D-dimensional Arrays of Josephson Junctions, Spin Glasses and q-deformed Harmonic Oscillators

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Abstract

We study the statistical mechanics of a D-dimensional array of Josephson junctions in presence of a magnetic field. In the high temperature region the thermodynamical properties can be computed in the limit $D \to \infty$, where the problem is simplified; this limit is taken in the framework of the mean field approximation. Close to the transition point the system behaves very similar to a particular form of spin glasses, i.e. to gauge glasses. We have noticed that in this limit the evaluation of the coefficients of the high temperature expansion may be mapped onto the computation of some matrix elements for the q-deformed harmonic oscillator.

1 Introduction

In this paper we are interested to study the statistical mechanics of arrays of Josephson junctions in D-dimensions in the limit where $D \to \infty$. We will construct here the solution of the mean field theory in the high temperature phase. We postpone to a later stage the computation of the corrections to the mean field approximation and the study of the low temperature phase. The model has been studied in two dimensions, especially in the low temperature region [1, 2], but no results are known in very high dimensions.

The model we consider is described by the Hamiltonian:

$$H = -c(D) \sum_{j,k} \overline{\phi}_j U_{j,k} \phi_k + h.c. \ . \tag{1}$$

Here $c(D)$ is a normalisation constant, which will be useful later to rescale the Hamiltonian in order to obtain a non trivial limit when D goes to infinity. The spins ϕ_j are defined on a D-dimensional hypercubic lattice.

We can consider three possibilities:

- The spins ϕ_j are constrained to be of modulus one.
- The spins ϕ_j have modulus one in the average at $\beta = 0$: in this limit they have a Gaussian distribution.
- The spins satisfy the constraint $\sum_i |\phi_i|^2 = N$. This is the spherical model which is intermediate among the two previous models.

In the limit where the dimension D goes to infinity the properties of the first model and of the third model can be obtained from that of the Gaussian model. We will concentrate our attention on the Gaussian case.

The couplings U are non zero only for nearest neighbour sites. They are complex numbers of modulus one and they satisfy the relation

$$U_{k,j} = \overline{U_{j,k}}. \tag{2}$$

In other words the couplings U are the links variables of an $U(1)$ lattice gauge field.

We will select the couplings U to give a constant magnetic field. Many different orientations of the magnetic field can be chosen. For simplicity we restrict our computation to the case where the flux through each elementary plaquette is given by B (or $-B$), independently from the plane to which the plaquette belongs. This corresponds to constant uniform frustration on all the plaquettes. In the extreme case ($B = \pi$) we obtain a fully frustrated model, while for $B = 0$ we recover the ferromagnetic case. Random point dependent B values correspond to a particular form of spin glasses, i.e. to gauge glasses [3]-[6].

More precisely we set

$$B_{\alpha,\beta} = S_{\alpha,\beta} B, \tag{3}$$

where $S_{\alpha,\beta}$ may take the values 1 or -1, $B_{\alpha,\beta}$ is the antisymmetric tensor corresponding to the magnetic field, which in the continuum limit is given by $\partial_\alpha A_\beta - \partial_\beta A_\alpha$. The ordered product of the four links of a plaquette in the α, β plane is equal to $\exp(i\ B_{\alpha,\beta})$.

We must now specify $S_{\alpha,\beta}$, i.e. the sign of $B_{\alpha,\beta}$. A possible choice would be to take

$$S_{\alpha,\beta} = 1 \quad for \quad \alpha > \beta, \tag{4}$$

which implies $B_{\alpha,\beta} = B$ for $\alpha > \beta$.

In two and in three dimensions this choice is equivalent to any other possible choice of the sign. In three dimension the magnetic field is a vector and all the vectors corresponding to different choices of the sign may be obtained one from the other with a rotation. The choice of S does not influence the thermodynamics.

In more than three dimensions different choices of the matrix S are not equivalent [1] and we must select one among all the possible ones. In this note we consider the case in which the matrix S is a generic one, i.e. the signs of B are randomly chosen. The system is translation invariant and the randomness appears in only in the relative orientation of the magnetic field with the crystal axis.

In the two dimensional case we recover the usual description for an XY system (or equivalently an array of Josephson junctions) in constant magnetic field.

The aim of this note is to compute the statistical properties of this model in the mean field approximation in the high temperature region. The first difficulty we face consists in finding the spectral properties of the lattice discretised Laplacian in presence of a magnetic field. The lattice Laplacian is defined as

$$(\Delta f)_j = \sum_k U_{j,k} f_k. \tag{5}$$

The spectral properties of the lattice Laplacian in two dimension have been carefully studied. They depend on the arithmetic properties of the B/π, i.e. different results are obtained for rational and irrational B/π [2].

The study of the lattice Laplacian in higher dimensions is much less developed. In any dimension the explicit construction of the field U show that for rational B/π, of the form $B = 2\pi r/s$, with both r and s integers, there is a gauge in which the U couplings are periodic functions of the position, with period s. In this case the spectrum of the Laplacian has the typical band form, the edges of the bands being related to the eigenvalues of a $s^D \times\ s^D$ matrix. When both s and D are large, a direct study of the eigenvalues is rather complex.

We will study this problem in the limit of an infinite number of dimensions . We cannot solve it in a completely satisfactory way, but we can put forward some educated guesses. We will find some unexpected relations with the properties of the q-deformed harmonic oscillator. At the end the behaviour of the model will come out very similar to that of spin glasses. The reader should notice that it is not clear how much of our results survives in a large, but finite, dimensions and that the properties of the model in high dimensions may be quite different from that of the two dimensional model.

In section II we present some general considerations. In the next section we show some general properties of the high temperature expansion in the limit $D \to \infty$. We consider in detail the ferromagnetic case, the spin glass case and the constant frustration model. In section IV we show the relation among the high temperature expansion for the constant frustration model in infinite dimension and the q-deformed harmonic oscillator. In the next section we study the behaviour of our model near the critical point and we find that it is very similar to that of spin glasses. In section VI we briefly discuss the problems related to the exchange of limits ($\beta \to \beta_c$ and $D \to \infty$). Finally (in the last section) we present our conclusions and express our points of view on the open problems. In the appendix we

[1] I am grateful to E. Marinari and F. Ritort for crucial discussions on this problem.

will describe some interesting features of the q-deformed harmonic oscillator, which shows an anomalous behaviour for $q = \exp(2\pi\ i\theta)$, when θ is rational.

2 General Considerations

There are two extreme cases for the U which are very well studied for the Hamiltonian (1):

- We set
$$U_{j,k} = 1. \tag{6}$$
In this way we obtain the usual ferromagnetic XY model. There is a ferromagnetic transition at $\beta = 1$ in the limit $D \to \infty$, if we set $c(D) = \frac{1}{2D}$, i.e. $c(D)$ has to be equal to the inverse of the coordination number of the hypercubic lattice.

- We set
$$U_{j,k} = \exp(ir_{j,k}), \tag{7}$$
where r are random numbers belonging to the interval $0-2\pi$, such that the symmetry condition eq. (2) is satisfied.

  In this way be obtain an spin glass model of the XY type, which is called a gauge glass. The transition temperature is $\beta = 1$ in the limit $D \to \infty$, if we set $c(D) = (2D)^{-1/2}$, i.e. [3, 4, 5] $c(D)$ is equal to the inverse of the square root of the coordination number.

The model we study is intermediate among the previous two problems. In order to define it properly, it is convenient to introduce the so called Wilson loop. Let us consider a closed oriented circuit (C) on the lattice, which goes from the point j to the same point j and let us define $W(C)$ as the product of the U's along the circuit. The Wilson loop $W(C)$ is a gauge invariant. The knowledge of $W(C)$ for any C gives all gauge invariant informations concerning the gauge field.

In the continuum limit we have
$$W(C) = \exp(i \int_C dx^\mu A_\mu(x)) = \exp(i\Phi(C)), \tag{8}$$
where $\Phi(C)$ is the magnetic flux entangled within C.

In 2 dimensions in presence of a constant magnetic field the Wilson loop is given by
$$W(C) = \exp(iBS(C)), \tag{9}$$
where $S(C)$ is the signed area of the loop C.

In D dimensions there are $D(D-1)/2$ planes oriented in the directions of the lattice. The choice of the magnetic field we study here is

$$\begin{gathered} W(C) = \exp(i\Phi(C)), \\ \Phi(C) = \sum_{\nu,\mu\ =\nu<\mu} S_{\nu,\mu}(C) B_{\nu,\mu}\ , \end{gathered} \tag{10}$$

where the indices ν and μ denote one of the D possible different directions and $S_{\nu,\mu}$ is the signed area of the projection of the curve C on the ν, μ plane.

As a consequence of gauge invariance there are infinite many choices of the U which correspond to these Wilson loops. All these choice are physically equivalent. In two dimensions we could set

$$U_1(j) = 1, \quad U_2(j) = \exp(iBj_1),$$

where j_ν is the $\nu^{\rm th}$ component of the vector j and we have introduced the short-handed notation

$$U_\nu(j) = U(j, j + n_\nu), \tag{11}$$

n_ν being the unit vector in the ν direction.

This construction can be generalised to the D-dimensional case. For example in 4 dimensions one obtains

$$\begin{aligned} U_1(j) &= 1, \quad U_2(j) = \exp(iB_{2,1}j_1), \\ U_3(j) &= \exp(i(B_{3,1}j_1 + B_{3,2}j_2)), \quad U_4(j) = \exp(i(B_{4,1}j_1 + B_{4,2}j_2 + B_{4,3}j_3)). \end{aligned} \tag{12}$$

Our main task will be the study of the associated Gaussian model, where the Hamiltonian is given

$$H = -c(D) \sum_{j,k} \overline{\phi}_j U_{j,k} \phi_k + h.c. - 1/2 \sum_k |\phi_k|^2. \tag{13}$$

The solution of this associated Gaussian model is a crucial step in the computation of the properties of the high temperature expansion.

3 The high temperature expansion

In the case of the Gaussian model the free energy density can be written as

$$\beta F(\beta) = \sum_C W(C)(\beta c(D))^{L(C)}/L(C), \tag{14}$$

where the sum is done over all the closed lattice circuits with given starting point; $L(C)$ is the length of the circuit [6].

In a model (like the present one) where gauge invariant quantities are translational invariant [7], we can chose the origin (and the end) of the circuit at an arbitrary point of the lattice. In other cases, like spin glasses, we must average over all the possible starting points [8].

The previous formula can also be written as

$$\beta F(\beta) = tr \ln(1 + c(D)\beta\Delta) = \sum_n \frac{(\beta c(D))^n}{n} \mathcal{N}(n) \langle W(C) \rangle_n \tag{15}$$

where by $\langle W(C) \rangle_n$ we denote the average over all the circuits of length n and by $\mathcal{N}(n)$ the number of (rooted) closed circuits.

Differentiating the previous formulae we obtain a similar result for the internal energy density:

$$2\beta c(D) U(\beta) = \sum_n (\beta c(D))^n \mathcal{N}(n) \langle W(C) \rangle_n. \tag{16}$$

Here the factor $1/n$ has disappeared.

3.1 The ferromagnetic case

This is the simplest case. We have only to compute $\mathcal{N}(n)$ since $\langle W(C)\rangle_n = 1$.

It is evident that $\mathcal{N}(n) = 0$ for odd n. The first non zero contributions for small n are

$$\mathcal{N}(2) = 2D, \qquad \mathcal{N}(4) = 6D(2D-1). \tag{17}$$

We could also compute $\mathcal{N}(n)$ using the representation

$$\mathcal{N}(n) = \int_B d^D p \; (\sum_{\mu=1,D} 2\cos(p_\mu))^n. \tag{18}$$

If we use the correct normalisation of $c(D)$, that gives the critical temperature at 1, we immediately find that when $D \to \infty$ all these contributions vanish. This is a well known fact: in the high temperature phase in the mean field approximation the internal energy of a ferromagnetic system is zero. The fluctuations contribute only in the subdominant terms of the large D expansion.

This behaviour implies that one should be careful in taking the limit $D \to \infty$. Indeed it is easy to check that in the limit where $n >> D$ one finds that [9]

$$c(D)^n \mathcal{N}(n) \propto n^{-D/2}, \tag{19}$$

but in the opposite limit $D >> n$ one gets

$$N(n) \sim (n-1)!!(2D)^{n/2}, \tag{20}$$

and therefore

$$c(D)^n \mathcal{N}(n) \sim \frac{(n-1)!!}{(2D)^{n/2}}. \tag{21}$$

The equation (20) is very simple to understand. In a closed circuit for each step in one direction there must be a step in the opposite direction. In infinite dimensions all the steps are taken in different directions (in a way compatible with this constraint) . The generic circuit will be thus identified by the directions in which these steps are done (we have to make a choice $n/2$ times between these directions) and by the locations of the steps at which two opposite directions are chosen. In high dimensions all the steps are done in different directions and in this way on obtains the previous formula, i.e. the number of pairing of n objects ($(n-1)!!$), multiplied by the number of choices for the directions ($(2D)^{n/2}$) .

If we were not aware of the correct normalisation factor and we had put $c(D) = (\frac{1}{2D})^{1/2}$ with the aim of obtaining a non trivial perturbative expansion, we would get the formula:

$$\beta U(\beta) = \sum_n (2n-1)!!(\beta)^{2n}. \tag{22}$$

We would have found in this way that the high temperature expansion has a zero radius of convergence. This is not a surprise[10] because in this scale the critical temperature is at $\beta = 0$ and any non zero value of β is already in the low temperature regime.

In the ferromagnetic case the singularity of the free energy disappears when $D \to \infty$ in the high temperature expansion with the correct $c(D)$. This effect can be easily explained. The ferromagnetic transition is characterised by the building up a singularity at momentum

$k = 0$ in the two point correlation function. The free energy in the high temperature phase is given by

$$f(\beta) \propto \int_B d^D k \ln(1 - \beta \sum_{\nu=1,D} \cos(k_\nu)/(2D)), \tag{23}$$

where the integral is done over the first Brillouin zone.

When $D \to \infty$ the region of momenta near the origin has a vanishing weight and its contribution to the singularity disappears. We can see a transition in the specific heat in the limit of infinite dimensions only if the directions of the most relevant modes are not orthogonal to the boundary of the Brillouin zone, where the measure is concentrated in momentum space.

3.2 Spin Glasses

In this case we will compute the spectrum of the random Laplacian. This can be done in the infinite dimensions limit since we recover the old problem of computing the spectrum of a random matrix, which is given by a semicircle law[2]. Instead of using directly this result we prefer to follow a diagrammatic approach.

In this case the U's have zero average and are random elements of the $U(1)$ group. After the average over all the possible starting points, $W(C)$ gets contributions only from those circuits for which for any step going from i to k there is a step going from k to i. In other words we must sum only over *backtracking* circuits.

Let us count the number of these circuits in infinite dimensions. We must to compute

$$G_{2n} = \lim_{D\to\infty} (2D)^{-n} \mathcal{N}(2n) \langle W \rangle_{2n}. \tag{24}$$

It is easy to check that for $n = 1$ we do not get any new contribution with respect to the previous case and $G_1 = 1$.

For larger values of n a more detailed computation must be done. At this end it is convenient to denote by a, b, c.. one of the different $2D$ possible directions in which a step could be done.

In the case $n = 2$ we have 3!! circuits which differs for the ordering possibilities:

$$aabb \; abba \; abab, \tag{25}$$

where it is implicit that the second identical letter denotes a back step in the opposite direction of the first one. We do not attach any meaning to the letters a or b: we could have written $aabb$ or $bbaa$ indifferently. In both case the second step and the fourth steps are in the opposite direction of the first and of the third step respectively.

Each of the 3!! choices correspond to $(2D)^2$ lattice circuits (we neglect subleading terms for large D). The first two are backtracking circuits the second is not. We thus find $G_2 = 2$.

In the case $n = 3$ we have 5!! circuits which differs for the ordering possibilities. We list here all the backtracking ones:

$$aabbcc \; abbcca \; abccba \; aabccb \; abccab. \tag{26}$$

Therefore $G_3 = 5$. It is easy to verify that a circuit is backtracking if and only if the corresponding word may be reduced to the null one by subsequent elimination of consecutive identical letters.

[2] One could also use the replica approach.

The computation of G_n can be thus cast under the following graphical form. For each given word, we put its $2n$ letters (two by two equal), on a circle starting from a given point, in the same order of the letters of the corresponding word. We connect those points which have identical letter by a line and we count the number of intersections of the lines. This number is topological invariant and it does not depend on the point where the letter have been put on the circle , but only on their order.

We can associate to each word the number of intersections. Let us call $I_n(m)$ the number of words which have m intersections ($m \leq n(n-1)/2$). It is easy to check that

$$I_n(0) = G_n. \tag{27}$$

Indeed only in the case in which the resulting diagram is planar, the diagram may be reduced to zero by removing consecutively equal letters.

The combinatorial problem of computing $I_n(0)$ has been solved [11] in the past[3]. After a short computation one finds:

$$I_n(0) = 4^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+2)} \tag{28}$$

The result of the computation can also be written is a slightly different form. We consider an Hilbert space, and a base ($|m\rangle$ on this Hilbert space, where m ranges in the interval $[0-\infty]$). We define on this space two shift operators $\mathcal{R}$ and $\mathcal{L}$:

$$\begin{aligned} \mathcal{R}|m\rangle &= |m+1\rangle \\ \mathcal{L}|m\rangle &= |m-1\rangle, \end{aligned} \tag{29}$$

where $|-1\rangle$ is identified with the null vector.

These two operators satisfy the relation,

$$\mathcal{L}\mathcal{R} = 1, \tag{30}$$

which is a particular case (for $q = 0$) of the q-deformed commutation relations [4]:

$$\mathcal{L}\mathcal{R} - q\mathcal{R}\mathcal{L} = 1. \tag{31}$$

It is easy to see that

$$G_n = \langle 0|(\mathcal{R}+\mathcal{L})^{2n}|0\rangle, \tag{32}$$

where the state $|0\rangle$ could also be characterised the condition

$$\mathcal{L}|0\rangle = 0. \tag{33}$$

The existence of these two other formulations should not be a surprise. The condition of zero intersection implies that the diagram is planar and the theory of random matrices may be reformulated in terms of planar diagrams. The theory of random matrices can also

[3] The result is a by-product of the formula relating the generating functionals of the connected and of the disconnected functions.

[4] In the case $q = 1$ we have Bosonic commutation relations, for $q = -1$ we have Fermionic commutation relations and for $q = \exp(i\theta)$ anionic commutation relations. Some applications of the anionic commutation relations can be found in [12, 13] and references therein.

be formulated in terms of the orthogonal polynomials respect to a given measure [14] and in this contest it is well known that the shift operators play a crucial role [15].

We finally find that

$$1+\beta U(\beta)=\sum_n (4\beta^2)^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+2)}=\frac{2}{\pi}\int_{-2}^{2} d\lambda \frac{(1-\lambda^2/4)^{1/2}}{(1+\beta\lambda)} \tag{34}$$

There is a transition at $\beta=1/2$, which is characterised a singularity of the specific heat of the form $(\beta_c-\beta)^{-1/2}$. In other words the critical exponent α is equal to $1/2$.

Equation (34) gives the result for spin glasses in the Gaussian approximation. Starting from it one can obtain the more familiar results for the Ising spin glass or for the spherical spin glass.

3.3 Josephson junctions in Magnetic Field

In this case we need at first to compute the function

$$G_n(B)=\lim_{D\to\infty}(2D)^{-n}\mathcal{N}(2n)\langle W\rangle_n. \tag{35}$$

We will follow the strategy of first dividing the circuits into classes corresponding to different words of $2n$ letters (as in the previous case) and to evaluate the contribution of each class.

Let us start by computing $G_2(B)$ (it is trivial that $G_1(B)=1$). The backtracking circuits which correspond to the *planar* diagrams, (the corresponding words are $aabb$ and $abba$) give a contribution 1 each. More generally we can define the area of a circuit as the minimal area of a surface of lattice plaquettes which have that circuit as boundary. Backtracking circuits can be characterised as area zero circuits.

For large D the word $abab$ corresponds to $(2D)^2$ circuits with area 1. For half of them the signed area (defined in eq. 10) $S(C)$ is equal to 1, for the other half is equal to -1. If we recall that $W(C)=\exp(i\Phi(C))$, the contribution of these circuits average to $\cos(B)$. We finally find

$$G_4(B)=2+q, \tag{36}$$

where

$$q=\cos(B). \tag{37}$$

Generally speaking each different word of length $2n$ is associated to $(2D)^n$ circuits having the same area. The signed area of these circuits having the same area (A) is different. In a large number of dimensions (in the generic case where all the independent steps are done in different directions) the projected signed areas $S_{\mu,\nu}$ take only the values 0 or ± 1 and

$$\sum |S_{\mu,\nu}|=A. \tag{38}$$

If we average over all the possible orientations of the lattice the contribution coming from the circuits having the same word, we find that the average value of $\langle W(C)\rangle$ depends only on A and it is given by

$$\langle W(C)\rangle_A=\left(\frac{\exp(iB)+\exp(-iB)}{2}\right)^A=q^A. \tag{39}$$

We finally find that

$$G_n(B) = \sum_w q^{A(w)}, \tag{40}$$

where the sum is taken over all words of $2n$ letters and $A(w)$ is the area associated to each of these words.

We now show that the area of a of the circuit is exactly equal to the number of intersections of the lines connecting equal letters in the corresponding diagram. We can decrease the area by an unity by interchanging two letters. For example

$$A(acdefbacdefb) = A(acdefabcdefb) + 1. \tag{41}$$

Indeed the area of the projection on the $a - b$ plane goes from 1 to 0 and the projected area on the other planes is the same in the two circuits corresponding to the two words. The same braiding operation decreases the number of intersections by 1. By subsequent operations of the previous kind we can arrive to the zero intersections case (planar diagrams) by decreasing each time both the area and the projection by an unity. We have already remarked the relation between the number of planar diagrams and the coefficient of the high temperature expansion for spin glasses ($G_n = G_n(0)$).

We have thus transformed the problem of computing the high temperature expansion into a combinatorial problem, although not very easy, which generalise the computation of planar diagrams. The solution of this problem will be presented in the next section.

4 The q-deformed harmonic oscillator plays a role

We have reduced the problem of evaluating the high temperature expansion for the Gaussian model in presence of a magnetic field to the computation of the number of words of $2n$ letters, two by two equal, such that the number of intersections in the corresponding diagram is equal to a given number.

We claim that

$$G_n(B) = \sum_w q^{A(w)} = \langle 0|X^{2n}|0\rangle, \tag{42}$$

where

$$X = \mathcal{R}_q + \mathcal{L}_q, \tag{43}$$

and the operators $\mathcal{L}$ and $\mathcal{R}$ satisfy the commutation relation of a q-deformed harmonic oscillator:

$$\mathcal{L}_q\mathcal{R}_q - q\mathcal{R}_q\mathcal{L}_q = 1. \tag{44}$$

Therefore $\mathcal{L}_q$ may be identified with the distruction operator and R_q with the creation operator for a q-deformed harmonic oscillator. For $q = 1$ we recover the ferromagnetic case, for $q = -1$ the fully frustrated case and for $q = 0$ the spin glass case.

These operators may be represented as:

$$\begin{aligned} \mathcal{R}_q|m\rangle &= [m]_q^{1/2}|m+1\rangle, \\ \mathcal{L}_q|m\rangle &= [m-1]_q^{1/2}|m-1\rangle, \end{aligned} \tag{45}$$

where

$$[m]_q = (1 - q^{m+1})/(1-q), \tag{46}$$

and m ranges in the interval $[0-\infty]$. In the limit $q \to 1$ we obtain the usual Bosonic oscillator and we recover the usual formulae.

It is a simple matter of computation to verify that eq.(42) gives

$$\begin{aligned} G_1(B) = 1,\ G_1(B) = 2+q,\ G_3(B) = 5+6q+3q^2+q^3, \\ G_4(B) = 14+28q+28q^2+20q^3+10q^4+4q^5+q^6. \end{aligned} \tag{47}$$

These results coincide with the output of an explicit enumeration of the diagrams.

We have not been able of finding a neat proof of eq.(42). However we have checked its validity in many special cases (large q, small q, $q=1$, $q=0$ and $q=-1$) and we are convinced of its validity.

Intuitively eq.(42) tells us that when we use the *Wick* theorem for q-deformed harmonic oscillators, we must bring together the different terms we contract and for each term we get a factor q to the power of the number of object we have to cross.

If we use this result, we finally find the quite simple formula:

$$1+\beta U(\beta) = \langle 0|\frac{1}{1+\beta X}|0\rangle_q, \tag{48}$$

which gives a remarkable connection among the high temperature behaviour of the Gaussian model and the q deformed harmonic oscillator.

In this way we have reduced the combinatorial problem of computing the high temperature expansion to an algebraic problem.

5 Near the critical transition

The problem now is reduced to the computation of the spectrum of the operator X of the q-deformed harmonic oscillator. The computation is apparently non trivial. We are however interested to the computation of the spectral density near the largest eigenvalues.

A simple case is $q=1$, where the operator X_q is not bounded and the high temperature expansion is divergent. In this case X has a continuum spectrum and the highest eigenvalues of X are concentrated in the large m region. Let us assume that this feature is valid for q inside the interval $[-1,1]$. One finds that

$$\mathcal{L}_q \sim (1-q)^{-1/2}\mathcal{L},\ \mathcal{R} \sim (1-q)^{-1/2}\mathcal{R}. \tag{49}$$

when the operator is applied to a state $|m\rangle$ in the region of large m. ($\mathcal{L}$ and $\mathcal{R}$ are the two shift operators for $q=0$ which are used in the planar case).

The difference among $\mathcal{L}_q$ and $(1-q)^{-1/2}\mathcal{L}$ can be seen only when the two operators act on a state of low m. It is very reasonable to assume that the spectral radius and the spectral density near the maximum eigenvalues is the same in the two case. We have verified numerically that this conjecture is consistent (at least for q not too close to 1) by estimating the spectral density of X_q in subspaces of various size ($m < M$, with M up to a 300).

We find therefore that the critical temperature is given by

$$\beta_c = \frac{(1-q)^{1/2}}{2}, \tag{50}$$

which is the inverse of the spectral value of X, i.e.

$$|X|^2 = \frac{4}{(1-q)}. \tag{51}$$

The behaviour of the spectral density near the edge is the same as for the random matrix model, i.e. in spin glass. In this way we find the same critical exponents as in spin glasses in the Gaussian approximation.

A possible physical interpretation is the following. In computing the internal energy one has to sum over all the closed circuits. Circuits with large physical area average to zero and only fattened backtracking circuits survive. The situation is very similar to spin glasses, where only backtracking circuits contribute, the only effect being a renormalization of the temperature[5]

6 The issue of exchanging limits

A very serious problems in assessing the relevance of these results is related to the exchange of the limits $D \to \infty$ and $\beta \to \beta_c$. If we exchange the limits we become blind to any singularity whose strength vanishes in the limit $D \to \infty$. Sometimes this exchange is quite justified, sometimes it leads to disaster [16],[17].

The cases $q = 1$ and $q = -1$ are particularly instructive. The case $q = 1$ has been already discussed. The case $q = -1$ is quite interesting. We notice the following facts.

- The spectrum of the lattice Laplacian for the fully frustated model is well known [17]. A simple way to compute it consists in using the relation among the Gaussian fully frustrated model and the naive Wilson Fermions on the lattice [18]. Indeed let us start from the Hamiltonian of the naive Wilson Fermions

  $$H = \sum_i (\sum_\mu (\beta(\overline{\psi}(i+\hat{\mu}) - \overline{\psi}(i-\hat{\mu}))\gamma_\mu \psi(i)) + \overline{\psi}(i)\psi(i)), \tag{52}$$

  where $\hat{\mu}$ is the versor in μ direction, the γ_μ are the appropriate Dirac gamma matrices in D dimensions (which satisfies the usual algebra) and the ψ are the spinors on which these matrices act. For even D the gamma matrices may be taken to have dimension $2^D/2$. In order to simplify the notation we have not indicated the spinorial indices. If we introduce the field

  $$\phi(i) = \prod_{\mu=1,D} \gamma_\mu^{i_\mu} \psi(i), \tag{53}$$

  it is well known fact that the lattice Dirac operator reduces to the Laplacian of a fully frustrated model.

- The previous remark implies that for $q = -1$ one has in the Gaussian approximation (with the appropriate rescaling of β):

  $$1 + \beta U(\beta) = \int_B d^D k \frac{1}{(1 - 2\beta^2 \sum_{\nu=1,D} \sin^2(k_\nu)/D)}, \tag{54}$$

  for all even values of the dimensions.

[5]The previous results imply that when n and m goes both to infinity at fixed ratio one finds $I_n(m) = I_n(0)\frac{(n+m)!}{n!m!} f(m/n)$. It is quite possible that this simple result has a direct proof.

- If we send D to infinity we find that

$$1+\beta U(\beta)=\frac{1}{1-\beta^2}, \tag{55}$$

in perfect agreement with the direct computation. (In this case the creation and annihilation operators act on a two dimensional Fermionic space.)

- In any finite dimensions [17] the closest singularity to the origin of the function $U(\beta)$ is located at $\beta^2 = 1/2$, which corresponds to the integration point where all the momenta are at the boundary of the Brillouin zone (i.e. $(k_\mu) = \pm\pi/2$).

- In infinite dimensions the function $\beta_c(q)$ is discontinuous at $q = -1$. Indeed

$$\lim_{q\to -1} \beta_c^2(q) = 1/2 \neq \beta_c^2(-1) = 1 \tag{56}$$

As already found in [17], at $q = -1$ the limit $D \to \infty$ of β_c is smaller by a factor 2 of the value of β_c obtained from the high temperature expansion computed directly ad $D = \infty$. However this difficulty seems to be confined at $q = -1$. If we first take the limit $D \to \infty$ at $q \neq 1$, we recover the correct critical point for the $q = 1$ case.

In other words, if we first compute the critical temperature at $D = \infty$ for $q \neq -1$, we obtain the correct value of the critical temperature at $q = -1$, while we would get the wrong results if we perform the limit $D \to \infty$ directly at $q = -1$. By consistency we find that the prefactor in front of the nearest discontinuity vanishes when $q \to -1$ so that for $q = -1$ this singularity disappears.

It seems that we are free to conjecture that (apart from two well understood problems at $q = -1$ [17] and $q = 1$ [10]) the correct value of the critical temperature is obtained when we send firstly D to infinity. A numerical verification of the validity of this conjecture may be attempted for $q = 0$ or $\pm\frac{1}{2}$, where β_c can be computed by diagonalizing matrices of size 2^D or 3^D respectively.

7 Open problems

Let us suppose that the difficulties discussed in the previous section are not serious. We still face the problem of presenting a full computation of the high temperature expansion in the XY model. We must include high order terms which come from the fact that the distribution of the spins is not Gaussian. In the case of spin glasses these corrections are relevant; however they are identical in the Ising, XY and spherical model. In this last case they can be computed by tuning the coefficient of the quadratic term in such way that the spherical constraint is satisfied.

We have not checked that this happens also in our case, but it seems rather plausible. If this argument is correct, the knowledge of the Gaussian propagator is sufficient to reconstruct the high temperature expansion.

What happens in a finite number of dimensions is not clear. The first step is to verify if the equality of the two model survives in perturbation theory. Also if this check is satisfied one should be very careful because of non perturbative effects. It seems to me rather likely, but I do not have solid arguments in this direction, that for rational B the critical theory should behave differently from spin glasses, and the only hope for having a spin glass like behaviour is for generic irrational B. It would be vry interesting to connect this approach with the results obtained in two dimensions, where quantum groups have been used to compute the spectrum[19].

The possibility of having a spin glass behaviour for this non random system [20] is fascinating and deserves more careful investigations.

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9 Appendix

In this short appendix I report on some numerical findings that I have obtained on the behaviour of the spectral radius of X as function of θ for $q = \exp(i2\pi\theta)$. In this case I find a function which is discontinuous at all the rational points, but the discontinuity vanishes when the rational point becomes irrational.

If we apply the previous formulae we find that the spectral radius of X^2 should be

$$\frac{4}{|1-q|} = \left(\frac{4}{sin^2(\pi\theta)}\right)^{1/2} \tag{57}$$

The argument breaks down for rational θ. Indeed if $\theta = r/s$, with both r and s integer (r and s are the smallest integer for which have this property) X reduces to a finite dimensional operator of size s. In this case the previous formula is not correct. However in the limit where s goes to infinity it seems to become correct again. This can be seen by considering the function $R(\theta)$, defined as

$$|X|(\theta)^4 = \frac{4}{sin^2(\pi\theta)}(1 - \frac{\pi^2}{2s^2}) + R(\theta) \tag{58}$$

The function $R(\theta)$ is the difference among the analytic continuation of the value of the spectral radius from $|q|$ less than 1 and the actual spectral radius (apart the presence of a multiplicative factor which goes to zero as s^{-2} when $s \to \infty$ at fixed θ).

I have computed the function $R(\theta)$ for all rational with $s \leq 21$ (70 cases) and I have found that goes to zero fast with s (quite likely as s^{-2}). It seems likely that the function $R(\theta)$ is discontinuous at rational points, but the value of the discontinuity goes to zero when the rational becomes irrational (i.e. when $s \to \infty$).

Unfortunately I am not aware of a physical interesting model in which the properties of X for complex q enter. This appendix should be considered as a curiosity.

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