The physical meaning of Replica Symmetry Breaking

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A short summary.

1. Complex Systems: many phases
2. The overlap and its probabilities
3. Stochastic Stability
4. Overlap Equivalence and Ultrametricity
5. The replica approach
6. Stochastic Stability and Dynamics

Dirac Medal Talk, ICTP Trieste, May 2000

Boltzmann statistical mechanics:

an example of a successful *redutionistic* program.

Emergent behaviour or collective behaviour $\longrightarrow$

Phase Transitions

Problem: Phase co-existence:

At the liquid-gas or at the liquid-solid line

$\longrightarrow$ two phases

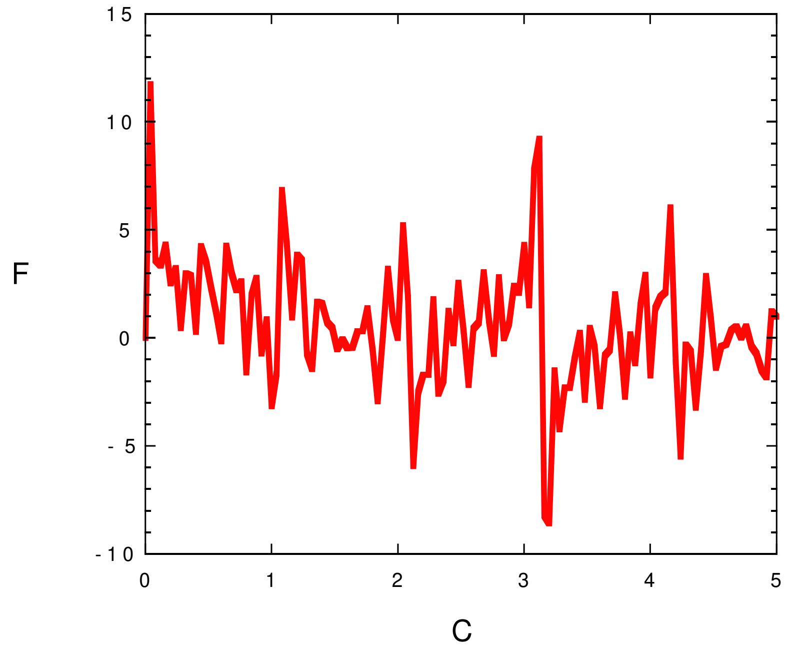
At the liquid-gas-solid triple point

$\longrightarrow$ three phases

Complex systems:

"Number of phases is very large (infinite) for generic choice of parameter".

Free energy landscape.



An artistic view of a free energy with many minima.

Phase democracy: translational invariant quantities (e.g. energy) have the same value in all the phases (apart from corrections proportional to $N^{-1/2}$).

We consider a large (*finite*) system. We want to decompose the phase state in valleys (phases, states) separated by barriers.

For simplicity we take a spin system with N points.

(spins are labeled by i, may be a lattice point)

States (labeled by α) are characterized by different local magnetizations:

$$m_\alpha(i) = \langle \sigma(i) \rangle_\alpha \ ,$$

where $\langle \cdot \rangle_\alpha$ is the expectation value

in the valley labeled by α.

$$\langle \cdot \rangle \approx \sum_\alpha w_\alpha \langle \cdot \rangle_\alpha$$

$$w_\alpha \propto \exp(-\beta F(\alpha)) \approx \exp(-\beta E(\alpha))$$

$F(\alpha)$ free energy,

which at small T becomes the internal energy $E(\alpha)$.

Quantities like internal energy or the squared magnetization $q_{EA}(\alpha) = N^{-1} \sum_{i=1,N} m_\alpha(i)^2$

do not depend on α.

We call J the control parameters of the systems

Average over J is a bar (e.g. $\overline{F}$)

J is the quenched disorder or the value of N.

Problem: to find those quantities which *do not* depend on J and to find the probability distribution of those quantities which *do* depend on J.

States may be separated by considering their mutual overlap:

$$q[\sigma,\tau] = \frac{1}{N}\sum_{i=1,N}\sigma(i)\tau(i)\ ,$$

$$q(\alpha,\gamma) = \frac{1}{N}\sum_{i=1,N} m_\alpha(i) m_\gamma(i) \approx q[\sigma,\tau]\ ,$$

where σ and τ belong to the states α and γ.

We define $P_J(q)$ as probability of q at given J,

i.e. the histogram of $q[\sigma,\tau]$ where σ and τ are two equilibrium configurations.

$$P_J(q) = \sum_{\alpha,\gamma} w_\alpha w_\gamma \delta(q - q_{\alpha,\gamma})\ .$$

If more than one state, $P_J(q)$ is not a single delta function

$$P_J(q) \neq \delta(q - q_{EA}) \; .$$

Models where this should happen:

one example is given by Ising spin glasses.

$$H = -\sum_{i,k} J_{i,k}\sigma_i\sigma_k - \sum_i h_i\sigma_i$$

$\sigma = \pm 1$ are the spins. The variables J are random couplings (e.g. Gaussian or ± 1)

Two different models for spin glasses:

1. Sherrington Kirkpatrick mode (infinite range): all N points connected $J = O(N^{-1/2})$.
   $N \; \to \infty$.
2. Short range models: i belongs to a L^D lattice. Nearest neighbour interaction
   $L \to \infty$ at fixed D (e.g. $D = 3$).

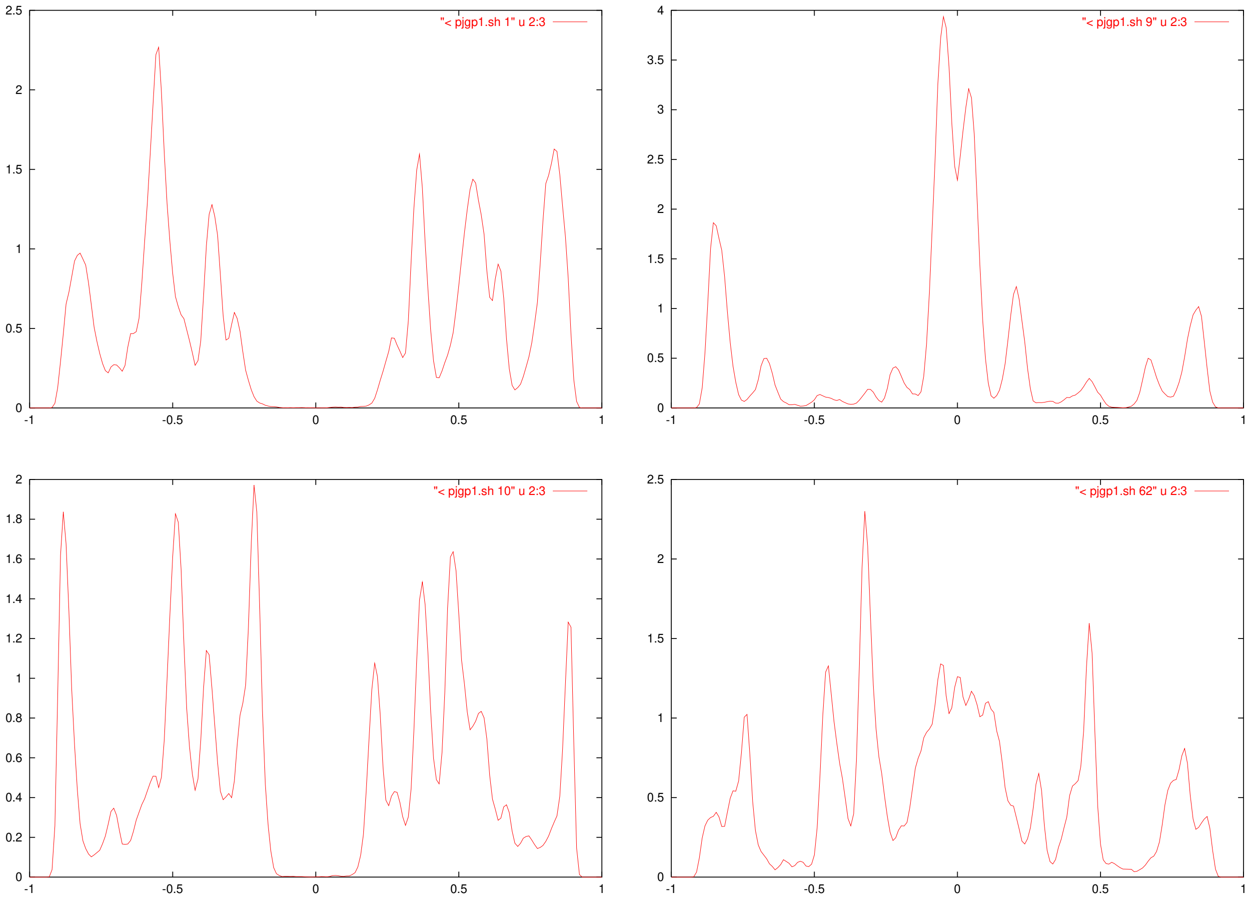
Is $P_J(q)$ trivial?

If not, replica symmetry is broken.

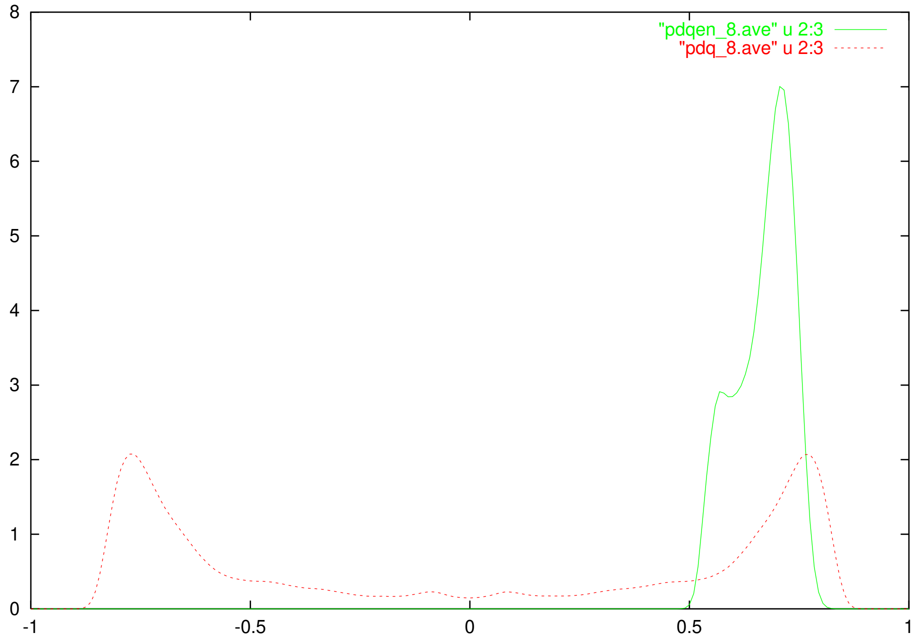
Let us see some data from numerical simulations.

A few samples (i.e different choices of J) for

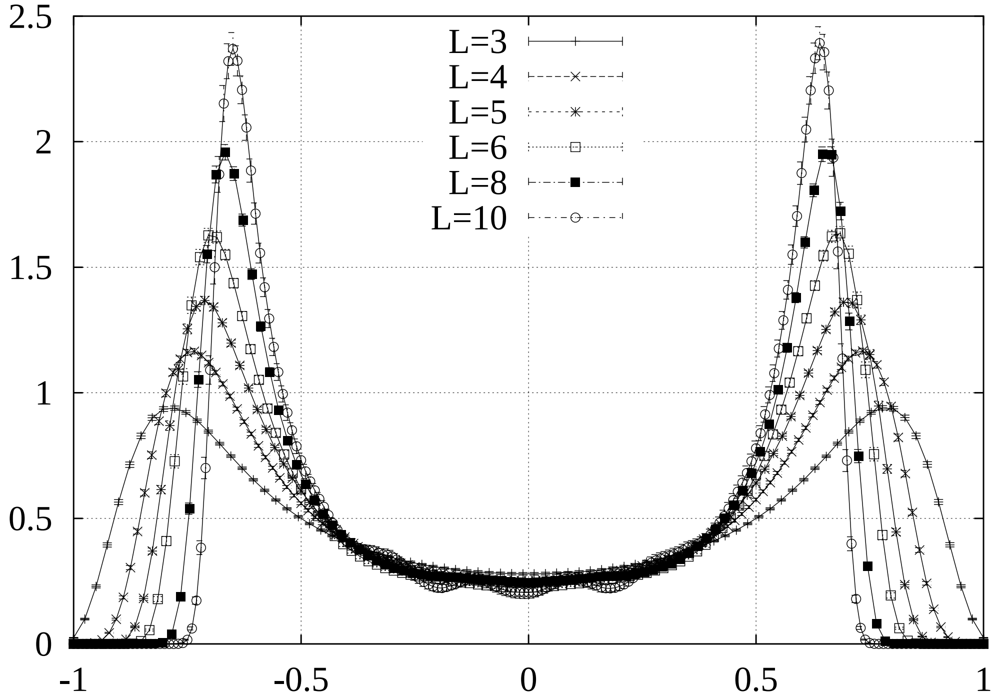
$D = 3 \quad L = 16$, 16^3 spins



After average over many samples (D=3, L=16)



After average over many samples (D=4, L=4...10)



$P_J(q)$ depends on J?

YES:

$$P(q) = \overline{P_J(q)} \qquad \overline{P_J(q_1)P_J(q_2)} \neq P(q_1)P(q_2)$$

Replica symmetry breaking affects equilibrium

$$H[\sigma] = H_0[\sigma] + \sum_i h\sigma_i .$$

Two susceptibilities which are different

as soon as replica symmetry is broken:

- In a valley $\chi_R = \beta(1 - q_{EA})$
- Total susceptibility $\chi_T = \beta \int dq \; P(q)(1-q)$

How to study this problem?

Main source of information:

- Mean field theory, which is valid at infinite D or infinite range models (analytic computations).
- Numerical computations in the previous case.
- Expansion around mean field for short range models: correlation function and loops corrections.
- Renormalization group.
- Numerical simulations for short range models.
- Experimental data

Theory: which is the probability distribution of the set of all $q_{\alpha,\gamma}$ and F_α (or equivalently w_α?

Given N and J we call $\mathbf{P}$ the set of all $q_{\alpha,\gamma}$ and F_α.

Problem: compute and evaluate the average for $\mathcal{P}(\mathbf{P})$.

High level (macroscopic) statistical mechanics,

opposite to low level (microscopic) statistical mechanics

Analytic computations using replicas.

The simplest theory is based on two principles:

- Stochastic Stability
- Overlap Equivalence.

Stochastic Stability is nearly automatically implemented in the replica approach.

Overlap Equivalence is usually implemented in the replica approach.

Stochastic stability

$$H(\epsilon) = H + \epsilon H_R$$

everything is smooth in ϵ.

Stochastic Stability says that the system behaves a generic random system.

Stochastic Stability is non-trivial: $\delta F_\alpha = O(\epsilon N^{1/2})$.

The union of two uncoupled stochastically stable systems is **not** stochastically stable.

Examples of perturbations (changing r):

$$H_R^{(r)} = N^{(r-1)/2} \sum_{i_1 \ldots i_r} J i_1 \ldots i_r \sigma(i_1) \ldots \sigma(i_r)$$

For $r = 1$ the perturbation corresponds to adding a random magnetic field.

An example of an application of stochastic stability.

In one-step replica-symmetry breaking:

$$q_{\alpha,\alpha} = q_{EA} \quad q_{\alpha,\gamma} = 0 \quad \text{for} \quad \alpha \neq \gamma$$

All states are at zero mutual overlap.

The only quantity to fix is the probability of free energies.

The states which contribute to the Gibbs measure have nearly degenerate free energies.

The probability of having a state with free energy in the interval $[F, F + dF]$ is

$$\rho(F)dF$$

Stochastic stability implies that

$$\rho(F) \propto \exp(\beta m(F - F_0)),$$

where F_0 a system dependent reference free energy, proportional to N.

As a byproduct the function $P(q)$ is given by

$$P(q) = m\delta(q - q_0) + (1 - m)\delta(q - q_1) \ .$$

Proof of the previous statement

Stochastic stability imposes that the form of the function $\rho(F)$ remains unchanged

(apart from a possible shift in F_0)

when one adds a small random perturbation.

Consider the effect of a perturbation of strength ϵ on the free energy of a state, say α. The unperturbed value of the free energy is denoted by F_α. The new value of the free energy G_α is given by

$$G_\alpha = F_\alpha + \epsilon r_\alpha$$

where r_α are identically distributed uncorrelated random numbers.

Stochastic stability implies that the distribution $\rho(G)$ is the same as $\rho(F)$.

Expanding to second order in ϵ we get:

$$\frac{d\rho}{dF} \propto \frac{d^2\rho}{dF^2}$$

Therefore $\rho(F) \propto \exp(\beta m(F - F_0))$.

QDE

In the general case we have that

$$P(q_1, q_2) \equiv \overline{P_J(q_1)P_J(q_2)} =$$

$$= \frac{2}{3}P(q_1)P(q_2) + \frac{1}{3}P(q_1)\delta(q_1 - q_2) \quad .$$

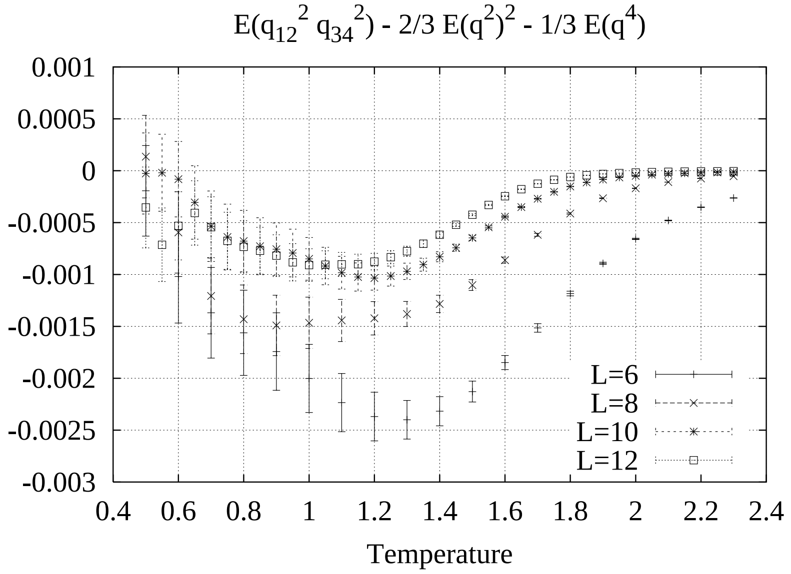
A particular case of the previous relation is the following one

$$\overline{\langle q^2 \rangle^2} = \frac{1}{3}\overline{\langle q^4 \rangle} + \frac{2}{3}\overline{\langle q^2 \rangle}^2$$

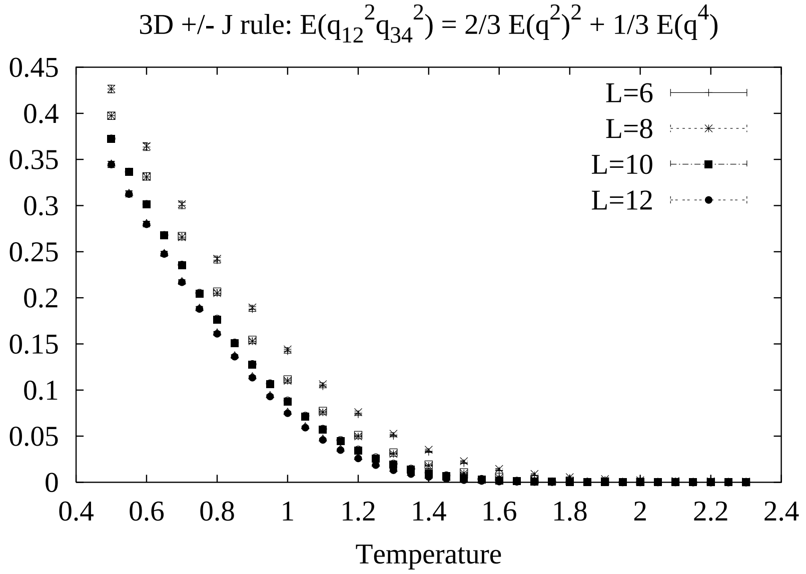
Is all this true?

Let us test the previous relations!

$$\overline{\langle q^2\rangle^2} - \frac{1}{3}\overline{\langle q^4\rangle} + \frac{2}{3}\overline{\langle q^2\rangle}^2$$



$$\overline{\langle q^2\rangle^2} \textbf{ and } \frac{1}{3}\overline{\langle q^4\rangle} + \frac{2}{3}\overline{\langle q^2\rangle}^2$$



How to be more quantitative?

We introduce a generalized overlap. $A(i)$ is a local quantity

$$q^A_{\alpha,\gamma} = N^{-1} \sum_i \langle A(i) \rangle_\alpha \langle A(i) \rangle_\gamma$$

Examples

$$A(i) = \sigma(i) \longrightarrow q^A = q \text{ (usual overlap)}$$

$$A(i) = H(i) \longrightarrow q^A = q^E. \text{ (energy overlap)}$$

Description of a system: all the w_α and all the $q^A_{\alpha,\gamma}$.

OVERLAP EQUIVALENCE

(only one significant overlap)

All overlaps (depending on the operator A) are given functions of the spin overlap.

$$\forall A \; \exists f^A(q) \text{ such that } q^A_{\alpha,\gamma} = f^A(q_{\alpha,\gamma})).$$

In general case an infinite number of quantities

(i.e. $q^A_{\alpha,\gamma} \; \forall A$)

characterizes the mutual relations among α and γ.

Here this infinite number is reduced to one.

We define the q-restricted ensemble

$$\langle f(\sigma,\tau)\rangle_q \propto \sum_{\sigma.\tau} f(\sigma,\tau)\delta(Nq - q[\sigma,\tau])$$

Overlap equivalence $\longleftrightarrow$

cluster decomposition in the q-restricted ensemble

Overlap equivalence seems to imply ultrametricity

$$q_{\alpha,\gamma} > \min(q_{\alpha,\beta}, q_{\beta,\gamma})\ \forall\beta.$$

Separability may be less compulsory of stochastic stability.

In the ultrametric Ansatz one finds that

$$P^{12,23,31}(q_{12}, q_{23}, q_{31}) = 0 \quad ,$$

as soon as the ultrametricity relations

$$q_{12} \geq \min(q_{23}, q_{31})\ , q_{23} \geq \min(q_{31}, q_{12})\ , q_{31} \geq \min(q_{12}, q_{23})\ ,$$

are not satisfied.

Moreover, given the $P(q)$, the ultrametric Ansatz completely determines the $P^{12,23,31}$, if we assume stochastic stability.

The replica approach

A compact algebraic way to code the previous information and to compute the free energy.

$Q_{a,b}$ are a family of $n \times n$ matrix which depends *analytically* and are defined for n multiple than M.

Probability (after average over the permutation of lines and columns of the matrix) that an element of the matrix $Q_{a,b}$ with $a \neq b$ is equal to q

$$\longrightarrow P(q)$$

Probability that an element of the matrix $Q_{a,b}$ is equal to q_1 and an other element of the matrix $Q_{c,d}$ is equal to q_2 (with a, b, c and d different)

$$\longrightarrow P(q_1, q_2) \equiv \overline{P_J(q_1) P_J(q_2)}$$

Probability statements become algebraic statements.

Stochastic stability $\longleftrightarrow$ One line of the matrix is a permutation of an other line of the matrix.

Overlap equivalence is a more complex statement.

Quite a different language !

Dynamics

What happens if we cool the system starting a random (high temperature) configuration at time zero?

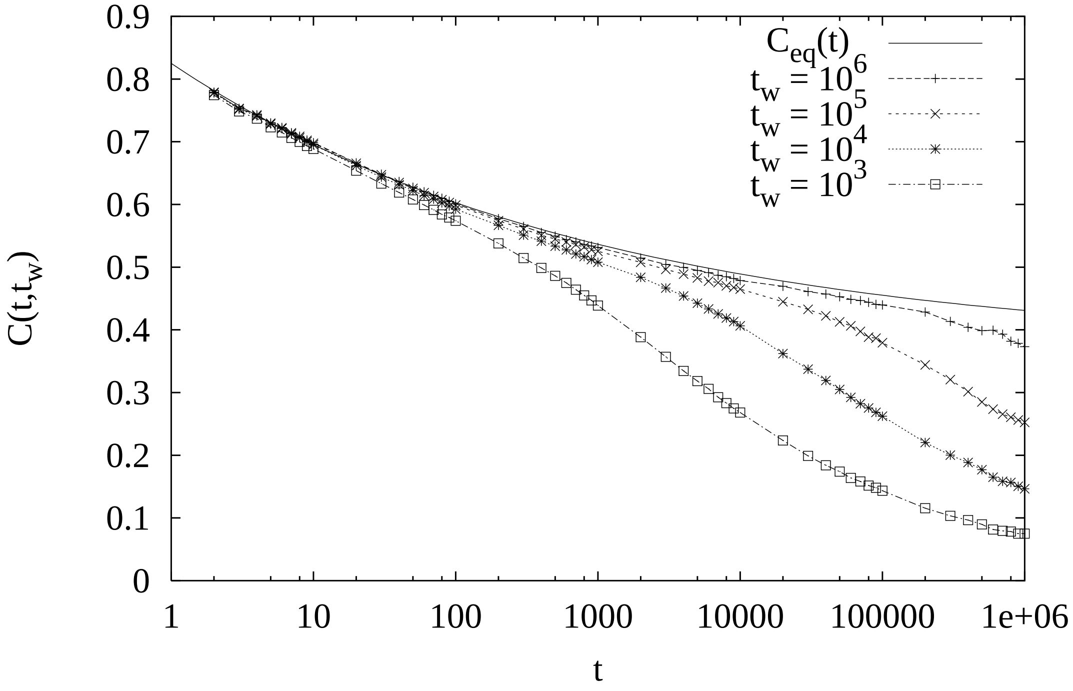
- Ageing
- Two time correlations and responses
- Scaling R versus C
- Fluctuation dissipation relations

Slightly out of equilibrium systems

We have the same results if we gently *shake* the systems or we have a force that has a small component that is not a gradient.

(we slightly violate the detailed balance condition)

$$C(t,t_w) \equiv \frac{1}{N}\sum_{i=1}^{N}\langle\sigma_i(t_w)\sigma_i(t_w+t)\rangle = q(t_q, t_w+t)$$



$t < t_w$: "quasi-equilibrium" regime, $C(t,t_w) \simeq C_{\rm eq}(t)$;

$t > t_w$: aging regime. If simple aging holds $C(t,t_w) \propto \mathcal{C}(t/t_w)$.

$$q_{\rm\ EA} \equiv \lim_{t\to\infty}\lim_{t_w\to\infty} C(t,t_w)$$

$$H = H_0 + \theta(t - t_w) \sum_i h_i \sigma_i$$

$$\beta R(t, t_w) \equiv \frac{1}{N} \sum_{i=1}^{N} \langle \frac{\partial \sigma_i(t_w + t)}{\partial h_i} \rangle$$

$\frac{dR}{dC}$ =-1 $\Longleftrightarrow$ Fluctuation Dissipation Theorem (FTD)

valid for $t << t_w$.

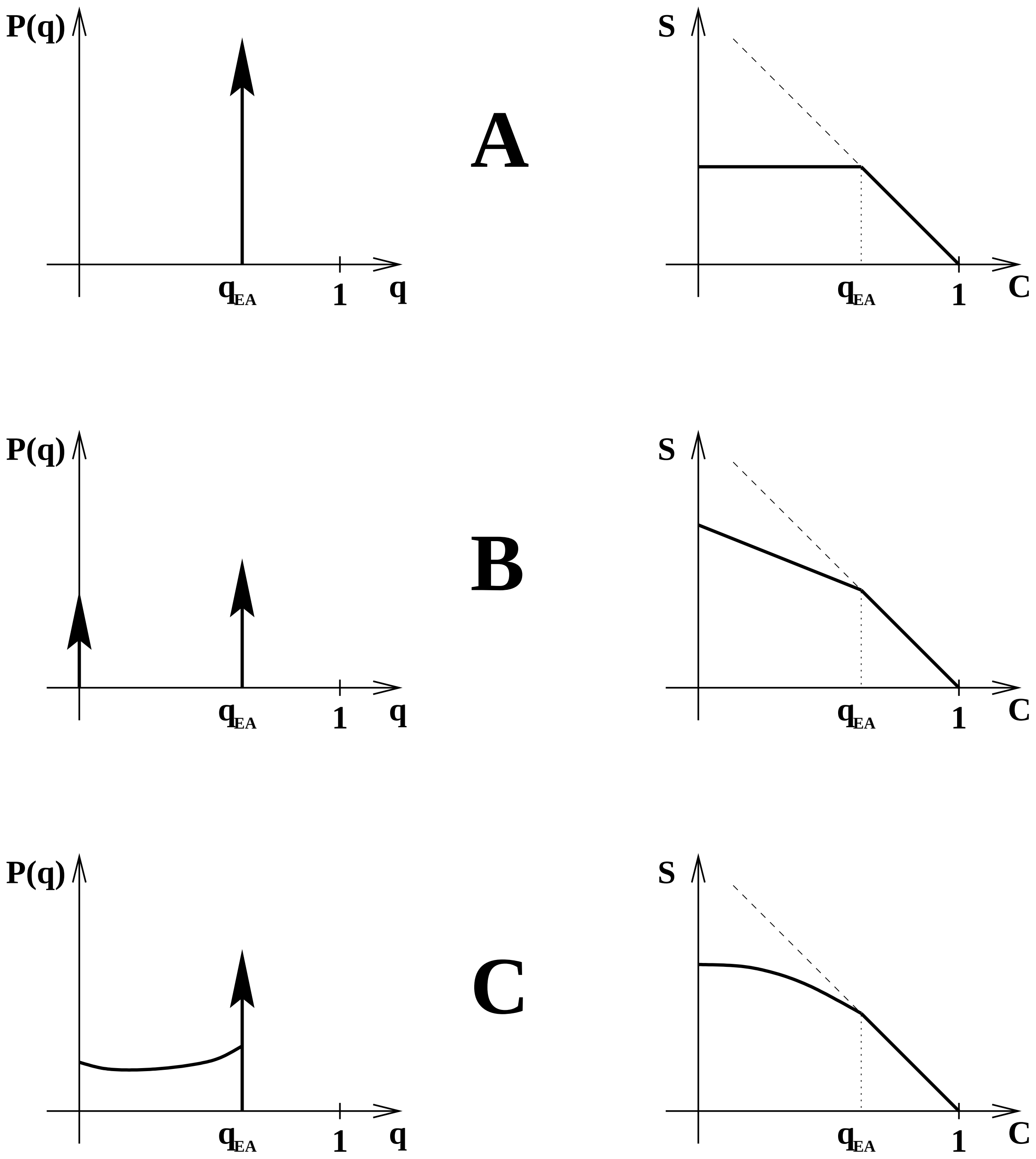
Classification

Stochastic stability implies a relation a among statics and dynamics:

R versus C $\Longleftrightarrow$ $P(q)$

$\frac{dR}{dC} = -\int_0^C dq P(q)$

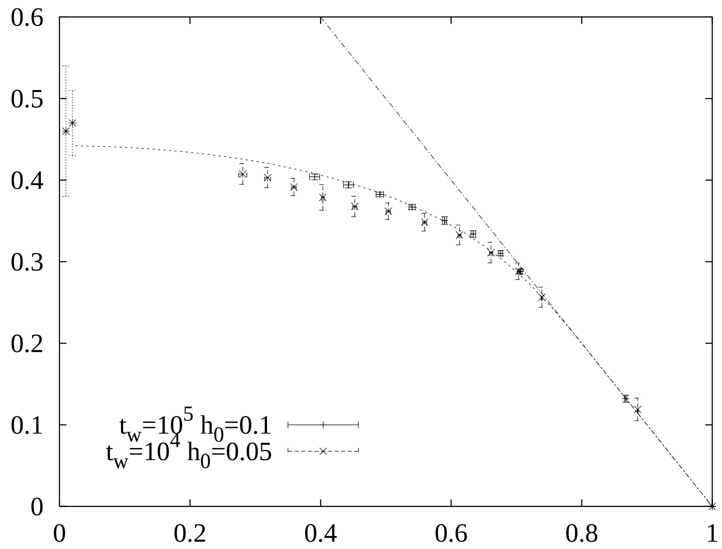
Three main kinds of dynamical response $R(C)$, that correspond to different shapes of the static $P(q)$.


P(q)
A
S
q_{EA}
1
q
C
P(q)
B
S
q_{EA}
1
q
C
P(q)
C
S
q_{EA}
1
q
C

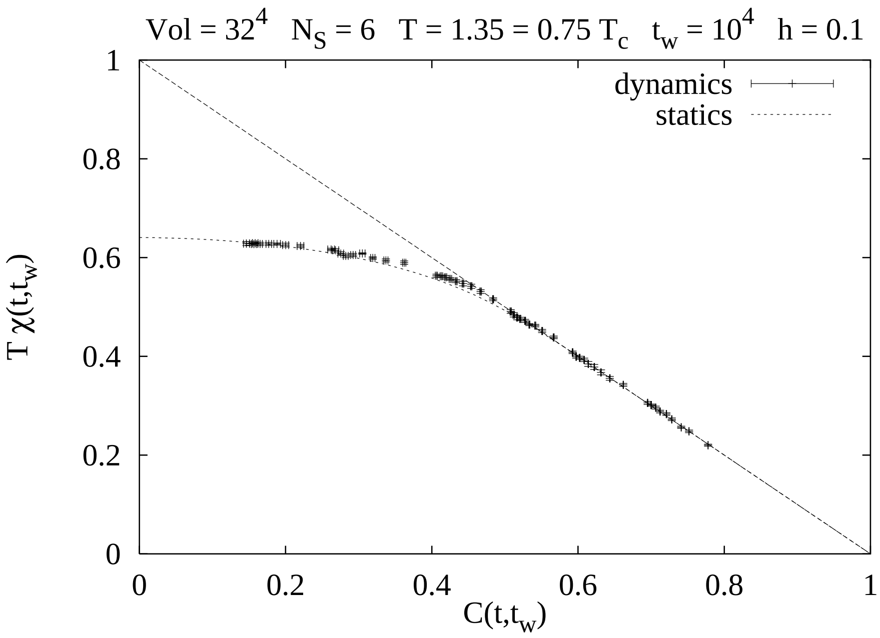
Numerical Results

Edwards-Anderson (EA) model

D=3 Volume=64^3 $T = 0.7 \simeq \frac{3}{4} T_c$



EA D=4



Direct measurements in experiments for glasses and spin glasses in progress

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