Statistical properties of Random Matrices and the replica method

Giorgio Parisi
Dipartimento di Fisica, Università La Sapienza *and INFN Sezione di Roma I*
Piazzale Aldo Moro, Rome 00185

Abstract

I present here some results on the statistical behaviour of large random matrices in an ensemble where the probability distribution is not a function of the eigenvalues only. The perturbative expansion can be cast in a closed form and the limits of validity of this expansion are carefully analyzed. A comparison is done with a similar model with quenched disorder, where the solution can be found by using the replica method. Finally I will apply these results to a model which should describe the liquid-glass transition in high dimensions.

1 Introduction

I cannot to work on random matrices without thinking to to Claude Itzykson. I started to study this subject with him together with Eduouard Brézin and Jean-Bernard Zuber [1]. I enjoyed very much to work in this group. I had collaborated before with each of them separately, but it was the first time I was working with all of them together. It was a very interesting and pleasant experience. I feel still nostalgia for the long afternoons spent observing my friends doing long and difficult computations at the blackboard practically without doing mistakes.

After that work our roads partially separated. My friends remained interested in the field [2,3], while I started to be interested in spin glasses. Claude and Jean-Bernard started to study the case with two interacting matrices and produced after much struggle the wonderful paper on the unitary integrals [2]. At that time I had frequent discussions with Claude; he was explaining me the difficulties they met with their problem and I was telling him those I had with spin glasses. We tried to make some progresses together.

This friendly exchange was quite fruitful [a]. Let me quote two examples.

The interpretation of the breaking of replica symmetry in terms of many pure states was also triggered by Claude's observation that

$$\sum_b Q^s_{a,b} = \frac{\prod_{i=1,s} \sum_{k_i} \langle \prod_{i=1,s} \sigma_{k_i} \rangle^2}{N^s}. \tag{1}$$

A that time this relation was quite mysterious but it turned out to be a crucial step for later developments [6,7].

During one of my visits to Saclay I found a simple derivation of a result of Bessis [8] on orthogonal polynomial. I showed it to Claude, who was very interested and started immediately to derive some of the formulae needed to cast the result in a more compact way. I was too busy with spin glasses so I didn't write it. The results were finally incorporated in the paper on the unitary integrals [2] and fortunately in this way they were saved from the oblivion.

Two years ago, when with Marinari and Ritort we started to apply the replica method to non random systems [9], it was a real pleasure for me to read again the paper of Claude of Jean-Bernard [2], which contained crucial results for our aims. At that time I was thinking that one day I will come to Saclay to explain our results to Claude and that I will hear his comments and suggestions, but sadly that day will never come.

The problem I will study here is the natural extension of that work of fifteen years ago [1] and it can be formulated as follows [10]. We have an $N \times N$ symmetric matrix M and we associate to it the following Hamiltonian:

$$H(M) = N\mathrm{Tr}(h(N^{-1/2}M) + \frac{1}{N}\sum_{i,k} f(M_{i,k}). \tag{2}$$

When N goes to infinity, the Hamiltonian becomes a quantity of order N. We will consider the expectation values of intensive observables $A(M)$, which have a well defined limit when $N \to \infty$, e.g. they have the same form of the Hamiltonian apart form a factor N.

[a] I use this opportunity to acknowledge an other debt I have with Claude. I was strongly influenced by his old paper with Abarbanel on the eikonal [4]. I read that paper when I was starting to work in physics. I learned from it the power of the functional method and the relation among classes of diagrams and the solution of the Schroedinger equation in a random Gaussian field. That paper was quite present in my mind when ten years later we started to work on what we called *quenched* gauge theories [5].

We are interested in the computation of the equilibrium quantities

$$\langle A \rangle_N \equiv \frac{\int dM \exp(-\beta H(M)) A(M)}{\int dM \exp(-\beta H(M))} \tag{3}$$

in the limit where $N \to \infty$.

We can also define dynamical expectation values: we introduce the Langevin equation

$$\frac{dM_{i,k}}{dt} = -\frac{\partial H}{\partial M_{i,k}} + \eta(t)_{i,k}, \tag{4}$$

where η is a random Gaussian distributed noise, uncorrelated for different pairs of indices, such that

$$\overline{\eta(t_1)_{i,k}\eta(t_2)_{i,k}} = \frac{2}{\beta}\delta(t_1 - t_2). \tag{5}$$

Here the bar denotes the average over different realization of the noise.

The dynamical expectation values are defined as

$$\langle A \rangle_D \equiv \lim_{t\to\infty} \lim_{N\to\infty} \overline{A(M(t))}. \tag{6}$$

The order of the limits matters. Indeed a well known theorem states that

$$\langle A \rangle \equiv \lim_{N\to\infty} \lim_{t\to\infty} \overline{A(M(t))}. \tag{7}$$

If the two limits do not commute when A is the energy density, metastable states are present. While metastable states with strictly infinite mean life cannot be present in systems with a short range interaction, the study of infinite range models where metastable states do exist, may be quite useful for understanding real systems in which there are metastable states with finite, but extremely large, mean life.

There are many motivations for studying this problem.

The problem is interesting *per se*. It is a non-trivial generalization of the original problem of random matrices [1] with the extra difficulty that the probability distribution of the matrices is no more invariant under the rotation group $O(N)$. This invariance was a crucial tool in the previous investigations and it is not present here.

In these models at low temperature the averages in the dynamic approach may not coincide with those of the equilibrium approach. Systems for which the two limits (infinite volume and time respectively) cannot be exchanged have a strong interest [11] and their properties may be investigated using the replica method [12].

While there are no real doubts that in the dynamic approach the limit ($N \to \infty$) of intensive quantities is unique, it is possible (and in my opinion it is quite likely) that one can finds two (or more) different sequences N_r (i.e. $N_r \to \infty$ when $r \to \infty$) such that the limit of the equilibrium quantities at low temperatures along the sequence depends on sequence. In similar cases the low temperature behaviour of equilibrium properties becomes linked with the solution of arithmetic problems [9,13].

The techniques we will use to study the model are based on the the replica method. The possibility of applying the replica method to non random systems is very interesting, especially because we hope that this extension may be useful in studying real glass.

Finally there is a model which is very interesting from the physical point: N interacting particles constrained to move on a sphere in αN dimension [14]. This model is the simplest non trivial model for interacting particles which should have a liquid-glass transition and (hopefully) can be solved.

This paper (which is based on the results of [9,10,14]) is organized as follows: in section 2 we present a diagrammatical analysis of the perturbative expansion which can be resummed in a compact way. In section 3 we present a model where the Hamiltonian is random (quenched disorder). Its properties coincide with those of our original model in the high temperature phase; the model can be solved analytically using the replica method. In section 4 we discuss the relations of the random model with our original model and we present and disprove some conjectures. In section 5 we present a physically motivated model [14] of interacting particles which can be written in the form given by eq. (2). Finally in section 6 we present some tentative conclusions.

2 The perturbative expansion

In this section we consider the perturbative expansion for the following model

$$H(M) = \frac{1}{2}\mathrm{Tr}(M^2) + \frac{g_E \mathrm{Tr}(M^4)}{N} + g_L \sum_{1,k} M_{i,k}^4 . \tag{8}$$

This Hamiltonian corresponds to the choice

$$\begin{aligned} h(x) &= (\frac{1}{2} + C)x^2 + g_E x^4 , \\ f(x) &= -Cx^2 + g_L x^4 . \end{aligned} \tag{9}$$

The value of C is arbitrary, indeed

$$\mathrm{Tr}(M^2) = \sum_{1,k} M_{i,k}^2 . \tag{10}$$

The same term in the previous equation can be represented in both forms. In the rest of the paper we will use $C = 0$; in this section will make the choice $\beta = 1$.

Our aim is to investigate at the diagrammatical level if there are some simplifications when $N \to \infty$. Indeed in this limit where $f = 0$ (i.e. $g_L = 0$) only planar diagrams survive. Here the situation is more complicated.

A careful diagrammatical analysis shows the following[10] results.

Let us consider the contributions to a connected Green function and let us exclude diagrams containing self energy corrections. In this case there are no mixed terms and the result is the sum of two functions, depending only on g_E and g_L respectively.

If we consider the self energy corrections in the internal lines, mixed terms may appear. Indeed the two point function has only one possible form, irrespectively from the type of vertices which contribute to it (in other words, there is only one quadratical invariant).

If we denote by G the value of the propagator (I do not indicate the dependance on the indices), we find that any Green function (Γ) can be written as

$$\Gamma = \Gamma_E(g_E, G) + \Gamma_L(g_L, G). \tag{11}$$

The same conclusion is valid for the self energy diagrams, which can be written as

$$\Sigma = \Sigma_E(g_E, G) + \Sigma_L(g_L, G). \tag{12}$$

The propagator is thus given by

$$G = \frac{1}{1 + \Sigma_E(g_E, G) + \Sigma_L(g_L, G)} . \tag{13}$$

The solution of this equation gives the value of the propagator G and from it we can reconstruct the other properties of the model.

These diagrammatical findings can be summarized in the following elegant way which generalizes to matrices the representation found in[9] for vectors.

We introduce two $N \times N$ matrices B and z. The total Hamiltonian for these two matrices is

$$H(B, z) = \mathrm{Tr}\left(N h(BN^{-1/2}) + \frac{R_E}{2} B^2 \right) + \sum_{i,k} \left(f(z_{i,k}) + \frac{R_L}{2} z_{i,k}^2 \right) , \tag{14}$$

where

$$R_L = 1 + \Sigma_L(g_L, G), \qquad R_E = \Sigma_E(g_L, G) \tag{15}$$

The previous conditions become now

$$G = \langle B_{i,k}^2 \rangle = \langle z_{i,k}^2 \rangle = \frac{1}{R_L + R_E} . \tag{16}$$

The expectation value of the energy in the original model can be finally written as

$$E = \langle \mathrm{Tr}\left(N h(BN^{-1/2}) \right) + \sum_{i,k} f(z_{i,k}) \rangle \tag{17}$$

More generally

$$\langle \mathrm{Tr}\left(N d_E(M N^{-1/2})\right) + \sum_{i,k} d_L(M_{i,k})\rangle =$$
$$\langle \mathrm{Tr}\left(N d_E(B N^{-1/2})\right) + \sum_{i,k} d_L(z_{i,k})\rangle \tag{18}$$

Under this form the result is quite compact and it can be generalized to other interactions which have higher powers of M.

Both the integrations on B and z can be done explicitly: The B integration can be done using the standard techniques, because the integrand is now rotational invariant. The z integration factorizes in the product of N^2 independent integrals. The values of R_L and of R_E can be found by solving the two equations (16) in two unknowns. All computations can be done explicitly.

In the nutshell the final formula corresponds to an Hartree-Fock type approximation or, better, to two parallel spherical approximations: the local interaction (f) in the eigenvalue representation (B) has the only effects of modifying the quadratical term and viceversa. This formula generalizes well known results:

- If $f=0$, the probability distribution of a given matrix element of the matrix M is Gaussian.
- If $h=0$, the probability distribution of the eigenvalues is the same of the Gaussian model

However we have not been able (with reasons) to obtain this result in a compact way. We may speculate that the result is not valid beyond perturbation theory. Indeed, if such a general result would be correct, there should be a non-diagrammatical proof.

Generally speaking a perturbative result may break for two different reasons:

- There are non analytic terms, e.g. of the form $\exp(-C/(g_E g_L))$.
- There is a first order phase transition [b] at a non zero value of $g_E g_L$.

We shall see later that the second possibility is exactly what happens. However, before finding the limitations of this perturbative result, it is convenient to consider an other case in which the same result is obtained.

3 A Random equivalent model

It is interesting to investigate if there are other models for which the previous formulae are exact and can be proved in a compact way. The model can be found among those with random quenched disorder. We consider the following Hamiltonian:

$$H_O(z) \equiv H(z, B(z)) = N \mathrm{Tr}\left(h(B N^{-1/2})\right) + \sum_{i,k} f(z_{i,k}), \tag{19}$$

where z is $N \times N$ matrix, $B(z)$ is a short notation for

$$B_{i,k} = \sum_{j,l=1,N} O_{i,k;j,l} z_{j,l}, \tag{20}$$

and O is an $N^2 \times N^2$ orthogonal matrix, where each pair of indices (i.e. i,k and j,l) plays the role of one index.

The Hamiltonian depends on O via the constraint in eq. (20). When the matrix O is the identity, we recover the previous model.

Here we consider the case in which O is a random orthogonal symmetric matrix. Our aim is to compute the following quantity in the large N limit

$$-N\beta F(\beta) = \int d\mu(O) \ln\left(\int dz \exp(-\beta H_O(z))\right). \tag{21}$$

[b] In this context we say that a phase transition is of first order if no precursor signs are present, i.e. its presence cannot be predicted by finding susceptibilities which diverges when approaching the transition.

At this end it is convenient to consider the quantity

$$-nN\beta F^{(n)}=\ln\left((\int d\mu(O)\left(\int dz\exp(-\beta H_O(z)\right)^n\right). \tag{22}$$

It is trivial to check that

$$\lim_{n\to 0}F^{(n)}(\beta)=F(\beta). \tag{23}$$

The replica method consists in computing $F^{(n)}$ for integer n and eventually continuing analytically the result to $n=0$.

The first step consist in writing

$$\int dz\exp(-\beta H_O(z))=\int dzdBd\lambda$$
$$\exp\left(-\beta H(z,B)+i\sum_{i,k}\lambda_{i,k}(B_{i,k}-\sum_{j,l=1,N}O_{i,k;j,l}z_{j,l})\right). \tag{24}$$

Now, a symmetric orthogonal matrix can be written as

$$O=VDV^* \tag{25}$$

where D is a diagonal matrix (in the generic case about half of the diagonal elements are equal to 1, the other to -1) and V is a generic orthogonal matrix. Using the formulae of reference[2] the integral over V can be done (if n remains finite when $N\to\infty$). In this way the constraint is integrated over and we remain with two separate integrals[c].

After a long computation, one finds that $F^{(n)}$ is given by the stationary point of $F^{(n)}(R_L,R_E)$. Here the R's are $n\times n$ matrices and

$$\exp\left(-nNF^{(n)}(R_L,R_E)\right)=\int dzdB\exp\left(-\beta H_{eff}(z,B,R_L,R_E)\right),$$
$$\beta H_{eff}(z,B,R_L,R_E)=\sum_{a=1,n}\beta H(z^a,B^a)+ \tag{26}$$
$$\frac{1}{2}\sum_{a,b=1,n}\left(R_L^{a,b}\mathrm{Tr}(z^az^b)+\mathrm{Tr}\ln(R_L+R_E)+R_E^{a,b}\mathrm{Tr}(B^aB^b)\right)$$

In the perturbative region where at least one of the two functions f and h is small, the matrices R's are diagonal and we recover the formulae of the previous chapter by looking to the stationary point of the free energy.

At low temperatures many new phenomena may appear. One of the most interesting is replica symmetry breaking. In this situation the saddle point is no more symmetric under the action of the replica group. The computations in the broken replica region are technically difficult, but they seem to be feasible and work in this direction is in progress[16].

It is interesting to recall that the the original model is one of those we are considering here, i.e. it corresponds to $O=1$.

4 A bold conjecture

4.1 A naive conjecture

The simplest conjecture would be to assume that the original formulae are always valid. In order to test this conjecture, we could study the model in some limiting case. It is convenient to consider a very simple non trivial case, i.e. matrices M constrained to have matrix elements of modulus 1;

$$M_{i,k}=\pm 1 \tag{27}$$

[c]The Itzykson-Zuber formula applies in the case of integral over unitary matrices for all N. Here we only need the leading term when N goes to infinity and we can this apply their final expression to orthogonal matrices; in both case only planar diagrams survive. The two results differ only to a crucial factor 2, which can be checked at the first non trivial order.

This can be realized in the previous setting by using a function h, such that

$$\exp(-h(z)) = \delta(z^2 - 1) \tag{28}$$

We can now set $g_E = 1$ and write the partition function of the model as

$$\sum_{M_{i,k}=\pm 1} \exp(-\frac{\beta}{N}\mathrm{Tr}(M^4)). \tag{29}$$

According to the previous considerations, we have to study an auxiliary model with partition function

$$\int dM \exp\left(-R\mathrm{Tr}(M^2) - \frac{\beta}{N}\mathrm{Tr}(M^4)\right) \tag{30}$$

where R has been chosen in such a way that $\langle B^2_{i,k}\rangle = 1$. In other words the spherical and the Ising model coincide in the high temperature region.

The solution of this problem can be found directly using the formulae of [1] for small positive β and their extension to the case of non-connected support of the eigenvalue distribution for large β.

Is this solution correct?

Certainly not for negative β. As noticed by Zinn-Justin, in this case we recover some form of Z_2 lattice gauge theory. For negative β the unconfining solution (i.e. $M_{i,k} \approx 1$) gives an energy proportional to N^2. This is not a serious contradiction, as far also the corresponding model eq. (30) does not exist for negative β.

The most serious troubles appear when β becomes large and positive. Here we can compute the free energy of the model $F(\beta)$ and from it we can recover other thermodynamical quantities. It is possible to compute the entropy.

One finds (as expected in a spherical model) that the entropy density $S(\beta)$ behaves at large β as

$$S(\beta) \approx -\ln(\beta) + const \tag{31}$$

The entropy becomes negative at sufficiently high β. This is impossible; the spherical and the Ising model cannot coincide in the low temperature region. Consequently, the solution of this *soluble model* must be wrong, in the same way as in the Sherrington-Kirkpatrick model [6].

Something better must be devised. Numerical simulations of the model show a perfect agreement for not too large values of β. The deviations appears suddenly at a value of β at which the specific heat is nearly discontinuous, strongly suggesting the presence of a phase transition.

4.2 The simplest reasonable conjecture

A better conjecture, which does not go against the positivity of the entropy, consists in assuming that the model with $O = 1$ coincides with the one with random O also in the low temperature phase. It is quite evident that in the case of random O the entropy cannot be negative and this observation is enough to remove the entropy crisis.

This conjecture may be extended also to the dynamic averages, which can now be computed by using an appropriate formulation of the replica method [12].

In this context, the analytic continuation of the free energy from the high temperature region has the meaning of the *annealed* value of the free energy, i.e. $F^{(1)}(\beta)$.

In principle, we can solve the random model by using the analytic formulae of the previous section. This is rather hard from the technical point of view. Detailed computations are in slow progress.

A trivial bound,

$$E(\beta) \geq 1, \tag{32}$$

comes from the inequality

$$N\mathrm{Tr}(M^4) \geq (\mathrm{Tr}(M^2))^2. \tag{33}$$

The same inequality tells us that $E = 1$ only if $\mathrm{Tr}(M^2)$.

However, some information can be obtained in a simple way: the annealed free energy is a lower bound to the quenched free energy and that the entropy is non negative. In this way one finds that

$$E(\beta) \geq 1.06 \tag{34}$$

An approximation which is often quite good for these systems [15], consists in assuming that the entropy is given by

$$\max(S_a(\beta), 0), \tag{35}$$

where $S_a(\beta)$ is the annealed free energy. In the framework of this approximation the previous bound is exact.

4.3 A counterexample

Can we test if the previous conjecture consequently is valid for all large N? Alas, yes and the answer is negative, at least for some N.

Indeed, if $N = 2^k$ for some integer k, it is possible to find a matrix M whose elements are all equals to ± 1 and which is the square root of the identity. This proves that (at least for this sequence of N) the energy $E(\beta)$ is equal to 1 at low temperature. The same result may be proved for many other values of N, but it is unclear what happens for the generic N [17].

In a related model [9,13] it seems that quite different results are obtained depending on the arithmetic properties of N and that there is a way of taking the infinite N limit (i.e. a sequence of appropriate values of N going to infinity) such that the model becomes equal to the random model.

The possibility of more than one thermodynamic limit for the same model (depending on the sequence) is quite interesting, but it is clear that there exist at least one way in which the conjecture we have proposed is not valid.

4.4 A more refined conjecture

The physical origin of the difference in the equilibrium properties of the random and of the original model is related to the existence of "small" region in the phase space of low energy. On the contrary, it is quite likely that the dynamical averages do coincide with those of the random model. These configurations ($C_{i,k}$) have such a low energy (they correspond in some sense to a crystal) that they cannot be reached in the natural evolution of a system arriving from the high temperature region due to the presence of infinitely high barriers.

At low temperature the equilibrium properties of this model have not been carefully studied. Here we follow the natural choice of generalizing the conjectures which have been done in a similar context on a simpler model which can be studied in a more effective way [9,13].

We suppose that the dynamic expectations values are the same as those in the random model (no contradiction is present). On the contrary the equilibrium expectations are definitely different from those of the random model. We suppose the system has a real first order transition (with latent heat) at some temperature T_C. At temperatures higher than T_C the two models are equivalent. At temperatures lower than T_C the phase equivalent to the random model still survive as a metastable phase.

We may think to stabilize this phase by adding a new term in the Hamiltonian, which has zero effect on the properties of the system in the *random* phase, but it strongly increases the energy of the *crystal* phase. An example of such a term could be

$$Nr\theta(\sum_{i,k}((M_{i,k} - C_{i,k})^2 - q)) \tag{36}$$

With an appropriate choice of C,r and q, it may be possible to kill the transition to the crystal phase without affecting the free energy in the other phase. We can thus modify the model adding a *small* perturbation to the Hamiltonian in such a way that the its free energy coincides with the free energy of the random model at all temperature.

5 Hard spheres on a sphere

In this section we show the existence of a physically interesting model which can be put in the form eq. (2). In this way we prove there is (al least) one non-trivial applications of the considerations presented here.

Let us consider the following Hamiltonian

$$H = \sum_{i,k=N} V(|x_i - x_k|^2) = \sum_{i,k=N} W(x_i \cdot x_k), \tag{37}$$

where the x's are N D-dimensional vectors which are confined on a the surface of sphere [d] of radius R. The quantity V is the interparticle potential as function of the distance squared and

$$W(S) = V(2(R^2 - S)) \tag{38}$$

is the potential as function of the scalar product.

The usual partition function of interacting particles in a $D-1$ dimensional flat space is obtained by sending $R \to \infty$, $N \to \infty$ at fixed density $\rho = NR^{-(D-1)}$. We may hope that for large dimensions the model will be soluble. Here we will study the case in which N, D and R goes to infinity together (and also V depends on D.

Generally speaking we can write the partition function under the following form

$$\int d\mu(S) \exp\left(-\beta \sum_{i,k=N} W(S_{i,k})\right), \tag{39}$$

where S is $N \times N$ matrix and

$$\mu(S) = \int \prod_i (dx_i) \prod_{i,k} \delta(S_{i,k} - x_i \cdot x_k). \tag{40}$$

We can now substitute the hard constraint $|x_i| = R$ with a term in the probability distribution of the x proportional to $\exp(-\omega/2|x_i|^2$, where ω is chosen in such a way that $\langle |x_i|^2 \rangle = R^2$.

The final model is slightly different form the previous one

$$Z = \int \prod_i (dx_i) \exp\left(-\beta H(x) - \omega/2 \sum_{i=1,N} |x_i|^2\right). \tag{41}$$

In sufficiently high dimensions the two models (37 and 41) should be the same, the measure on x should become concentrated on a sphere of fixed radius which depends on ω.

Now the measure $\mu(S)$ becomes

$$\mu(S) = \int \prod_i (dx_i) \prod_{i,k} \delta(S_{i,k} - x_i \cdot x_k) \exp\left(-\omega/2 \sum_{i=1,N} |x_i|^2\right), \tag{42}$$

which is invariant under $O(N)$ rotations as can be seen by an explicit computation.

The model is thus reduced to the original form eq.(2), we have just to obtain the correct scaling for large N. In other terms we have to choose the dependance of the various term on N is such a way to obtain the needed result.

It is possible to prove by an explicit computation that the *right* scaling is obtained when $N \to \infty$ and $D \to \infty$ together at fixed $\alpha \equiv N/D$. The potential V should also scale in an appropriate way in this limit.

After doing the appropriate computations one arrives to a model which can be explicitly solved in the low density case and that behaves in a quite similar model to the $M_{i,k} = \pm 1$. The details of the computation can be found on the original paper [14].

6 Conclusions

We can summarize these findings in the following way.

- The matrix model is soluble in the high temperature phase.
- In the high temperature phase the matrix model is equivalent to a random model which displays a replica breaking transition to a glassy phase.
- The random model describes the equilibrium and the dynamical properties of the model both in the high temperature phase and in the glassy phase.

[d] We could also put the particle in cubic box, but we choose a sphere for technical reasons.

- For some values of N there is a transition to a crystal phase at low temperature. The properties of the crystal phase cannot easily investigated: they depend on the arithmetic properties of N.
- It may be possible to destroy the crystal phase with a *small* perturbation which does not affect the properties in the other phase.

Some of these results are only conjectural, some have a more solid basis and some have been numerically verified in simpler models [9]. At the present moment we have a coherent picture of the behaviour of the model. it would be extremely interesting to find out if we can collect more evidence for its correctness.

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