EXACT SOLUTIONS FOR SPIN MODELS ON DILUTED HYPERGRAPHS

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- Definition of the models on random hypergraphs
- Some useful notions about hypergraphs
- Zero-temperature phase diagrams
- Graph theory tools: leaf removal
- Rigorous derivation (no replicas) of some results: clustering and complexity

Diluted 3-spin models

Defined by the following Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(\gamma N - \sum_{\{i,j,k\}\in G} J_{ijk}\, s_i\, s_j\, s_k\right)$$

G is a set of γN random triples (**the hypergraph**).

Two kinds of hypergraphs:

- fixed connectivity C
  every index must appear C times and $\gamma = \frac{C}{3}$
- fluctuating connectivity (Poisson distrib.)
  every plaquette is chosen with prob. $\frac{\gamma N}{\binom{N}{3}}$

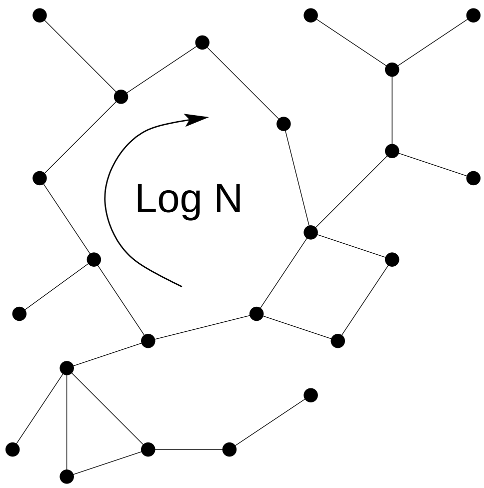
The model and the results can be generalized to any connectivity distribution and to any p-spin interacting terms (with $p > 2$).

Two versions of the model:

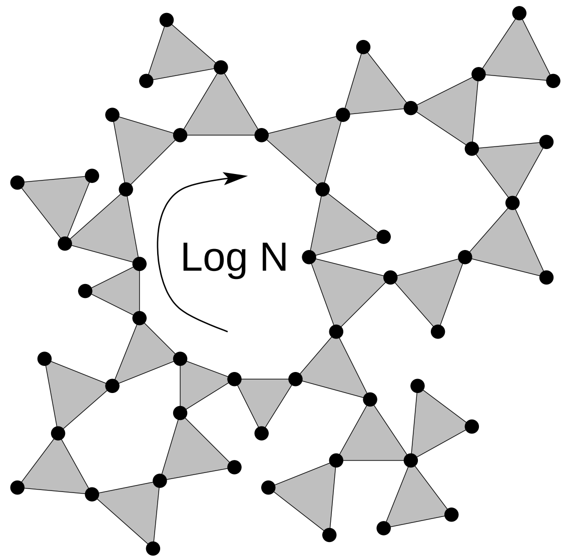
- unfrustrated, ferromagnetic: $J_{ijk} = 1$
  $\rightarrow$ 1^{st} order ferromagnetic transition
- frustrated, spin glass: $J_{ijk} = \pm 1$
  $\rightarrow$ SAT/UNSAT transition

Both versions have a glassy phase!

Graphs and Hypergraphs



Graph (2−spin interactions)

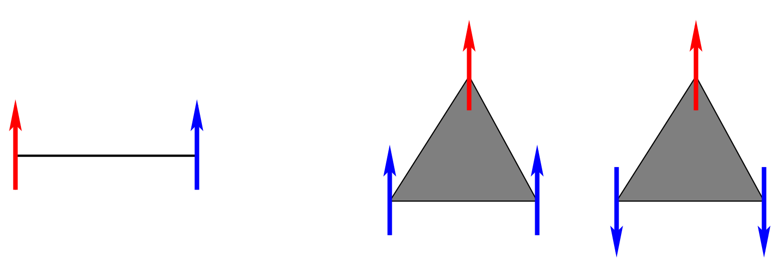


Hypergraph (3−spin interactions)

Typically they are non planar and long range: typical loops are of order $\ln(N)$

Problem: A connected (hyper)graph G, a set of interactions $\{J\}$ and the value of a spin are given. It is known that a spin configuration which satisfies all the interactions exists. Find that configuration.

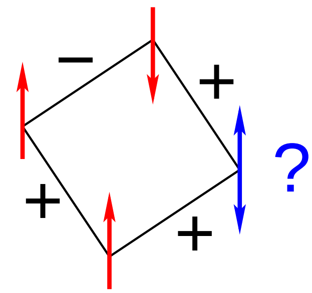
Propagating the information along a ferromagnetic (hyper)link

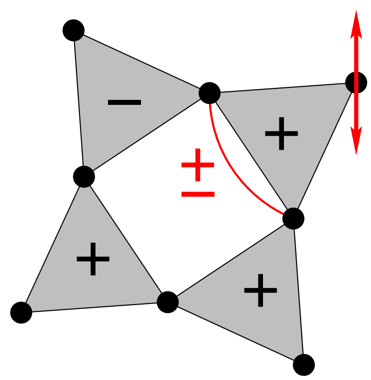


Conclusion: Finding the ground state of an unfrustrated spin model is an easy problem on a graph, but it may be hard on a hypergraph

Frustration $\leftrightarrow$ $J = \pm 1$

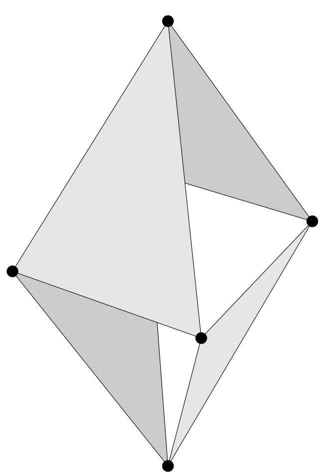
On a graph the frustration arises with loops at the percolation point ($\gamma_p = 1/2$). On a hypergraph the loops arising a the percolation point ($\gamma_p = 1/6$) give no frustration.





Spin on dangling ends can be freely fixed in order to change the effective interaction between the other two spins. Only **hyperloops** can generate frustration on a hypergraph.

Definition of hyperloop: a non-empty set $\mathcal{S}$ of hyperlinks $\{i,j,k\}$ such that every node (spin) appears in $\mathcal{S}$ an even number of times (zero is even).



hyperloops $\Rightarrow$ frustration

satisfied interaction: $J_{ijk} s_i s_j s_k = 1$

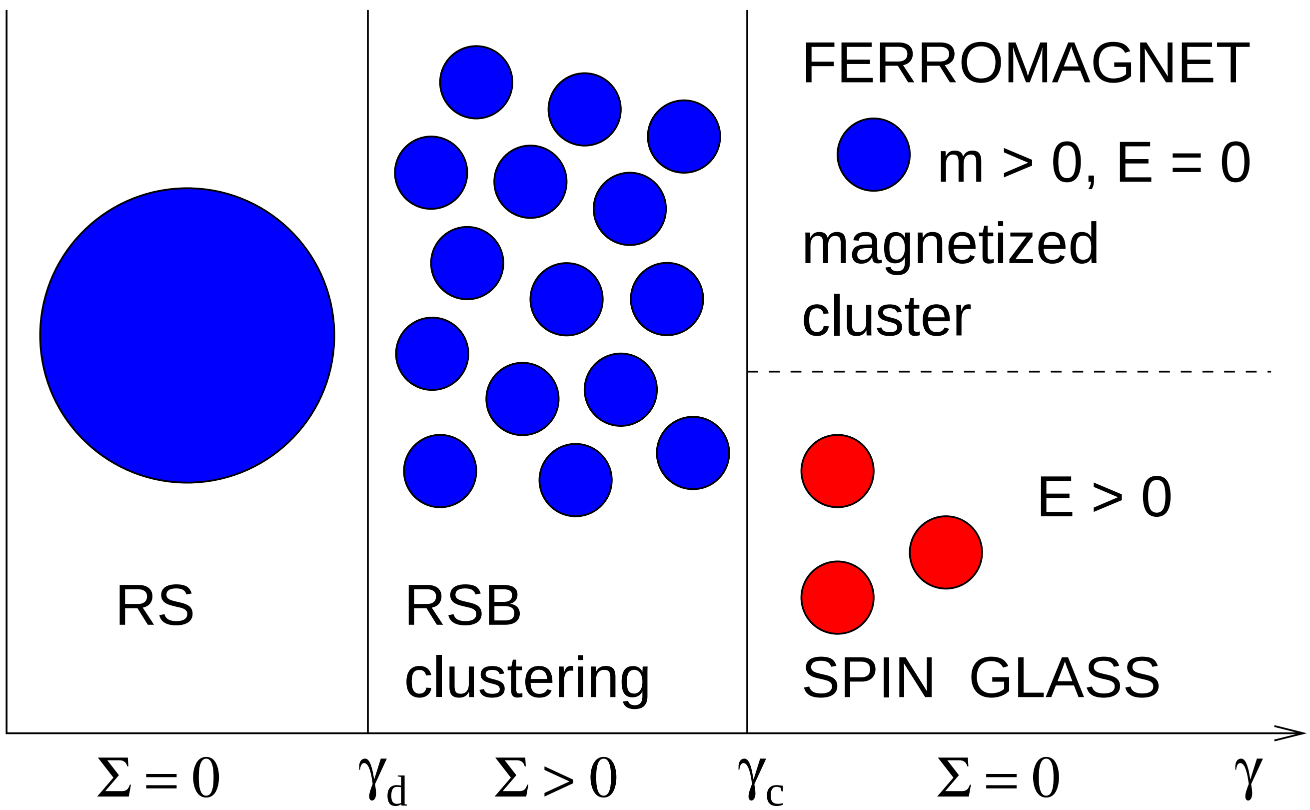
$$1 = \prod_{\{i,j,k\}\in\mathcal{S}} J_{ijk}\, s_i\, s_j\, s_k =$$

$$= \prod_{\{i,j,k\}\in\mathcal{S}} J_{ijk} = \begin{cases} 1 & p = 1/2 \\ -1 & p = 1/2 \end{cases}$$

Zero-temperature phase diagram

Any $p > 2$ and fluctuating connectivity hypegraphs.

Analytic solution and numerics:
if $E = 0$ (no frustration) $\rightarrow$ Gaussian elimination
if $E > 0$ $\rightarrow$ exhaustive enumerations

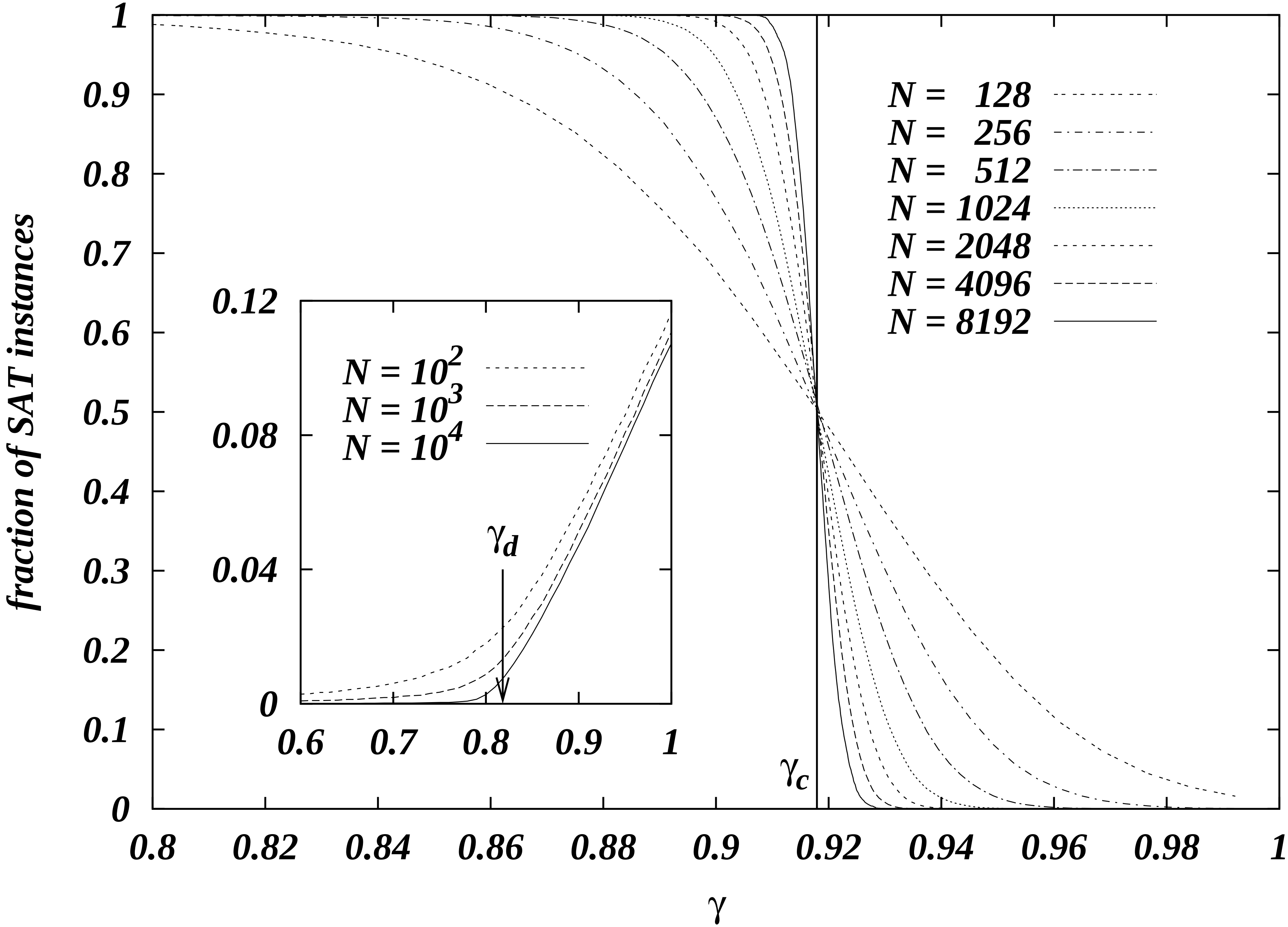


Configurational entropy: $\Sigma(\gamma) = \frac{1}{N} \ln(\#\ \text{clusters})$

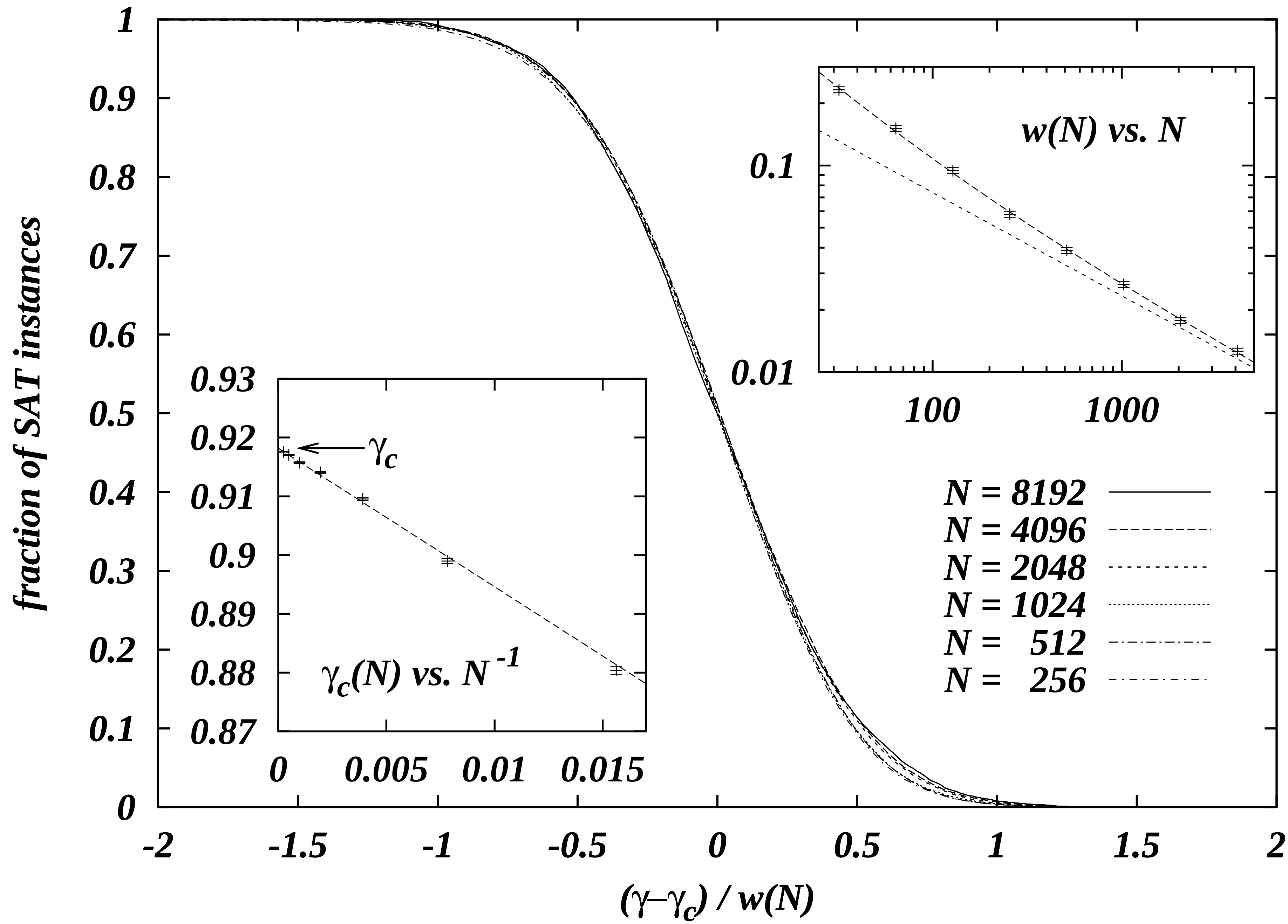
A common problem

- diluted p-spin glass at $T = 0$
- random p-XOR-SAT §
- low density Parity Check codes
- random linear systems in finite fields (GF[2])

§ considered an open problem in theoretical computer science



fraction of SAT instances
γ
γc
γd
N = 128
N = 256
N = 512
N = 1024
N = 2048
N = 4096
N = 8192
N = 10^2
N = 10^3
N = 10^4



fraction of SAT instances
(γ–γc) / w(N)
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
N = 8192
N = 4096
N = 2048
N = 1024
N = 512
N = 256
w(N) vs. N
0.1
0.01
100
1000
γc(N) vs. N -1
γc
0.93
0.92
0.91
0.9
0.89
0.88
0.87
0
0.005
0.01
0.015

The unfrustrated model: $J = 1$

The existence of a ferromagnetic solution follows from a simple argument

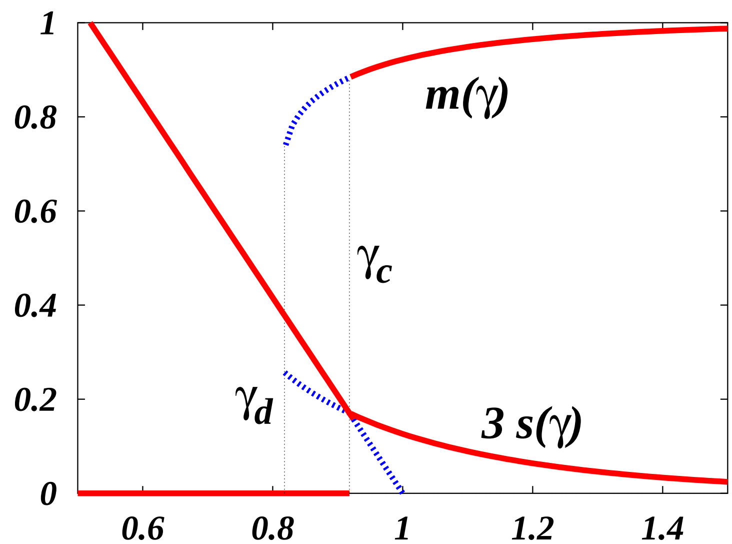
$$1 - m = \sum_c e^{-3\gamma} \frac{(3\gamma)^c}{c!} \frac{c}{3\gamma} (1 - m^2)^{c-1} = e^{-3\gamma m^2}$$

Up to $\gamma_d = 0.818$ the hypergraph is like a tree

$m(\gamma)$ is the fraction of frozen spins, i.e. the magnetization

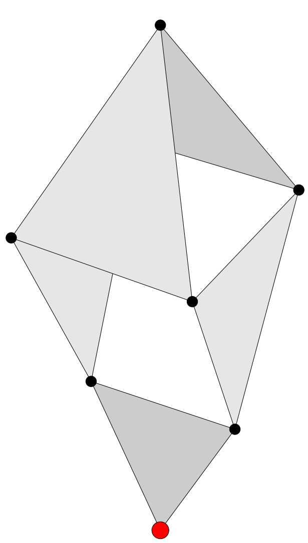
$s(\gamma)$ is the entropy

$\gamma_c = 0.918$



The difficult task is to calculate $s(\gamma) \to$ replicas!

$$s(\gamma) = \ln(2)[1 - \gamma - m + 3\gamma m^2 (1 - m) + \gamma m^3]$$



At γ_c percolate hyperloops and **hyperfields $\Rightarrow$ magnetization**

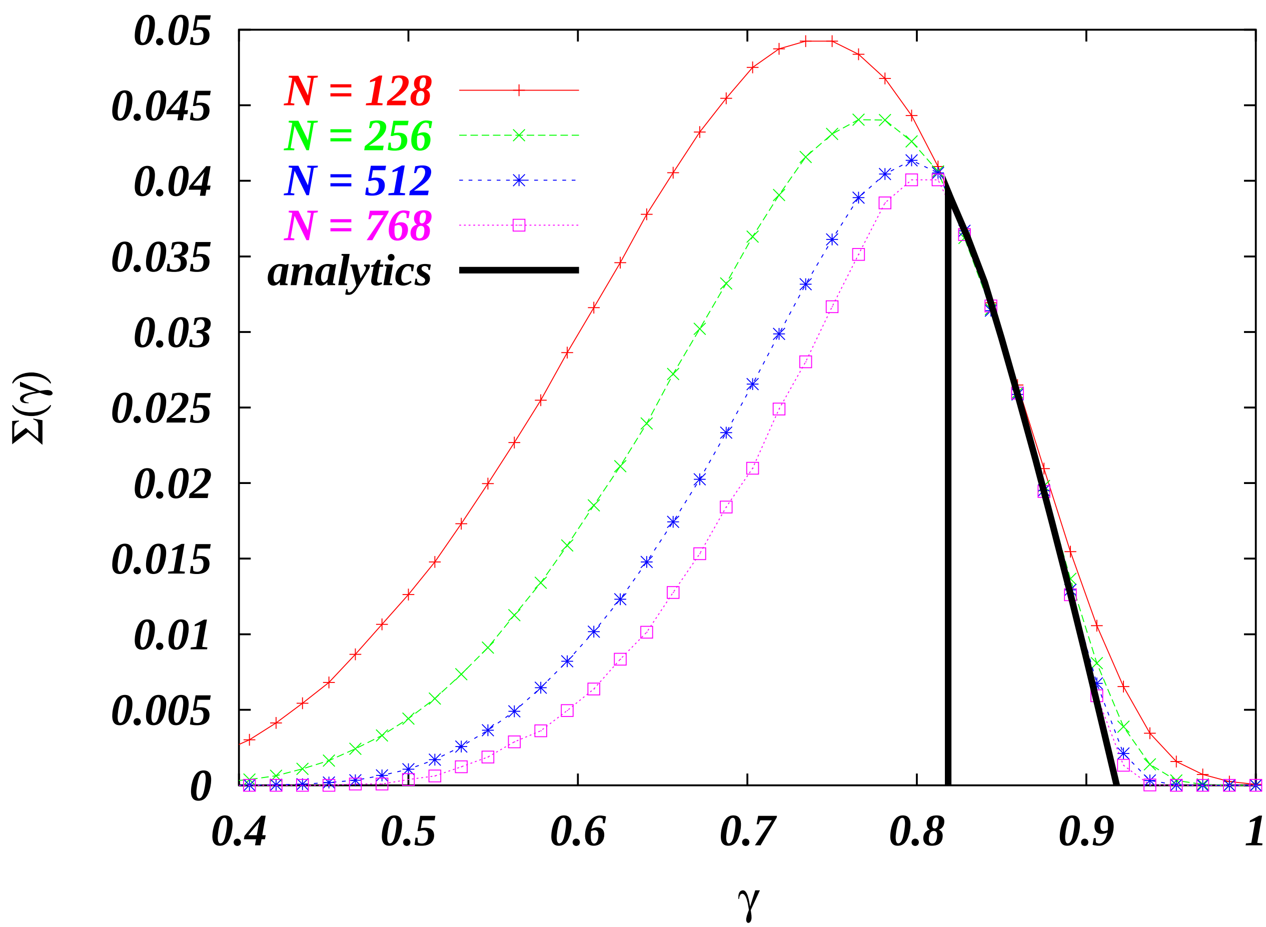
$$s_\ell = \prod_{\{i,j,k\} \in \mathcal{T}} s_i \, s_j \, s_k = \prod_{\{i,j,k\} \in \mathcal{T}} J_{ijk} = 1$$

The structure of the configurational space

Configurational entropy: $\Sigma(\gamma) = \frac{1}{N} \ln(\# \text{ clusters})$
Since all clusters are equal

$$\begin{aligned} \Sigma(\gamma) &= \ln(2)(1-\gamma) - s(\gamma) = \\ &= \ln(2)\left[r - 3\gamma r^2(1-r) - \gamma r^3\right] \end{aligned}$$

with $1 - r = \exp(-3\gamma r^2)$, is exact !



$\Sigma(\gamma) > 0$ for $\gamma \in [\gamma_d, \gamma_c]$ ⇒

⇒ clustering and search algorithms slowing down
⇒ γ_c = SAT/UNSAT threshold for 3-XOR-SAT
⇒ dynamical transition in memory requirements
[$\mathcal{O}(N) \to \mathcal{O}(N^2)$] solving linear systems in GF[2]

Equivalence with linear equations in GF[2]

As long as the ground state energy is 0 (no frustration) all the ground states can be found by solving the following set of linear equations modulus 2.

$$s_i = (-1)^{x_i} \qquad J_{ijk} = (-1)^{\xi_{ijk}}$$

$$\gamma N \text{ eq.} \begin{cases} x_i + x_j + x_k = \xi_{ijk} \ \text{ mod } 2 \\ \cdots \end{cases}$$

An example for the ferromagnetic model
with $N = 7$ and $\gamma = 1$

$$\begin{cases} x_1 + x_5 + x_3 = 0 \\ x_4 + x_7 + x_6 = 0 \\ x_2 + x_4 + x_6 = 0 \\ x_1 + x_3 + x_4 = 0 \\ x_7 + x_5 + x_2 = 0 \\ x_2 + x_7 + x_3 = 0 \\ x_5 + x_1 + x_6 = 0 \end{cases}$$

Gauge transformation to hidden the ground state

Choose a random configuration τ_i and set the interactions to $J_{ijk} = \tau_i \, \tau_j \, \tau_k$
By construction, $s_i = \tau_i$ is a ground state, but now the interactions look like random variables.

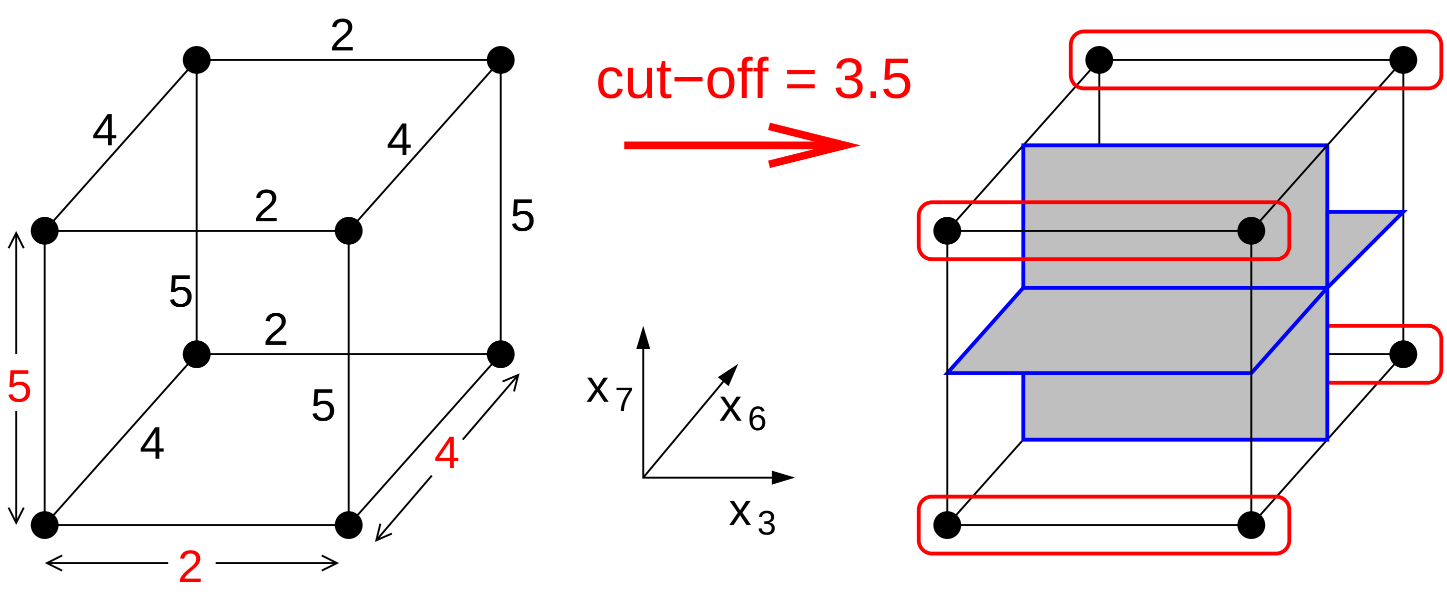
The structure of the solutions

After having solved γN equations there are still $2^{N(1-\gamma)}$ solutions. How are they organized?

$$\gamma N \text{ eq.} \begin{cases} x_1 = x_3 + x_6 + x_7 \\ x_4 = x_6 + x_7 \\ x_2 = x_7 \\ x_5 = x_6 + x_7 \end{cases}$$

- the blue ones are γN dependent variables
- the red ones are $(1-\gamma)N$ independent variables

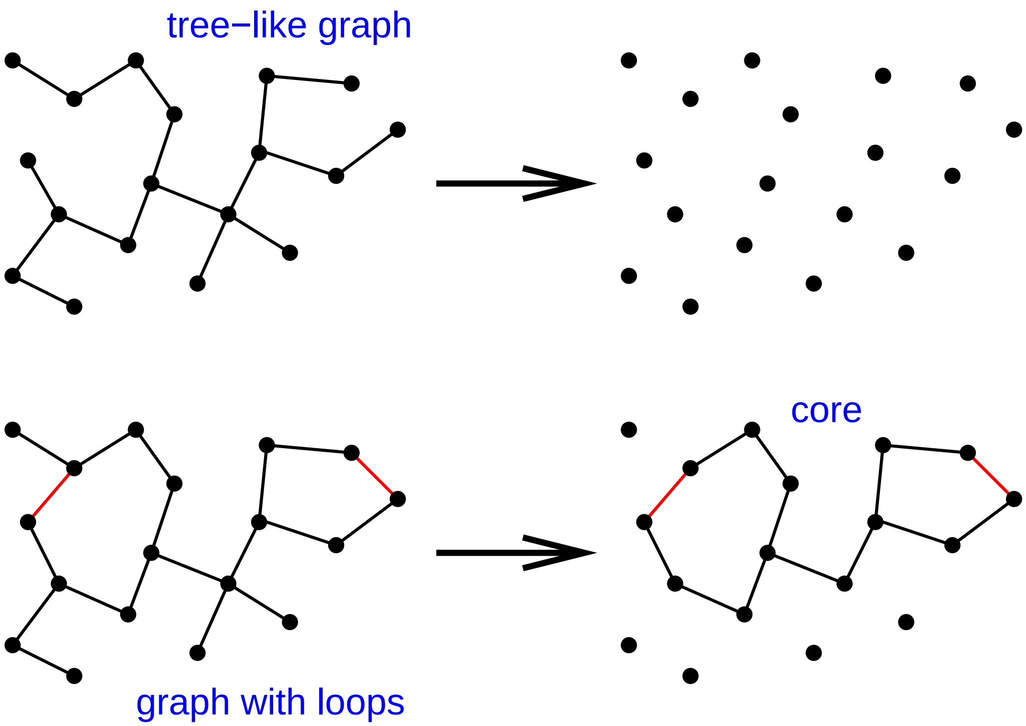
Here we have 8 solutions organized as follows



Clusters are of the same size!

Graph theory: Leaf removal

Rule: As long as there are nodes of connectivity 1 remove their (hyper-)links.

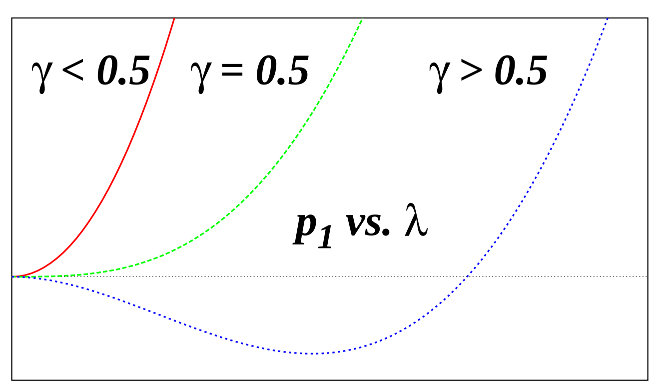


Eqs. for $p_1(t)$ and $p_k(t) = e^{-\lambda(t)} \frac{\lambda(t)^k}{k!}$ $(k \geq 2)$, with init. cond. $p_k(0) = e^{-2\gamma} \frac{(2\gamma)^k}{k!}$, give

$$\lambda(t) = 2\sqrt{\gamma(\gamma - t)} \quad p_1(t) = \lambda(t) \left(e^{-\lambda(t)} - 1 + \frac{\lambda(t)}{2\gamma} \right)$$

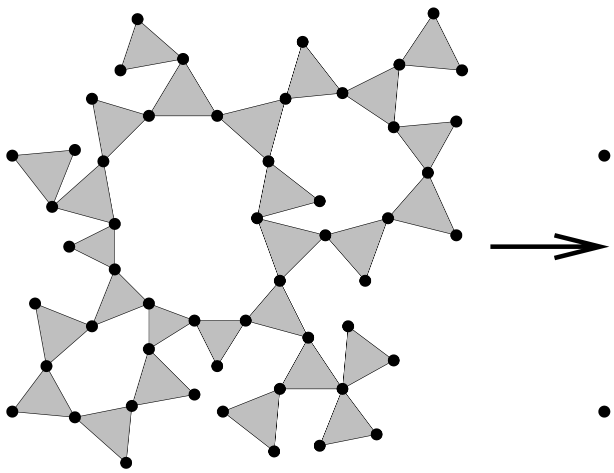
Second order transition at the percolation point $\gamma = \frac{1}{2}$

Core size $\propto (\gamma - \frac{1}{2})^2$



Leaf removal on hypergraphs

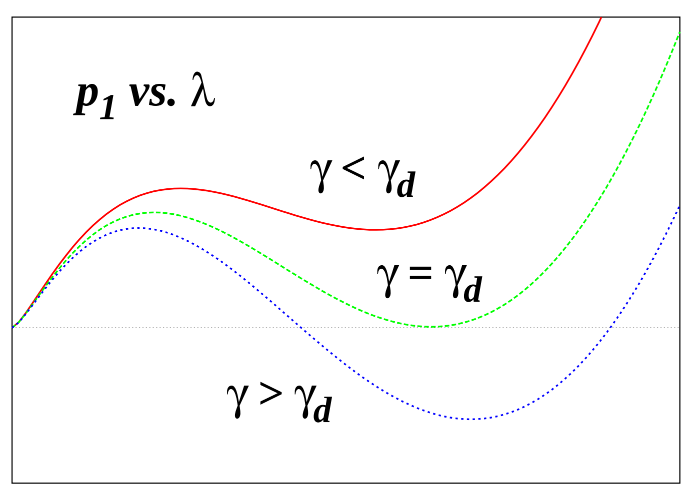
The percolation transition at $\gamma_p = \frac{1}{6}$ does not affect the leaf removal process, which can completely remove also percolating hypergraphs.



$$\lambda(t) = 3\sqrt[3]{\gamma(\gamma - t)^2} \qquad p_1(\lambda) = \lambda\left(e^{-\lambda} - 1 + \sqrt{\frac{\lambda}{3\gamma}}\right)$$

First order transition at $\gamma = \gamma_d = 0.818$

The core appears in a discontinuous way



Counting solutions in the core for $\gamma_d < \gamma < \gamma_c$

Once the leaf removal process stops, the core is made of N^* nodes and M^* hyperlinks:

$$\gamma^* \equiv \frac{M^*}{N^*} \qquad S^* \equiv \frac{N^*}{N}(1-\gamma^*)\log(2)$$

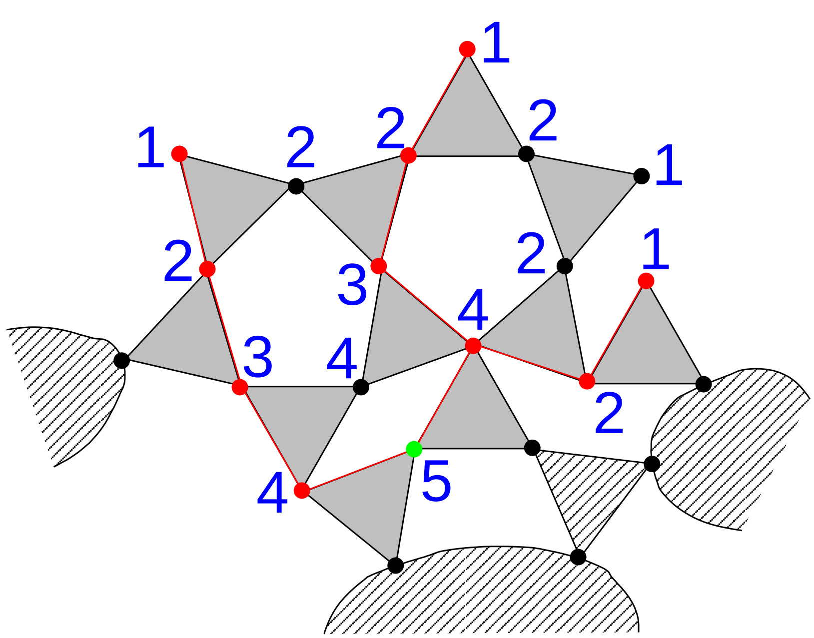
The number of solutions is $2^{N^*-M^*}$ and hyperloops are still exponentially rare.

$$S^*(\gamma) = \Sigma(\gamma)$$

The distances between solutions in the core are all of $\mathcal{O}(N)$. Moreover $\gamma_c \Leftrightarrow \gamma^* = 1$.

The intra-cluster entropy $s(\gamma)$ is given by the non-core part of the system (unfrozen spins).

A finite fraction of spins in the non-core part can be flipped by adjusting $\mathcal{O}(1)$ other spins.



Summary

- Spin models on random hypergraphs with finite connectivity present a very rich behavior, even at $T = 0$.
- The ferromagnetic versions have a glassy phase.
- $T = 0$ properties directly depend on the topology of the hypergraph (independently of the coupling for $\gamma < \gamma_c$).
- Critical points correspond to percolation transitions.
- RSB at $T = 0 \Leftrightarrow$ clustering.
- The configuration entropy can be obtain without replicas (and the results coincide!).

Some references

- F. Ricci-Tersenghi, M. Weigt and R. Zecchina, Phys. Rev. E **63**, 026702 (2001).
- M. Leone, F. Ricci-Tersenghi and R. Zecchina, J. Phys. A **34**, 4615 (2001).
- S. Franz, M. Mézard, F. Ricci-Tersenghi, M. Weigt and R. Zecchina, Europhys. Lett. 55, 465 (2001).
- S. Franz, M. Leone, F. Ricci-Tersenghi and R. Zecchina, preprint `cond-mat/0103328`, to appear in Phys. Rev. Lett.
- S. Franz, M. Leone, F. Ricci-Tersenghi and R. Zecchina, in preparation.