On the Replica Approach to Spin Glass
Talk given at Fifth International Workshop on Disordered Systems, Andalo, February 1994

Giorgio Parisi
Dipartimento di Fisica
Piazzale Aldo Moro, Roma 00187

December 2, 1994

Abstract

In this talk I will review the approach to spin glasses based on the spontaneously broken replica symmetry. I will concentrate my attention mostly on more general ideas, skipping technical details and stressing the characteristic predictions of this approach. After the introduction of the replica method, the predicted structure of states is investigated in details, paying a particular attention to the local overlaps and to the structure of the clusters. I will finally study the behaviour of the system near the lower critical dimension and I will show that the technique of coupling real replicas is able to give relevant information.

1 Introduction

The aim of this talk is to present some considerations on spin glasses in the framework of the broken replica approach.

There are many characteristic predictions of the replica approach that are in very good agreement with the experiment. The most interesting prediction (in spectacular agreement with the experiments) is the following: the field cooled susceptibility in the low temperature glassy region is independent from the temperature, on the contrary the alternate susceptibility at fixed frequency goes to zero with the temperature.

However we would like to obtain more detailed predictions from the theory and to compare them with the very interesting experiments that have been done. Generally speaking, if we consider a realistic finite dimensional spin glass, we can firstly study it in the mean field approximation and later on to compute the corrections to it. While in many cases the fist step has been done, especially for the static, the second one has not yet completed, so that many results are still conjectural.

In this note we will consider mainly models with the Hamiltonian of this form [1]:

$$H = -\sum_{i,k} J_{i,k}\sigma_i\sigma_k - \sum_i h\sigma_i. \tag{1}$$

In the mean field approach [2, 3], which should be valid in the limit of infinite number of neighbours [1], the solution of a spin glass model is given by the broken replica approach with the usual ultrametric hypothesis.

In this note we shall firstly recall the results obtained in the mean field approach and we shall later discuss how these results may extended to the short range case. In the second section we present a general introduction to the replica method and to the connection between the breaking of the replica symmetry and the existence of many equilibrium states. In the third section we study the structure of the equilibrium states in greater details. In section four we investigate the properties of the local overlaps, which give very crucial information on the properties of the system. In section five we introduce the concept of cluster of spins and we study its properties. In section six we present some simple predictions on the behaviour of the system near the lower critical dimension, which is not very far from three. In section seven we show how the technique of coupling real replicas may be used to characterise the behaviour of the system. Finally in the last section we present our conclusions.

2 The General Setting

In mean field approach there is an infinite number of stable states, which are labelled by an index α. They are characterised by a weight $w_\alpha = \exp(-\beta f_\alpha)$, f_α being (a part from

[1] More generally one neglects the existence of closed paths of finite length on the lattice, which implies that $\sum_{k,m} J_{i,k}J_{k,m}J_{m,i}$ is much smaller than $(\sum_k J_{i,k}^2)^{3/2}$.

an addictive constant) the total free energy of the state α. We will denote by $<>_\alpha$ the expectation in the state α. The weights are normalised by the condition $\sum_\alpha w_\alpha = 1$.

It is usual to introduce their overlap and the energy overlap (in zero magnetic field:

$$
\begin{aligned}
q_{\alpha,\gamma} &= \sum_i < \sigma_i >_\alpha < \sigma_i >_\alpha \ , \\
q^e_{\alpha,\gamma} &= \sum_{i,k} J^2_{i,k} < \sigma_i \sigma_k >_\alpha \ < \sigma_i \sigma_k >_\gamma \ \ .
\end{aligned}
\tag{2}
$$

For each given sample one can define the distribution probability of the overlaps

$$
\begin{aligned}
P(q) &= \sum_{\alpha,\gamma} w_a w_g q_{\alpha,\gamma}, \\
P^e(q^e) &= \sum_{\alpha.\gamma} w_a w_g q^e_{\alpha,\gamma}.
\end{aligned}
\tag{3}
$$

If replica symmetry is broken, the function $P(q)$ is non trivial.

At low temperatures the function $P(q)$ becomes concentrated at $q = q_M$. For $q \neq q_M$ the function $P(q)$ goes to zero at zero temperature and it proportional to T. Indeed two different states can contribute to the statistical sum only if they differ in energy of a quantity proportional to T; if their free energies are randomly distributed, this event happens with a probability which is linear in T.

There are two possible forms of the function $P(q)$ which correspond to quite different situations. In the first case the function $P(q)$ is the sum of two delta functions (i.e. $P(q) = w_m\delta(q - q_m) + w_M\delta(q - q_M)$, while in the second case the function $P(q)$ contains a smooth region between the two deltas. For the moment we will restrict the discussion to only this last second case, in which all possible values of q among q_m and q_M are allowed.

The predictions I will present are obtained in the mean field approximation. The main physical feature is the existence of different equilibrium states, which are not related locally by any symmetry operation, which have the same free energy density [2]. The existence of states with free energy difference of order 1 is a prediction which is valid in the mean field theory and could disappear in more realistic short range model (although there is no evidence for such an effect).

Moreover no experiment or computer simulation on very large size system may be sensitive to these difference, so that it interesting to find out methods which test the structure of replica symmetry breaking, without to be sensitive to these details. This point will be discussed in the next sections.

3 The Microstructure of the States

In the mean field approach one finds that there is a strict correlation between the different kinds of overlaps one can define. In particular in the case of the energy overlap one finds

[2]These states have remarkable properties, the most well known being ultrametricity [4, 5].

that

$$q^e = Aq^2 + B. \tag{4}$$

Consequently the probability distribution of the energy is given by

$$P^e(q^e) = \left(\frac{q^e - B}{A}\right)^{-1/2} P\left(\frac{(q^e - B)}{A}\right). \tag{5}$$

In other words different equilibrium states do not differ only by the reverse of large droplets of spins [10]. If two states differ only by the reverse of large droplets of spins, the energy overlap would be the same as that of a state with itself, apart from the contributions of the interface among different clusters [9].

In the picture which emerge from replica symmetry breaking there are clusters of spins which are strongly coupled. These clusters have a surface which is proportional to the volume so that the change in energy overlap, which is proportional to the surface, is comparable to the change in overlap, which is proportional to the volume. Of course we could call a fractal object a droplet, but this would be a linguistic abuse.

In principle we can get information on the form of the clusters from the analysis of the correlations fucntions. The situation is rather complex [3] and we shall only stress some features.

In order to define the correlation functions it may be useful to consider two real replicas of the same system: the spins will be denoted σ and τ respectively. One can introduce the overlap $q_i = \sigma_i \tau_i$. This two replicas system can be characterised by the value of $q = 1/V \sum_i q_i$. Different values of q in the interval $q_m - q_M$ are allowed. One can consider the connected correlation function restricted on configurations with a given overlap q:

$$G_q(x) = \frac{1}{V} \sum_i (< q_{i+x} q_i >_q - q^2) \equiv \frac{1}{V} \sum_i (< q_{i+x} >_q < q_i >_q - q^2). \tag{6}$$

The correlation function decreases in the mean field approximation in dimensions d as [7]

$$x^{-d+D}, \tag{7}$$

where $D = 3$, if $0 < q < q_M$, and $D = 4$, if $q = 0$ (this is possible only if $q_m = 0$, which happens in zero magnetic field).

These results are supposed to be valid at all orders in the perturbative expansion above 6 dimensions. The exponent D may change for less that 6 dimensions (D at $q = 0$ does actually change when $d < 6$ [8]).

Numerical simulations in 4 dimensions [12, 13] strongly suggest the correctness of the broken replica picture and no anomaly is observed.

[3] See for example the description in the very interesting paper [4]

4 A Simple Model for the Clusters

In order to interpret the previous results geometrically, we consider a simple model of spin glasses in which we find explicitly the existence of infinite number of equilibrium states [14, 15]. The model is described by the usual Hamiltonian, with an unusual distribution of the couplings: if we order the coupling in decreasing strength we must have that

$$|J_a| > \sum_{b>a} |J_b|. \tag{8}$$

This can be realised by setting $J_{i,k} = (R_{i,k})^{g(L)}$, where $R_{i,k}$ is a random number uniformly distributed in the interval $[-1, 1]$ and $g(L)$ is an integer function (which takes odd values) which diverges sufficiently fast when the size L of the box goes to infinity.

We will consider this model at zero temperature. If we consider the system in a box of size L with assigned boundary conditions, there will be clusters of rigidly connected spins which are controlled by the spins at the boundary. If we send the volume to infinity, we find that the ground state depends on the boundary conditions.

The properties of the model may be found by mapping it on invasion percolation. In dimensions $d > 8$ one finds that these clusters of rigidly connected spinshave Hausdorff dimension 4 and the system is filled by an infinite number of them. If we consider two systems with random (and different) boundary conditions, spins in different clusters will be random oriented so that the global overlap will be zero.

The correlation function is given by the probability that two spins at distance x belong to the same cluster. This probability goes to zero as x^{-d+D_c}, where D_c is the Hausdorff dimension of the cluster. If we stick to this interpretation, one finds that the Hausdorff dimension of the clusters obtained from mean field theory is 4, which is the same value as in this simple model.

In the real world things are more complicated. Clusters are not really independent, they interact one with the othe. This phenomenon may explain the shift of D from 4 to 3 when we go to non zero q. Lot of theoretical work is needed to present a full understanding of the properties of the clusters of strongly connected spins in the framework of the broken replica approach.

5 Local Overlaps

Generally speaking, the function $P(q)$ alone is a rather delicate object because its computation involves the differences among total free energies which are of order 1. We could consider situations in which the free energy differences are of order L^λ with $\lambda < d$. In this a way the free energy differences diverge, but the free energy densities are the same. In this situation most of the conclusions based on broken replica symmetry would be valid, but the function $P(q)$ would be trivial.

It should be clear that a non trivial form of the function $P(q)$ at zero magnetic field is not sufficient to rule out the droplet model. Indeed in the framework of the droplet model there could be a few compact independent droplets which can be flipped independently. Compact droplets are characterised by the following conditions: the radius R goes to infinity the volume af a compact droplet goes as R^d and the surface as $R^{d-1+\omega}$ with $\omega < 1$. If $\omega > 0$ the droplets have a fractal surface. If $\omega = 1$ they are no more compact and in this limit we recover the broken replica approach.

In the droplet model the function $P(q)$ can be non trivial, but the function $P^e(q^e)$ must go to a delta function acquiring a width of order $L^{-\omega}$.

The differences among the broken replica and droplet model become more sharp if consider the probability distribution $P_R(q_R|q)$ of the local overlap q_R defined as

$$q_R = |\frac{1}{R^d}\sum_k q_k|, \tag{9}$$

where the sum is done in a box of side R centred around and arbitrary point i[4]. The probability distribution may be obtained for a given sample by changing the point i, keeping the constraint of total overlap q.

The quantity q_R is invariant under a reversal of spins in a cluster that contains the whole region of size R. In the droplet model the function $P_R(q_R|q)$, should not depend on q in the infinite volume limit: indeed when we reverse a large droplet, the overlap q_R changes only for those points near the boundary of the droplet, so that the dependence of $P_R(q_R|q)$ on q should vanish as $(\frac{R}{L})^{-1+\omega}$ when the size of system L goes to infinity.

6 The Lower Critical Dimension

In testing spin glass theories in finite dimensions a crucial issue is the validity of the mean field picture. Now it seems that the mean field theory gives the correct exponents in dimensions (d) greater than 6, while some corrections are expected in dimensions less than 6. Unfortunately the computation of the corrections to mean field theory is rather complex and the one loop contribution has not been fully computed so that no precise conclusions are available.

The most critical issue is if the behaviour of the propagator at $q \neq 0$ (i.e. $1/k^3$ in momentum space) remains unchanged. This behaviour arises as a consequence of the breaking of replica symmetry in a continuos way. The corresponding Ward identities has not been carefully spelled out, so that it is not evident if such a relation is renormalized (the $1/k^4$ behaviour at $q = 0$ apparently changes in less that 6 dimensions).

If this behaviour would be not renormalized, it would be an easy guess to suggest that the lower critical dimension of the system is 3 in the same way as the lower critical dimension for breaking of continuos symmetries is 2. A direct computation of the free energy of an interface among states at different values of q gives rather different results, i.e. a lower critical dimension of 2.5, which is by *accident* the same obtained in the framework of the Migdal Kadanoff approximation [18].

[4] I am grateful to J.P. Bouchaud for stressing this point to me.

The discrepancy among the two different results has an unclear origin, which we hope to be able to understand in the next future, when the evaluation of the loop corrections to various different quantities will be available. In any case 3 it is not very far from being the lower critical dimension so that it is reasonable to investigate what should happen near the critical dimension and to compare with the experimental (or numerical results).

The renormalization group scheme for the behaviour at the lower critical dimension has not yet developed. Let us try do some educated guesses. We suppose that for a system with continuos distribution of the couplings J [5], there is only one relevant quantity that describe the behaviour of the system at scale L and it satisfies a renormalization group equation

$$\frac{dT}{dt} = \beta(T(t)). \tag{10}$$

The quantity $T(t)$ can be identified with the effective temperature at scale $L = \ln(t)$ and the function $\beta(T)$ should behave as a power of T as small T.

One readily finds that if

$$\beta(T) \propto T^{\lambda}, \tag{11}$$

the correlation function diverges as

$$\xi \propto \exp(a/T^{(\lambda-1)}). \tag{12}$$

In the Migdal Kadanoff one finds that approximation $\lambda = 3$.

In the same way we expect that the field q has an effective dimension ($\psi(T)$) which is proportional to T, and therefore the value of q, observed on a scale of size L, should decrease (roughly speaking) as

$$q(L) \propto L^{-\psi(T(\ln(L))}. \tag{13}$$

In a given range of scales, which may be quite wide at low temperatures if λ is large, $T(t)$ may be approximated with a constant. If this happens in the low temperature region, where the correlation length ξ is much greater that the side of the system, the correlation function should decrease with temperature dependent power law. The range of validity of these approximate power law depends on the form of the function $\beta(T)$. In the two dimension ferromagnetic $x - y$ model the β function is identically zero so that a power behaviour is exact.

Under the approximation of a constant $T(t)$, the function $P(q)$ in a box of size L is a function of $qL^{-\psi(T)}$, the space correlation function of two overlaps ($\langle q(x)q(0)\rangle$) decays as $x^{-2\psi(T)}$ and the time correlation function of two spins ($\langle\sigma(0,t)\sigma(0,0)\rangle$ decays as $t^{\psi(T)/z(T)}$, where $z(T)$ is a temperature dependent dynamic exponent. Some numerical indications of such a behaviour in three dimensions are already been obtained and further work is needed to distinguish the case of a system at the lower critical dimension or slightly above it [19, 20].

[5] Anomalous behaviour may be present if the couplings are discrete when $T \to 0$, due to the degeneracy of the ground state

These predictions can be tested relatively easily within a dynamic study. Let us consider the dynamics of two replicas starting by two different boundary conditions. The two replicas will have zero overlap for times not very large compared with the volume. In this way the dynamics automatically selects the state $q = 0$. In the same way, by observing the same single system at two large times t_1 and t_2, such that $\frac{|t_1-t_2|}{t1+t_2|} << 1$, the two configurations at different times should behaviour as two generic configurations a $q = q_M$ [21].

Large scale simulations have been recently done on a three dimensional system and high precision data have been obtained in the high temperature region. One finds out that the data are compatible both with a phase transition at a non zero temperature and with a no transition scenario , altough this last scenario seems more consistent. [23].

There are some disturbing feature in the framework of the transition scenario. Indeed one could look (in a two replica system) to the value of the Binder cumulant

$$g = \frac{K-3}{2}; \qquad K = \frac{< q^4 >}{< q^2 >^2}, \tag{14}$$

where the average is also done over the different realisation of the systems.

The Binder cumulant can be computed for different sizes and the value where the different curves intersect, characterises the critical temperature. The value of g at the critical point should be universal. As far as the curves for the Binder cumulant collapse at a given temperature (as predicted from replica symmetry breaking), it is difficult to establish the existence of a point in which the curves merges, because a small shift is enough to avoid the intersection.

There is good numerical evidence that the value of g at the apparent intersection point is strongly dependent on the choice of the probability distribution for the couplings J [23].

This dependence of g on the probability distribution of the couplings strongly suggest that 3 is the lower critical dimension or it is very near to it and the real critical temperature, if any, is far from the apparent point where different cumulants merge.

It should also be said that simulations (and experiments) at temperature smaller that the apparent critical temperature are well compatible with the possibility that $D = 3$ is the lower critical dimension.

Very large scale simulations with a dedicated computer [22] have shown a power law decay of the correlation of the same spin at different times: $C(t) \propto t^{-\delta}$. Real experimental data show a behaviour similar behaviour for the remanent magnetization as function of the waiting time. They are compatible with the law $M(t, t_W) = t^{-\delta} f(t/t_w)$. The presence of a non zero value of δ can be considered as an effect of having an anomalous dimension for the q field.

Other simulations for the static properties done in the low temperature region seem to be well compatible with the scenario where no transition is present and that q has a temperature dependent dimension.

7 Coupled Replicas

Some of the previously discussed points become clear if we consider the following Hamiltonian [9, 17]:

$$H = -\sum_{i,k} J_{i,k}(\sigma_i\sigma_k + \tau_i\tau_k) - h\sum_i(\sigma_i + \tau_i) - \epsilon\sum_i \sigma_i\tau_i. \tag{15}$$

The term proportional to ϵ has the tendency to align the two replicas in the same direction for positive ϵ and in the opposite direction for negative ϵ. Therefore we expect that

$$\begin{aligned}\lim_{\epsilon\to 0^+} q(\epsilon) = q_M \\ \lim_{\epsilon\to 0^-} q(\epsilon) = q_m\end{aligned} \tag{16}$$

An unexpected result of a detailed computation [17] show that in the replica approach

$$\begin{aligned}q(\epsilon) = q_M + A\epsilon^{1/2} \quad for \quad \epsilon > 0 \\ q(\epsilon) = q_m - B\epsilon^{1/2} \quad for \quad \epsilon > 0.\end{aligned} \tag{17}$$

In other words both the spin glass susceptibility and the $q-q$ correlation length diverge when $\epsilon \to 0$.

This prediction of the replica approach is very interesting because it can be well studied in computer simulations as far as the function $q(\epsilon)$ can be computed with a reasonable amount of time, at leas for not infinitesimal ϵ. The correlation function for q is finite for non zero ϵ. In other words the new term forces the two systems to be in the same state and also if the state is not the real equilibrium one its properties should no be crucial dependent on this fact.

The divergence of the correlation length when $\epsilon \to 0$ implies the possibility of changing larger and larger clusters of spins and the consequent divergence of the correlation time.

The naive replica predictions seems to be approximately well satisfied in 4 dimensions [12, 13], while they are far from being correct in the three dimensions [23]. In this case at zero magnetic field the preliminary data are compatible with a law of the type

$$q = \epsilon^{\lambda(T)}, \tag{18}$$

where the exponent $\lambda(T)$ is related to the exponent $\psi(T)$ of section (5) by

$$(\lambda + 1)\psi(T) = 1. \tag{19}$$

8 Conclusions

In the recent years there have been both theoretically and experimentally progresses in the study of spin glasses. For short range models the theoretical situation is still not completely clear because we do not known how to compute systematically the corrections to the mean field theory (or better we know how to do, but the computations are too complex).

Experimentally (from numerical simulations) it seems that the mean field picture is in very nice agreement with the results in four dimensions and in three dimensions for not too large size.

An open possibility is that the critical dimension is near to 3. In this case there would be a wide temperature range where the correlation length is very large, but finite. The predictions from replica symmetry breaking on the global order parameters should apply to systems of size smaller then the correlation length. All the data I know are consistent with this possibility, which should be better investigated numerically and theoretically. I hope that a strong progress will be done in the next future on this particular point.

9 Acknowledgements

It is pleasure for me to thank for very useful discussions and fruitful collaboration on these problems L. Cugliandolo, S. Franz, J. Kurchan, E. Marinari, F. Ritort and M. Virasoro.

References

[1] S.F.Edwards and P.W. Anderson, J. Phys. F5, 965 (1975).

[2] M. Mézard, G. Parisi and M.A. Virasoro, *Spin glass theory and beyond*, World Scientific (Singapore 1987).

[3] G. Parisi, *Field Theory, Disorder and Simulations*, World Scientific, (Singapore 1992).

[4] M. Mézard and M.A. Virasoro, J. Phys. 46, 1293 (1986).

[5] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse and M. Virasoro, J. Phys. 45, 843 (1984).

[6] R. Rammal, G. Toulouse and M.A. Virasoro, Rev. Mod. Phys. 58, 765 (1986).

[7] C. De Dominicis and I. Kondor, Europhys. Lett. 2, 617 (1986).

[8] C. De Dominicis and I. Kondor, J. Physique Lett. 46, L1037 (1985).

[9] S. Caracciolo, G. Parisi, S. Paternello and N. Sourlas, Europhys. Lett. 11, 783 (1990).

[10] D.S. Fisher and D. Huse, Phys. Rev. Lett. 83, 83 (1983).

[11] D. Badoni, J.C. Ciria, G. Parisi, J. Pech, F Ritort and J.J Ruiz Lorenzo Europhys. Lett. 21 (1993) 493.

[12] G. Parisi and F Ritort, J. Phys. A 26, (1993) 6711.

[13] J. C. Ciria, G. Parisi and F Ritort, J. Phys. A 26 (1993) 6730.

[14] M. Cieplak, A. Maritan and J.R Banavar, Phys. Rev. Lett. **72** 2330 (1994)

[15] E. Neuman and F. Stein, preprint (1994)

[16] L.F. Cugliandolo and J. Kurchan. Phys. Rev. Lett. 71, 1 (1993).

[17] S. Franz, G. Parisi, M.V. Virasoro, J. Phys. A 18, 72 (1992).

[18] S. Franz, G. Parisi, M.V. Virasoro, Interfaces and Lower Critical Dimension in a Spin Glass Model, Nordita Prerprint 94/21 S.

[19] E. Marinari, G. Parisi, F. Ritort, J. Phys. A 27 (1994) 2687.

[20] J J. Ruiz-Lorenzo, E. Marinari, G. Parisi, F. Ritort, (in preparation).

[21] L. F. Cugliandolo, J. Kurchan and G. Parisi, *Off equilibrium dynamics and aging in unfrustrated systems*, cond-mat preprint (1994).

[22] A. T. Ogielsky, Phys. Rev. B 11, 7383, (1985).

[23] E. Marinari, G. Parisi, F. Ritort, (in preparation).