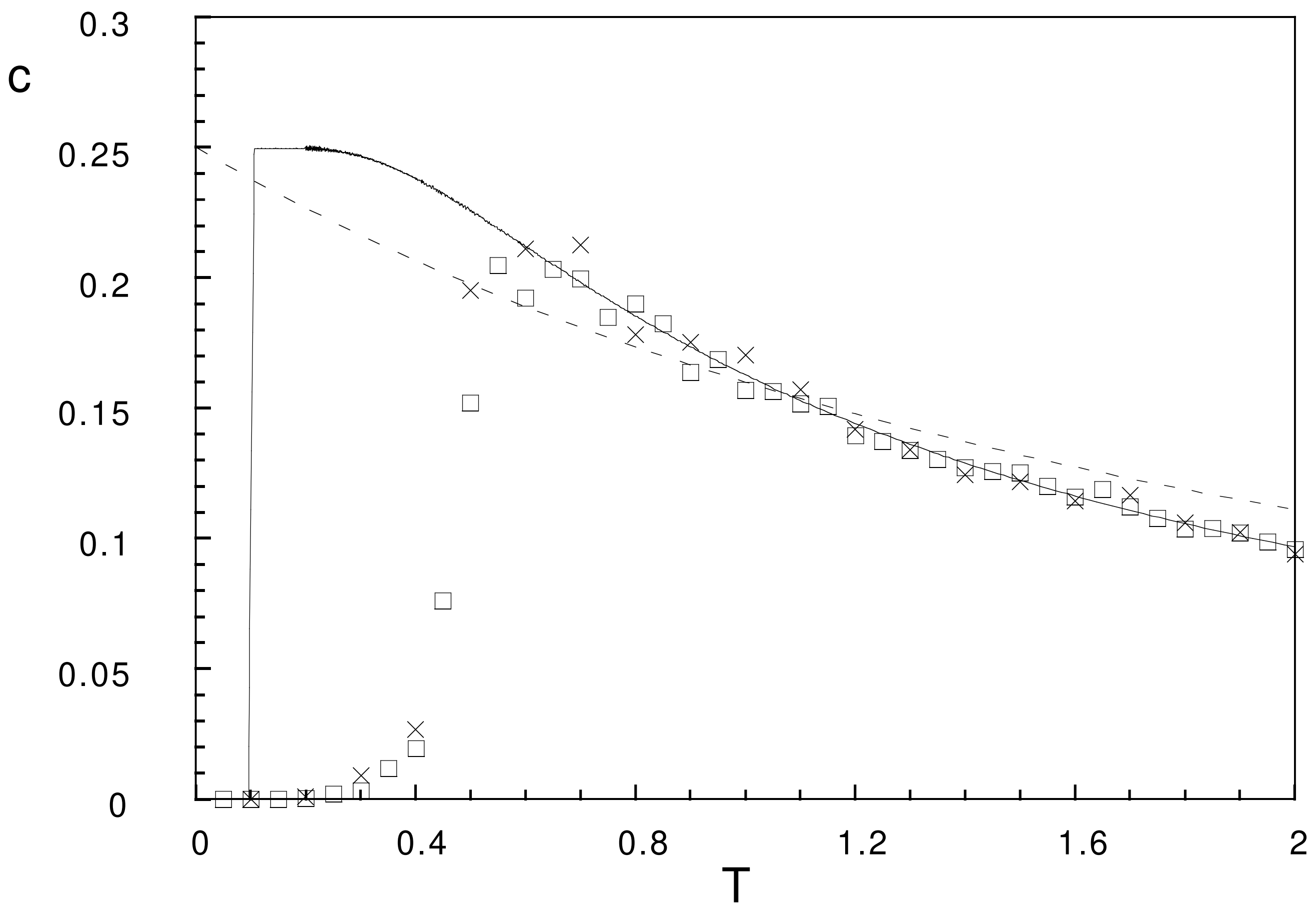

Figure 3: Specific heat of the periodic model versus temperature. The continuous line is the high-temperature result eq.(6). The dashed line is the GB approximation. Simulation results are for $N = 100$ (squares) and $N = 500$ (crosses).

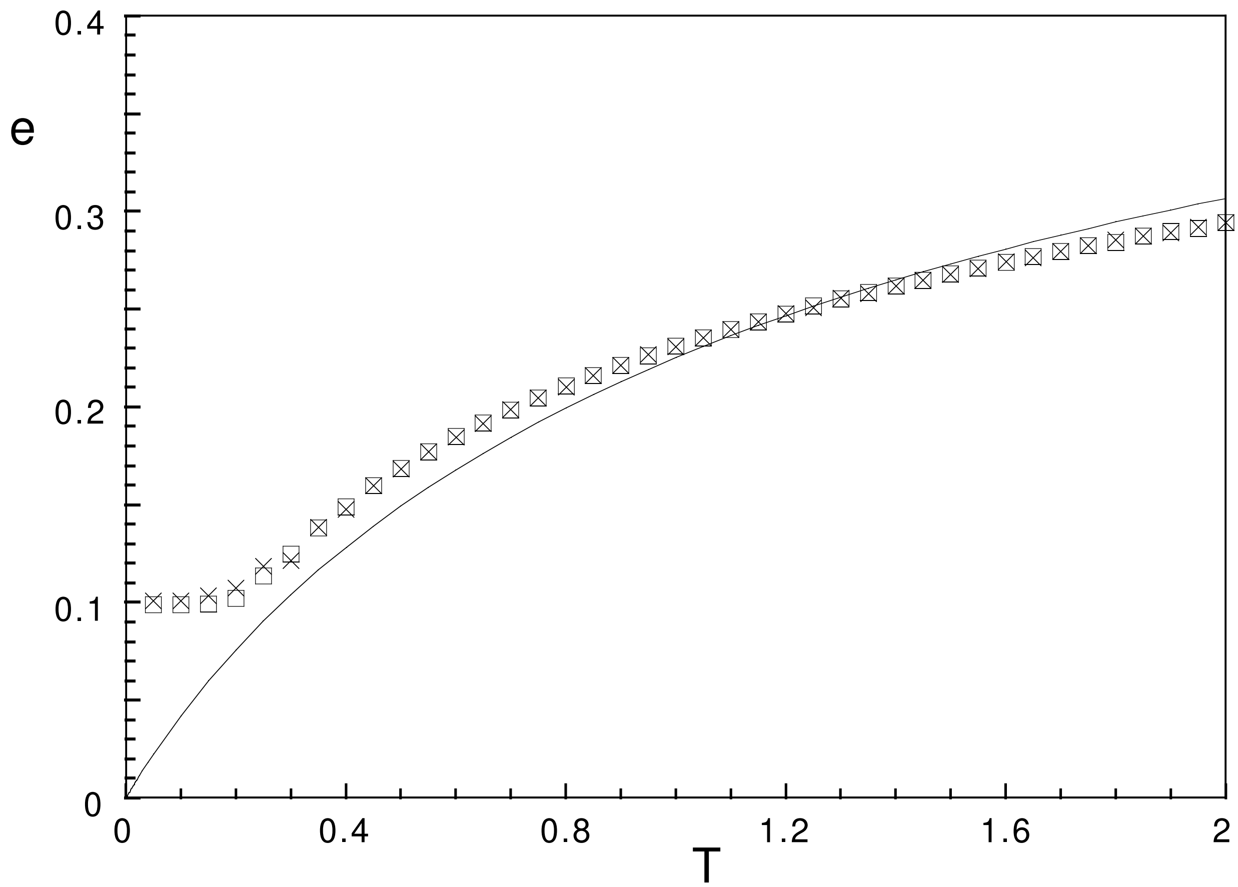

Figure 2: Energy of the open model versus temperature. The dashed line is the GB approximation. Simulation results are for $N = 500$ (squares) and $N = 1000$ (crosses).

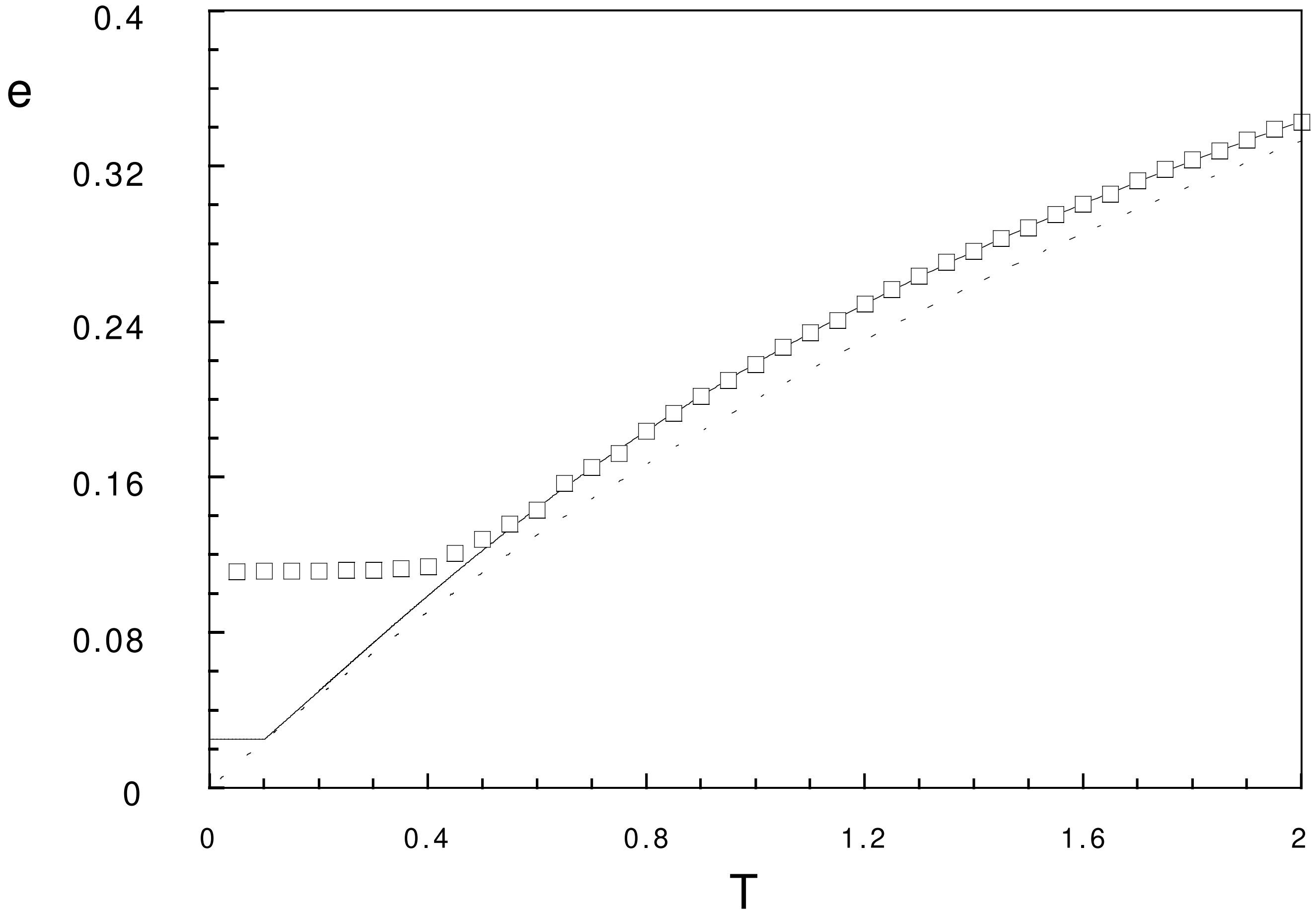

Figure 1: Energy of the periodic model versus temperature. The continuous line is the high-temperature result eq.(6). The dashed line is the GB approximation. Simulation results are for $N = 100$

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and should be applicable to a large variety of systems where disorder is not present. In particular we think that the behaviour of $c(t_w, 2t_w)$ as a function of the temperature is well suited in order to characterize the transition. It remains to be understood to what extent the results obtained in this work are generic for real glasses and to what extent short-range interactions can modify the main conclusions of this work.

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exponent γ increases as the dimensionality decreases. This happens in case of Ising spin glasses where the product $z\nu$ ranges from 2 in mean-field theory to 6 in three dimensions [33] (in case there is a true phase transition [34])

5 Conclusions

Low autocorrelation models display a dynamical behaviour very similar to disordered spin glasses. The reason for this similarity is that these models (and more generally, glasses) do have a broad distribution of higher free energy metastable states like happens in case of spin glasses [7].

The feeling which emerges from recent studies by several groups is that deterministic models display a glassy behaviour of a purely dynamical nature. This glassy behaviour seems to be associated to spin glass models with one step of replica symmetry breaking [17]. In the case of models with an infinite number of breakings like the SK Model [18] the situation is different [19].

We have also seen that the open case and the periodic case behave very similarly. We have studied the relaxation of magnitudes like the internal energy, specific heat and magnetic susceptibility. More interestingly, the Binder parameter associated to the magnetization has a non Gaussian shape even for very large temperatures above the glass transition. This result should apply very probably also for any other Fourier component B_p of the configuration of the spins.

According to this result we have seen that well above the glass transition the dynamical correlation functions decay exponentially fast to a small non zero value. The system is not fully ergodic because it has some memory about the previous configurations it has visited. We have given an explanation to this fact but it remains unsolved what is the real nature of this high-T phase. Above the glass transition temperature finite-time scaling analysis has revealed a good technique in order to locate the transition and the dynamical exponents. For the open and periodic models we obtain the equivalent of the product exponents $z\nu$ of the critical dynamical theory. Values close to 2 are obtained. Compared to experimental values obtained in case of real glasses these are small. But this could be an artifact of the long-ranged interactions of the low autocorrelation models.

We have also investigated the dynamics below the glass transition where aging phenomena is present. This is one of the main features in spin glasses. At the glass transition, where the effect of the traps is not very strong, we have found that the scaling law eq.(21) is well reproduced.

We conclude by saying that the techniques developed in this work are very general different temperatures. Then, for each temperature, we computed $C(t_w, 2t_w)$ averaging the correlation function in a logarithmic scale. We proceeded in this way in order to get smooth values of the correlation $C(t_w, 2t_w)$ as a function of the temperature and the waiting time. We have done this numerical analysis for different values of $t_w = 100, 300, 1000$ in the periodic model and $t_w = 100, 300, 1000, 3000$ for the open model. Figures 11 and 12 show the results for the periodic and open model respectively. From this data we can see clearly the discontinuity because the predicted value of q_1 is very close to 1 just below T_G.

In order to obtain T_G we have performed a usual finite-time scaling analysis. To this end we have measured the relaxation curves above the glass transition and also above the temperature at which finite-size effects are negligible (approximately $T = 0.25$ for the open model and $T = 0.55$ in the periodic case). Correlation functions decay exponentially and one can estimate the relaxation time τ as a function of T. In this range of temperatures we expect the correlation time will diverge as a power law singularity of the type

$$\tau \sim (T - T_G)^{-\gamma} \tag{22}$$

where γ is a dynamical exponent. We note that this kind of divergence is typical also of disordered systems with long or short-range interactions. In case of frustrated models without disorder the situation can be different depending on the range of the interaction. Low autocorrelation models are of the long-ranged type. It is possible that for more realistic models of glasses the dynamics will be much more complex and strongly different relaxation behaviours, like the Arrhenius or the Vogel-Fulcher law, could take place. Now we want to observe that usual critical dynamics works well in the case of low autocorrelation models. This is not surprising if (as we have seen in this work) glasses and spin glasses do have so much in common [32]. We have fitted the correlation functions in the high T regime with a scaling law of the type

$$c(t) \sim f(t/\tau) \tag{23}$$

where τ is given in eq.(22). The scaling behaviour is shown in figures 13 and 14 for the open model and the periodic model respectively. Good fits are obtained with $T_G \sim 0.21 \pm 0.02$ and $\gamma \sim 2 \pm 0.5$ for the open model and $T_G \sim 0.43 \pm 0.2$, $\gamma \sim 2 \pm 0.5$ for the periodic model. The scaling function $f(t/\tau)$ is nearly an exponential in both cases. The exponent γ is the equivalent of the product of exponents $z\nu$ for usual critical dynamics and it is certainly much lower than known values in realistic glasses (tipically these are of order 10, see [32]). As we have already indicated, low autocorrelation models are long-ranged models. Realistic glasses are not of the long range type and it could well be that the

the shape of the correlation function is strongly dependent on the noise realization. To get smooth correlation functions one should average over a very large number of trajectories and this demands for a lot of computer time. From this considerations it emerges that a scaling law of the type

$$C(t_w, t+t_w) \sim f(t/t_w) \tag{21}$$

is very difficult to observe in a small number of decades of time. This scaling law has been obtained by Bouchaud in his phenomenological approach to the off-equilibrium dynamics [27]. Cugliandolo and Kurchan [15] have explicitely shown that this is indeed a solution of the off-equilibrium equations in case of the p-spin spherical spin-glass model and the Potts model [28]. These models do have a spin-glass phase with one step of replica symmetry breaking. It is reasonable to suppose that the scaling law eq.(21) also applies in case of low autocorrelation models for which a REM-like transition describes well the low T behaviour. We should also note that the dynamical behaviour we are observing in these models is strongly different from the dynamical relaxation of the SK model [29] or short-ranged Ising spin glasses [30]. In that case, one does not has a first order dynamical transition and the free energy landscape is not so rough. The system is not trapped in the metastable states and correlation functions decay to zero smoothly without apparent jumps [19]. When a strong metastability is present (like in low autocorrelation models) new numerical techniques like those recently developed by Krauth and Pluchery [22] and Krauth and Mezard [21] are very useful. If one wants to observe smooth aging in a reasonable scale of time, it is mandatory to go to higher temperatures. Precisely, at the glass temperature, we expect that the system will display nice aging and the scaling law eq.(21) will be satisfied for enough large sizes. This is shown in figure 10 where we have measured the aging at a temperature $T_G \sim 0.19$ for the open model and a very large size $N = 10000$. The inset of figure 10 shows the scaling law eq.(21).

Now we want to show how aging allows for a nice confirmation of the first order nature of the glass transition. This is one of the main results of this work. Because the nature of this glass transition is purely dynamical we can use the relations eq.(19) and eq.(20) in order to find the temperature at which the discontinuity of the order parameter appears. A similar technique could be used by coupling two replicas like has been done in case of the p-spherical spin glass model [31]. Nevertheless, we think that our dynamical technique is more direct because we do not need the introduction of an extra coupling parameter in the model.

We have computed the correlation function for different waiting times t_w and also

[24] or Potts glasses [25] and low autocorrelation models, the last ones show a very large discontinuity in the value of q_1). Because the value of q_1 jumps from zero above T_G to a finite value below T_G the transition is of a discontinuous type. Before showing our dynamical results in case of low autocorrelation binary sequences we would like to note that, as regards to the dynamical experiments, deterministic models are much more suited than disordered models. Because our model is ordered, we do not need to save the realization of the random couplings. The number of random couplings, in case of a long range model, can be very large and this sets a limit on the maximal size one is able to memoryze in the computer. The major part of the numerical results we will show correspond to $N = 5000$ in both models (open as well as in the periodic case).

The existence of aging is one of the most outstanding features of spin glasses [26]. Now we are going to show that also low autocorrelation models also exhibit these phenomena just below the glass transition. Because the results we have obtained for the periodic and the open case are very similar, we will present in some cases the results only for the open case. Figure 8 shows the correlation function eq.(18) for the open model, for different values of the waiting time above the glass transition T_G (as estimated in the previous section). The data in this case corresponds to a temperature $T = 0.45$. This figure shows that above the glass transition the aging effects are absent (i.e., the correlation functions do not depend on the value of t_w). Also, all correlation functions decay very fast with the time. Surprisingly (as shown in the figure 8) they do not decay always to zero. In some cases, they decay to a small finite value (for the suspicious reader we will note that this value is fairly large than the standard deviation $\frac{1}{\sqrt{N}}$). This means that, well above the glass transition, the system preserves a small temporal correlation with previous configurations. As discussed in the previous section, this is strongly related to the non gaussianity of the fluctuations (for instance, this was shown in the case of the magnetization). This behaviour is far from being paramagnetic. It is not clear to us what is the real dynamical nature of this high-T region.

As soon as we go below T_G the dynamics slows down dramatically. The system remains trapped in metastable states and it takes a very long time for the system to overcome the barriers and explore new configurations. This is clearly seen in the results of figure 9 where we show the correlation function below the glass transition at $T = 0.1$ for one realization of the noise for the open model. Aging effects are present and we expect correlation functions to depend mainly on the ratio t/t_w, for large enough values of t_w. Some comments are now in order. As shown in figure 9 the correlation function stays very close to 1 during a time of order of 10^4 Monte Carlo steps for all different waiting times. This is because for enough low temperatures the system is able to surmount only a few number of states and

Below the glass transition the time behaviour of the correlation and response function drastically change and, for instance, time correlations decay very slowly in time. In this low T regime the time homogeneity hypothesis is lost and strong aging effects start to appear. Then the decay of the correlation functions depends on the previous history of the system. More concretely, it depends on the time t_1 at which the spin configuration is memorized (in case of the correlation functions) or on the time t_2 at which the magnetic field is switched off (in case of the response function).

For simplicity reasons we have focused our research on the two-time correlation function (one could also perform aging experiments measuring the remanent magnetization). In this case we have measured the time-time correlation function between the spins configuration at time t_w and the configuration at time $t_w + t$,

$$C(t_w, t_w + t) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t_w)\, \sigma_i(t_w + t) \tag{18}$$

Above the glass transition temperature we expect time homogeneity applies (this means that $C(t_w, t_w + t)$ only depends on t) and time correlation functions should decay very fast to zero. The following condition holds

$$\lim_{t_w \to \infty} C(t_w, 2t_w) = 0 \ . \tag{19}$$

Just below T_G the correlation function decays very slowly in time to a finite value q_1. This finite value q_1 is positive and smaller than the static Edwards-Anderson order parameter at the static transition point. This value q_1 is zero above T_G and is very close to 1 just below T_G and increases as the temperature decreases (linearly with T at low temperatures). We have to call the attention of the reader to the fact that this value is physically related to the local order parameter associated to the metastable states and this is smaller than the local overlap associated to the true equilibrium configurations (the static Edwards-Anderson order parameter). The procedure in order to determine the value of q_1 has been applied recently to a particular deterministic model (see (II)) and corresponds to the replica order parameter within the same block at one step replica of replica symmetry breaking. This is evaluated at the dynamical transition point where the free energy is maximized according to the marginality condition. More precisely, we can write (for an infinite system)

$$\lim_{t_w \to \infty} C(t_w, 2t_w) = q_1 \tag{20}$$

where q_1 depends on the temperature. For low autocorrelation models we know the value of q_1 is very close to 1 (for instance, this is the greatest difference between p-spin glasses section, this result has strong implications for the dynamics. We expect that well above T_G the spin-spin correlation function $\langle\sigma(t_0)\sigma(t)\rangle$ decays to zero very fast but the system can preserve a certain memory of the configuration at time t_0. In fact, if the $P(M)$ is so much flattened around $M = 0$, the system can need a very large time to reach configurations completely uncorrelated from the memorized configuration at t_0.

Let us summarize the results of this section. Doing annealings, starting from large temperatures down to the low T region, we observe a glass transition where the energy freezes and fluctuations vanish. This temperature is several times larger than that predicted by the statics and this is related to the peculiar structure of the high energy metastable states which the systems explores during the relaxation. More concretely, we have learned that the glassy temperature occurs at $T_G^C \simeq 0.45$ for the periodic model and $T_G^O \simeq 0.2$ for the open case. In the next section we will confirm this results by studying the off-equilibrium dynamics of these models. In particular, aging phenomena will appear as a nice method to determine the glass transition.

4 Aging and the first order nature of the dynamical transition

As we said in the last section this transition is of first order nature in the dynamical order parameter. In principle the dynamics is described by the two-time correlation functions $C(t_1, t_2)$ and the response functions $G(t_1, t_2)$. They are defined as usual by:

$$\begin{aligned} C(t_1,t_2) &= \langle\sigma_i(t_1)\sigma_i(t_2)\rangle \\ G(t_1,t_2) &= \frac{\delta\langle\sigma_i(t_1)\rangle}{\delta h(t_2)} \qquad t_2 < t_1 \qquad (16) \\ & \qquad (17) \end{aligned}$$

where $\langle ... \rangle$ is the usual time average over different noise realizations in the dynamics and $h(t_2)$ is the magnetic field applied to the system at the time t_2. We have performed discrete Monte Carlo dynamics which we expect to give similar results as well as a usual Langevin dynamical process.

In the high-temperature regime, above the glass transition, we expect that the correlation and the response functions are related one to the other by the fluctuation-dissipation theorem. Also in this high T region the correlation and the response functions satisfy the time-homogeneity hypothesis, i.e. the functions $C(t_1, t_2)$ and $G(t_1, t_2)$ depend only on the time difference $t_1 - t_2$. Both functions decay very fast in time.

and the B_p is that these last Fourier components are complex while the magnetization is real. According to eq.(10) the effective probability distribution of the magnetization is given by

$$P(M) \sim \exp(-\beta M^4 - \mu M^2) \tag{13}$$

We immediately observe that only at infinite temperature the probability distribution will be a Gaussian and at finite β non Gaussian corrections can be very strong (the same discussion is valid for any Fourier component B_p). This result was numerically observed by Migliorini studying the local field distribution [20] in the open model. Using this approximation and equation (8) for the periodic model one gets

$$\chi = \beta \tag{14}$$

for the linear susceptibility of the periodic model. Figures 5 and 6 show the linear susceptibility obtained during an annealing process. Figure 5 also shows the prediction eq.(14) for the periodic model. The values obtained for the glass transition from the discontinuity of the linear susceptibility agree with those obtained measuring the specific heat (figures 3 and 4).

In case of the Binder parameter we use eq.(9) and we can obtain it in terms of the internal energy (now one has to be a little bit careful and realize that the integral of the fourth power of the magnetization, which is a real variable, over the probability distribution eq.(13) is $\frac{3}{2}$ times the integral of the fourth power of any complex Fourier component B_p over the effective Hamiltonian eq.(10)). One gets the result,

$$g = \frac{3}{4}(1 - e) \tag{15}$$

We show in figure 7 the behaviour of the Binder parameter associated to the magnetization for the periodic model (Monte Carlo results are also shown for the open case). It is shown up to $T = 2$ (five times the predicted glass temperature of the periodic model). For very large temperatures the Binder parameter should vanish because the magnetization distribution becomes a Gaussian. In our case it decays very slowly to zero which indicates that well above the glass temperature fluctuations in the magnetization are large. Also from figure 7 we can observe a a jump for the Binder parameter at the glass transition to a value close to 1. One comment about the high value of g above T_G is now appropriate. This large value of the curtosis parameter means that the probability distribution of the magnetization is far from being a Gaussian. It is a symmetric distribution very flat close to $M = 0$ and possibly with two peaks symmetrically distributed. As we will see in the next

Figures 3 and 4 show the behaviour of the specific heat for the periodic and open model respectively. Also in this cases we plot the results for the GB approximation and, in case of the periodic model, we plot also the high-temperature prediction eq.(6). In both cases we observe a discontinuous jump of the specific heat. It appears at a temperature $T_G^C \sim 0.45$ for the periodic model and $T_G^O \sim 0.2$ for the open case. We have to note that this energy and specific heat in the low T phase are purely dynamical. Anyway they satisfy fluctuation dissipation theorems like the relation $C = \frac{\partial e}{\partial T}$ where C is the specific heat and e is the internal energy.

We have also measured the magnetization and its fluctuations. The global magnetization is zero above T_G and below that temperature remains stacked to a certain small non zero value (of the order of the standard mean deviation $1/\sqrt{N}$). Valuable information can be obtained from its fluctuations like the linear susceptibility and the Binder parameter. If $P(M)$ is the probability distribution of the magnetization, we expect it will be a Gaussian at very large temperatures and become more and more flattened as the glass transition is approached. We are going to show that this is really the case and that fluctuations are very large even if we stay at high temperatures. In plain words, the linear susceptibility and the Binder parameter are the variance and the curtosis of the probability distribution $P(M)$. The linear susceptibility is given by

$$\chi = \beta(\langle M^2 \rangle - \langle M \rangle^2) \tag{11}$$

where M is the global magnetization and we recall the fact that one factor N has been absorbed in the temperature in order to have an appropriate thermodynamic limit. The Binder parameter is given by

$$g = \frac{1}{2}(3 - \frac{\langle M^4 \rangle}{\langle M^2 \rangle^2}) \tag{12}$$

Now we would like to compute approximately this quantities in the high T phase above the glass transition. From eq.(5) we observe that the Hamiltonian is the sum of $\frac{N}{2}$ Fourier components $B(p)$. We can suppose that these Fourier components are independent at least in the high T phase (in some sense this is the original idea of Golay in order to resumme the high T series). One can soon realize that this approximation has to fail because the total number of Fourier components is too large (it diverges with N). But this is the easiest approximation one can do. In order to reach the correct expression it should be necessary to solve the low autocorrelation models in a magnetic field. Within this approximation and using the Hamiltonian eq.(5) we observe that the zero momentum term $|B(0)|^4$ corresponds to the fourth power of the magnetization. The only difference between the magnetization conclusions will be the same for the open and the periodic case. Because the open model has historically received more attention than the periodic version we will present more dynamical results in the former case.

3 A first determination of the glass transition

The main property of the glass transition in low autocorrelation models regards the first order nature of this dynamical transition. From the thermodynamic point of view this transition is second order. So, for instance, the energy and the entropy are continuous while the specific heat experiences a jump. Because the transition is purely dynamical, this implies a transition for the correlation and response functions. In this section we will explore the behaviour of thermodynamic quantities leaving the discontinuous feature of the order parameter to the next section. For the periodic and open models we have done the same kind of studies. In fact, we have discovered that they are strongly similar except for the fact that the periodic model is analytically solved in the high T phase and displays an explicit ground state for chain lengths N such that N is prime and of the type $4k+3$, k being an integer.

Starting from a random initial configuration in the high-temperature region we have decreased progressively the temperature in a Monte Carlo annealing. We have simulated several sizes up to $N=1000$ (because it is a long range problem the number of bit operations in a Monte Carlo updating procedure grows very fast with the size of the system). We have also tested that finite-size corrections are negligible and different initial conditions give the same result. As said in the introduction, we have now only one realization of disorder on which we have to do simulations. We have computed the main thermodynamical observables like the energy, magnetization and their associated dissipative quantities like specific heat and magnetic susceptibility. The behaviour of the energy are shown in figures 1 and 2 respectively for the periodic and open model. The energy decreases down to a certain temperature where it remains constant. This is much similar to what happens in the REM [8]. The dashed line in figures 1 and 2 corresponds to the GB approximation and the continuous line (only for the periodic model) corresponds to the correct high-temperature prediction eq.(6) which is in agreement with the data. As has been already commented, the glass transition is higher than the static transition (close to 0.1 in the periodic model). Figure 1 shows where the entropy of the high-temperature expression of eq.(6) vanishes. Curiously it does at the same temperature as that given by the GB approximation. We have no explanation for this result. If this were true also in the opened case one would be tempted to state that the GB approximation is enough to predict the static transition.

where the value of μ is determined by the equation

$$\int_0^\infty r^3 \exp(-\beta r^4 - \mu r^2)\, dr = \int_0^\infty r \exp(-\beta r^4 - \mu r^2)\, dr \tag{7}$$

The last condition corresponds to the closure condition

$$\sum_{p=1}^{\frac{N}{2}} \langle |B_p|^2 \rangle = 1 \ , \tag{8}$$

the internal energy is given by

$$e = \frac{\partial \beta f}{\partial \beta} = \sum_{p=1}^{\frac{N}{2}} \langle |B_p|^4 \rangle - 1 \tag{9}$$

and the mean values $\langle ... \rangle$ are evaluated using the effective Hamiltonian

$$\mathcal{H}(\{B_p\}) = -\beta \sum_p |B_p|^4 - \mu \sum_p |B_p|^2 \ . \tag{10}$$

The integration variables B_p are complex variables and the mean values $\langle ... \rangle$ are obtained integrating over the real and imaginary part of the B_p.

These expressions are valid for the periodic model down to the temperature at which the entropy vanishes, which is $T^C_{RSB} \sim 0.1$ (superindex C means for periodic model and O for the open case). This result is surprisingly close to that given by the GB approximation. We indicate that temperature with the subindex RSB because at that temperature replica symmetry is broken. Below T^C_{RSB} the entropy is nearly zero and the energy is constant, a situation indeed very similar to that of spin-glasses with one step of replica symmetry breaking [8, 17]. Obviously the previous expression eq.(6) is not valid for the open model for which a high-temperature resummation is still lacking. Anyway, using the Golay approximation we can estimate this transition to be close to $T^O_{RSB} \sim 0.047$.

As we will see in the following sections the transition T_{RSB} for the periodic and for the open model are not the true *glass transition*. As discussed in the introduction, the true glass transition corresponds to the transition where dynamical stability is lost (ie. the temperature given by the marginality condition) and it can be several times larger than the corresponding transition given by the statics. This result was already known in spin-glasses but it is new to know that this result also applies in case of deterministic models. The following sections are devoted to the numerical determination of this transition in the open and periodic case. We will see also how off-equilibrium phenomena and particularly the property of aging yield a very nice preciser to determine this glass transition temperature. Regarding the dynamical behaviour of both models we can advance that the main of the periodic model it can be shown that for prime values of N of the type $(4k+3)$, k being an integer, there exists an explicit ground state of finite global energy (and energy per spin zero in the thermodynamic limit because eq.(1) has to be normalized by N). This construction does not exist in the open case. This ground state has a very low entropy up to a finite temperature where the entropy experiences a sudden jump. This finite temperature is the crystallization transition. Starting at zero temperature from the ground state configuration and slowly increasing the temperature the entropy increases also very slowly (remaining always very close to zero). At the cristallization temperature the entropy jumps to a finite value and the system enters into the high-temperature regime. To account for this situation sometimes it is said that the phase space has *'golf course'* like properties. As regards to the dynamical behaviour of these models, the existence of a ground state of very low energy is of no relevance because we are interested in the behaviour of large systems. In fact, during a usual dynamical relaxation process, the system is unable to find the ground state because this state has very low entropy. In this situation, the particular mathematical features of the selected number N (prime of the type $4k+3$ or not) are irrelevant. As has been shown in (I) the high-temperature phase of the periodic model can be exactly solved. Due to the translational symmetry of the model one can write the Hamiltonian in terms of the Fourier space components

$$B(p) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp(\frac{2\pi i j p}{N}) \sigma_j \tag{4}$$

where i stands for the imaginary unit. The Hamiltonian eq.(2) now reads

$$H = \frac{2}{N} \sum_{p=1}^{\frac{N}{2}} |B(p)|^4 \; . \tag{5}$$

Because the σ_i are real functions (i.e., $B(p) = \overline{B(-p)}$) half of the Fourier components can be neglected. Writing the Hamiltonian in the Fourier space one can show that only certain kind of connected diagrams contribute to the free energy allowing for a Hartree-Fock resummation of the full series. In another way one can demonstrate, by introducing in eq.(4) a generic unitary matrix, that the replica approach can be used to find the free energy of the model. In the replica symmetric approximation one recovers the Hartree-Fock resummation. For our purposes it is important to note that the free energy of the periodic model is given by

$$f = \frac{1}{\beta} \log \int_0^{\infty} r \exp(-\beta r^4 - \mu r^2) dr \; - \frac{1}{\beta} \log(2) - 1 \tag{6}$$

The work is divides as follows. Section 2 introduces the low autocorrelation models we will investigate and presents the main theoretical results in this case. Section 3 is devoted to the study of different thermodynamical quantities during an annealing cooling process which clearly display the existence of the glassy transition. Section 4 is the main nucleus of our work which is the study of off-equilibrium properties of this models and particularly of aging. The discontinuous nature of the glass transition will enable us the use of this property for an accurate prediction of the glass temperature.

2 Low autocorrelation models

By low autocorrelation models we denote a large class of deterministic one dimensional models with long range interactions. In this work we have focused our interest on the periodic and open models (so called depending on the type of boundary conditions). These models have their own interest as optimization problems in the field of communication systems. Let us suppose a one dimensional chain of Ising spins $\{\sigma_i; i = 1, N\}$ which can take the values ± 1 and the following Hamiltonian

$$H = \frac{1}{N} \sum_{k=1}^{N} C_k^{2\nu} \tag{1}$$

where the C_k are correlation functions which connect spins at distance k. The case $\nu = 1$ is the problem on is generally interested but nothing prevents of considering different models for a generic value of ν. For the periodic model we have,

$$C_k = \sum_{i=1}^{N} \sigma_i \sigma_{i+k} \tag{2}$$

and in case of the open model,

$$C_k = \sum_{i=1}^{N-k} \sigma_i \sigma_{i+k} \tag{3}$$

so in this case there is not translational symmetry in the model.

It was shown by Golay [23] and after by Bernasconi [10] that one could approximate the thermodynamics of the open model by supposing that the different correlation functions C_k are uncorrelated gaussian distributed random variables. This is the Golay-Bernasconi (GB) approximation and predicts the existence of a phase transition at a low temperature where the entropy vanishes. The same conclusion is valid for the periodic model where one expects the existence of a phase transition at low temperatures.

The interested reader can find most of the results of this section for the periodic model in reference (I). Now we will recall some of the main results obtained in that work. In case in the static case) such that one particular eigenvalue of the stability matrix vanishes (the so called replicon eigenvalue). This condition corresponds to the temperature at which dynamical stability disappears. The dynamical transition temperature has been obtained by several autors using the dynamic mean-field theory of spin glasses initially studied by Sompolinsky and Zippelius for the study of the SK model [12]. Always in the framework of disordered systems, studies of Kirkpatrick and collaborators on the p-spin Ising models [13] and the Potts glass [14] showed that this dynamical temperature is above that predicted by the statics. Recent studies of the off-equilibrium dynamics of the p-spherical spin-glass model by Cugliandolo and Kurchan have shown that the energy of the dynamics in the low temperature phase, below the dynamical transition, is higher than that predicted by the statics [15]. As the dynamical transition temperature is approached the off-equilibrium dynamics slows down and aging effects start to appear. Similar aging phenomena have been found in the context of random manifolds [16].

In some cases the glass transition for the models without disorder can be also predicted using the replica approach. A concrete test of all these theoretical results for a deterministic model and its disordered version (defined as the disordered model which has the same high-T expansion as the deterministic model) has been performed very recently for the sine model [2] (hereafter referred as paper II). In this case, the dynamical transition can be exactly computed and compared to the numerical results. We stress on the fact that this dynamical transition in the context of disordered systems corresponds to the glass transition for the deterministic case.

The purpose of this work is to show how several numerical techniques in spin glasses can be used for the determination of the glass transition temperature for deterministic models. Because this glassy transition is, as we have already indicated, purely dynamical this will be also the main spirit of the techniques we will use. Now the reader will realize that the main advantage of the study of deterministic models relies on the fact that one does not need to average over different realizations of disorder. Because of the dynamical nature of the transition one should average over different initial conditions. Anyway, comparing to the spin-glass case we have eliminated one source of strong fluctuations. In this work we will concentrate in the case of low autocorrelation binary sequences. These models have received a lot of attention very recently because they are the simplest prototype of ordered systems with a very complex energy landscape. We adress the reader to the differents works in this subject. Migliorini has performed extensive numerical simulations using the tempering method [20] and Krauth and Mezard [21] and Krauth and Pluchery [22] have applied a modified version of BKL algorithm which allows to investigate dynamical properties for very large times.

1 Introduction

There has been much recent interest in the field of condensed matter physics in the study of frustrated models without explicit disorder [1, 2, 3, 4, 5]. These models (also called deterministic models) show a very similar behaviour to spin glasses [6], i.e. there exist a very large number of metastable configurations where the system remains trapped and it is very difficult to reach the global equilibrium state in a dynamical process starting from a random initial configuration.

The main difference between these frustrated models and spin glasses is that in case of deterministic models the quenched disorder is not present. Because some symmetries are preserved in the deterministic model, it is possible in some cases to explicitely construct the ground state. This possibility is generally forbidden in disordered systems because no symmetry is preserved. Also, in disordered systems, each realization of the randomness yields a different ground state implying that there is much difficulty in devising any kind of algorithm to identify the ground state.

Recently it has been shown that the application of techniques initially devised for this random systems promises to be a powerful tool in the understanding of the deterministic models [1]. In particular, much effort has been recently devoted to the study of the Bernasconi model [10]. This is an optimization problem in which one searches for strings of binary digits with minimal autocorrelation. The high-temperature phase of this model has been exactly solved in the particular case of periodic boundary conditions ([1] hereafter referred as paper I). The system shows a static transition to a frozen phase where the entropy is nearly zero. In the original Bernasconi model with open boundary conditions an exact solution for the high T phase is still lacking but some approximations suggest that a similar static transition takes also place in that case. This static transition is different from the dynamical transition one observes in a real system. The dynamical transition is higher than the static transition and corresponds to the situation in which the system remains trapped in metastable configurations. Below this dynamical transition temperature, thermal fluctuations are very small and reminds a lot of what happens in real glasses [9]. In the context of models without explicit disorder this transition has been called the *glass transition*. Starting a dynamical evolution from the high-temperature region an enough large system is unable to see the static transition because it gets trapped in the metastable phase at a higher temperature. For all purposes, it is always this higher temperature transition which governs the dynamics. Within the realm of disordered systems this dynamic transition can be computed using the *marginality condition* [11]. This condition corresponds to the search for certain saddle points of the free energy (not true maxima like

Dynamical Behaviour of Low Autocorrelation Models

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Abstract

We have investigated the nature of the dynamical behaviour in low autocorrelation binary sequences. These models do have a glass transition T_G of a purely dynamical nature. Above the glass transition the dynamics is not fully ergodic and relaxation times diverge like a power law $\tau \sim (T - T_G)^{-\gamma}$ with γ close to 2. Approaching the glass transition the relaxation slows down in agreement with the first order nature of the dynamical transition. Below the glass transition the system exhibits aging phenomena like in disordered spin glasses. We propose the aging phenomena as a precise method to determine the glass transition and its first order nature.

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