Scaling above the upper critical dimension in Ising Models

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Abstract

We rederive the finite size scaling formula for the apparent critical temperature by using Mean Field Theory for the Ising Model above the upper critical dimension. We have also performed numerical simulations in five dimensions and our numerical data are in a good agreement with the Mean Field theoretical predictions, in particular, with the finite size exponent of the connected susceptibility and with the value of the Binder cumulant.

1 Introduction

In this letter we will study the scaling above the upper critical dimension for Ising Models (where Mean Field holds). This issue was been controversial because the numerical simulations do not support the Mean Field predictions [1]. In particular neither the scaling exponent of the apparent critical temperature, nor the scaling exponents of the connected susceptibility, nor the value of the Binder cumulant were those of the Mean Field [2, 3]. The first discrepancy has been solved using the renormalization group [4] and we will show that it can also be explained using Mean Field Theory. We have also made numerical simulations in five dimensions and we see that our numerical values are in a good agreement with the Mean Field predictions and hence there is no discrepancy.

The plan of this letter is the following. In the next section we will made some remarks on the different limits of the Binder cumulant. We will then derive the scaling formula for the apparent critical temperature using Mean Field Theory. Next we show the numerical results for the five dimensional Ising model and finally we will report the conclusions.

2 Limits of the Binder Cumulant

The problem of the triviality of the field theory can be understood studying the value of the renormalized coupling constant. We center the following discussion using one component spin models.

If we renormalize the theory at zero momenta it can be shown that the renormalized constant, g_R, obeys:

$$g_R = \left(\frac{L}{\xi_\infty}\right)^d B, \tag{1}$$

where ξ_∞ is the correlation length defined through the decay of the correlation function, d is the dimension, and B, the Binder cumulant, defined as:

$$B = \frac{1}{2}\left(3 - \frac{\langle M^4\rangle}{\langle M^2\rangle^2}\right), \tag{2}$$

where M is the total magnetization. With this definition if the distribution of M is Gaussian B vanishes.

From this point of view, triviality is equivalent to have a renormalized coupling constant, g_R, which vanishes in the thermodynamical limit at the critical point (where we define the field theory). But g_R equal to zero implies that the Binder cumulant is also zero. This corresponds to requiring that the limiting (infinite volume) distribution of the total magnetization is Gaussian, as happens, for instance, in the high temperature phase. In a non trivial theory, the limiting probability distribution of the magnetization is not Gaussian, and the degree of the non-gaussianity is measured by the value of the renormalized constant. In the low temperature phase the limiting distribution is the sum of two Dirac deltas (one with positive magnetization and the another with negative) and then the Binder cumulant is one [5].

In this context is clear that we can use the value of g_R to classify the universality class and since L/ξ_L is a constant at β_c we could also use the Binder cumulant for the same purpose.

The problem is that we measure the Binder cumulant on a lattice L at temperature $1/\beta$, and so we obtain a function $B(L,\beta)$. The finite size scaling behavior of the Binder cumulant is

$$B(L,\beta) = f(L^{1/\nu}(\beta-\beta_c)). \tag{3}$$

The scaling function has the following limits:

- β is fixed to the critical value β_c and then we take the infinite volume limit:

$$B_1 \equiv \lim_{L\to\infty} \lim_{\beta\to\beta_c} B(L,\beta). \tag{4}$$

- We take the value of the Binder cumulant at the apparent critical temperature, denoted as $\beta_c(L)$ (defined, for example, as the maximum of the connected susceptibility, obviously $\beta_c = \lim_{L\to\infty}\beta_c(L)$), and we consider the limit:

$$B_2 \equiv \lim_{L\to\infty} B(L,\beta_c(L)). \tag{5}$$

- First take $L\to\infty$ and then $\beta\to\beta_c$:

$$B_3 \equiv \lim_{\beta\to\beta_c} \lim_{L\to\infty} B(L,\beta). \tag{6}$$

In general, these limits are all different:

$$B_1 \neq B_2 \neq B_3. \tag{7}$$

Assuming that we have the following scaling for the apparent critical point, $\beta_c(L)$

$$\beta_c(L) = \beta_c + aL^{1/\nu}, \tag{8}$$

where a is dependent on the precise definition of $\beta_c(L)$ (for example the specific heat or the susceptibility). Then we have:

$$B_2 = f(a).$$

Obviously B_2 depends on the observable used to define the apparent critical temperature. This limit can be related to the usual calculations of the Binder cumulant at $L/\xi_L = \text{const}$: at the maximum of the connected susceptibility in a finite size we have that the ratio between ξ_L and L is constant, independent of the size.

We remark that if we rewrite equation (3) as

$$B(L,\beta) = \tilde{f}(L/\xi_L(\beta)), \tag{9}$$

and calculate the Binder cumulant for a set of β's and L's such that $L/\xi_L(\beta) = \text{const}$, the value of this Binder cumulant will be independent of the β's and L's in the scaling region (defined as the region where the formula (3), or equivalently (9), holds).

For completeness:

$$B_1 = f(0) \quad , \quad B_3 = f(\infty). \tag{10}$$

B_1 and B_2 are both size independent and observable independent: are the zero value and the infinite value, respectively, of a universal function.

In next section we will show how B_2 can be calculated using the connected susceptibility to define $\beta_c(L)$.

3 Above the critical dimension.

We will follow the notation of reference [3] in our presentation. Above the critical dimension we can study the system considering an effective action with only one degree of freedom

$$S_{\text{eff}} = L^d \left(\frac{1}{2} t \phi^2 + \frac{1}{4!} u \phi^4 \right), \tag{11}$$

where t is the reduced temperature and we need to calculate:

$$m_{2p} = \frac{1}{\mathcal{Z}} \int \mathrm{d}\phi \; \phi^{2p} \exp(-S_{\text{eff}}), \tag{12}$$

where

$$\mathcal{Z} = \int \mathrm{d}\phi \exp(-S_{\text{eff}}). \tag{13}$$

In this notation the Binder cumulant is simply

$$B = (3 - m_4/m_2^2)/2. \tag{14}$$

We perform the following change of variables:

$$\Phi = (uL^d)^{-1/4} \phi, \tag{15}$$

and the action becomes

$$S_{\text{eff}} = \frac{1}{2} x(t, L) \Phi^2 + \frac{1}{4!} \Phi^4, \tag{16}$$

where $x(t, L) = tL^{d/2} u^{-1/2}$.

We remark that at this moment we are in a finite size and at a generic temperature, different from the critical one.

If we calculate the value of $x(t, L)$ at the maximum of the connected susceptibility $(< \phi^2 > - < |\phi| >^2)$ with the action of the equation (16) (in other words, we calculate the apparent critical reduced temperature $t(L)$) we obtain, evaluating the integrals numerically:

$$x(t, L)_{\max} = -0.44015 = t(L) L^{d/2} u^{-1/2}. \tag{17}$$

and hence the scaling of the apparent reduced temperature is

$$t(L) \sim L^{-d/2}, \tag{18}$$

as was previously found in reference [4] and numerically in [1]. The subdominant term L^{2-d} follows from the subsequent analysis of Brezin and Zinn-Justin [2]. The analysis of Brezin and Zinn-Justin was done with $t = 0$ and they miss the dominant term $L^{-d/2}$ because it is proportional to t.

We can evaluate the Binder cumulant, eq. (14), at this apparent critical temperature with the action (11) obtaining

$$B_2 = B(L, \beta_c(L)) = 0.742137. \tag{19}$$

This value is independent of the dimension, size and of the coupling u but depends of the observable used (in this case the connected susceptibility). The generalization to O(N) models is straightforward.

4 Numerical Results in $d = 5$.

We have performed numerical simulations in order to check the discrepancy between previous quoted values [1] and the theoretical prediction.

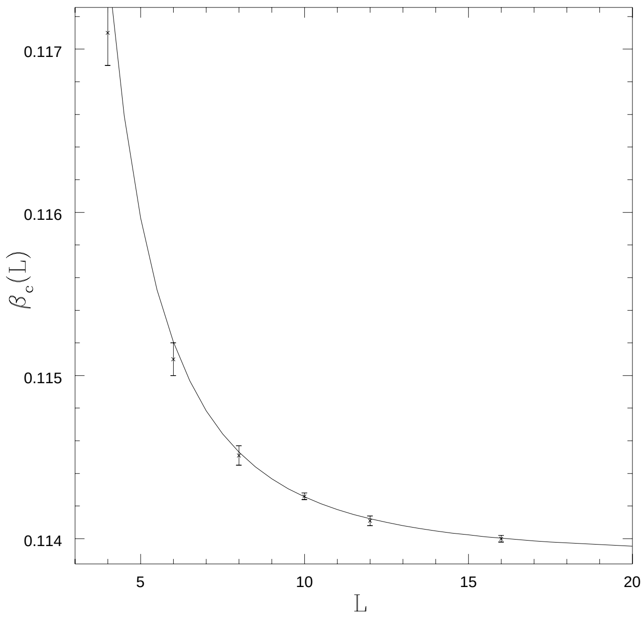

Figure 1: Effective inverse temperature, $\beta_c(L)$, against L for six different sizes.The continuous line is the global fit supposing that the exponent is 2.5 (as described in the text).

To do this we have simulated the five dimensional Ising Model using the Wolff single cluster algorithm. We have simulated $L = 4, 6, 8, 10, 12$ and 16 for a total number of cluster sweeps of the order 6.5 million. In the analysis of the data we have used the spectral density method [7] and the jack-knife method to evaluate the statistical error.

The shift of the critical temperature, calculated as the maximum of the connected susceptibility, follows (using all the sizes in the fit)

$$\beta_c(L) = 0.11387(4) + 0.0798(20)L^{-2.32(16)}, \tag{20}$$

L	$B(L,\beta_c)$	$B(L,\beta_c(L))$
4	0.485(5)	0.739(3)
6	0.475(8)	0.744(4)
8	0.450(2)	0.739(7)
10	0.425(5)	0.746(4)
12	0.39(2)	0.73(1)
16	0.40(2)	0.74(1)

Table 1: $B(L,\beta_c)$ and $B(L,\beta_c(L))$ for different sizes. The Mean Field theoretical value is 0.40578 for the first Binder cumulant and 0.742 for the second one.

with a χ^2/d.o.f ≈ 0.2 and we write the errors inside of the parenthesis. Hence we obtain a shift compatible with the theoretical prediction 5/2. This value was also reported in [1]. We have repeated the global fit with $L \geq 6$ the numbers are very similar ($\beta_c = 0.11386(6)$) but the errors are large.

Fixing the scaling exponent to the theoretical value of eq. (2.5) and using $L \geq 8$, we obtain:

$$\beta_c(L) = 0.11388(3) + 0.12(1)L^{-2.5} \tag{21}$$

with a good χ^2/d.o.f. ≈ 0.15. In Fig. 1 we plot the numerical values of $\beta_c(L)$ and plot, signed by a continuous line, the fit (21). We remark that the estimation of Rickwardt et al [1] was $\beta_c = 0.113929(45)$ that is a 1.5 standard deviations of our value.

Using $\beta_c = 0.11388$ we can monitor $B(L,\beta_c(\infty))$ that we plot in Table 1. For the large lattices the value is compatible with that reported by Brezin and Zinn-Justin in reference [2], $B = 0.40578$. Also we have written in this Table the values of $B(L,\beta_c(L)$ and the agreement is again very good with our previous calculated value (19).

Analyzing the maximum value of the connected susceptibility we found the following power law using all the lattices in the fit:

$$\chi_{\max} \sim L^{2.687(6)}, \tag{22}$$

but the fit is very poor (χ^2/d.o.f. = 7). However fitting only the lattices with $L \geq 8$ we found

$$\chi_{\max} \sim L^{2.48(4)}, \tag{23}$$

with χ^2/d.o.f. $\approx 1.95/2$, and so the agreement is very good with the theoretical prediction 5/2. We plot the numerical data with the fit (23) in Fig. 2.

We finally report in Fig. 3 the different curves of the Binder cumulant near the critical temperature. Using the critical temperature of reference [1] we have found, using our $L = 16$ data, a value of the Binder cumulant $B = 0.52(2)$ that is a two standard deviations of $B = 0.479(25)$, the value reported by this group [1].

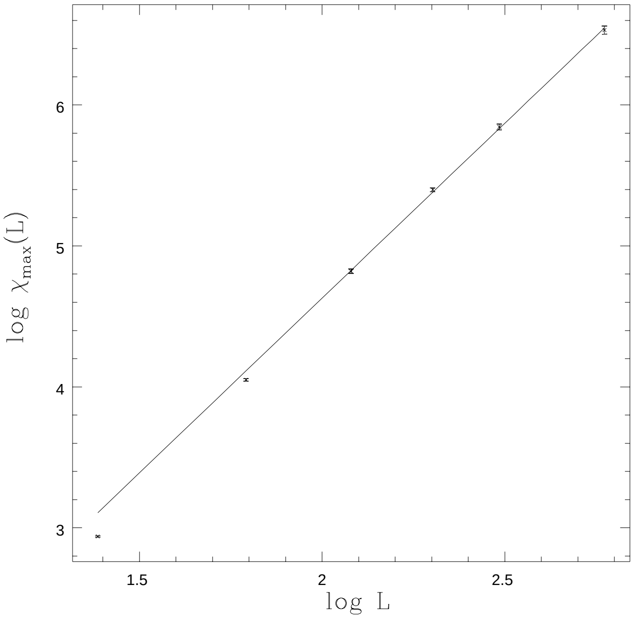

Figure 2: Scaling of the connected susceptibility for six different sizes. The continuous line is the fit using only the points $L \geq 8$, as described in the text.

5 Conclusions.

We have derived using Mean Field the finite size shift of the apparent critical temperature for the Ising model above the upper critical dimension. We also have shown that the numerical data for the five dimensional Ising model are compatible with the Mean Field predictions, in particular, the value of the Binder cumulant at criticality, the finite size exponent of the connected susceptibility and the shift of the apparent critical temperature agree very well with Mean Field Theory.

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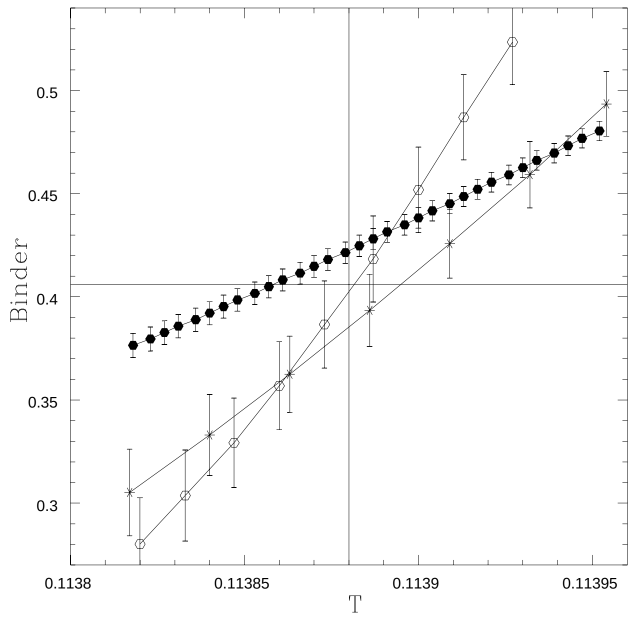

Figure 3: We plot the curves of the Binder cumulant for $L = 10$ ($\bullet$), $L = 12$ ($*$) and $L = 16$ ($\circ$). The horizontal line is the Mean Field prediction (B=0.406). The vertical line is our estimation of the inverse critical temperature.

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